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Perfect powers in linear recurring sequences

by

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1. Introduction. Let $A_1, \ldots, A_k$ and $G_0, G_1, \ldots, G_{k-1}$ be algebraic numbers over the rationals and let (G_n) be a kth order linear *recurring sequence* given by

(1) $$G_n = A_1 G_{n-1} + \ldots + A_k G_{n-k} \quad \text{for } n = k, k+1, \ldots$$

Let $\alpha_1, \ldots, \alpha_t$ be the distinct roots of the corresponding characteristic polynomial

(2) $$X^k - A_1 X^{k-1} - \ldots - A_k.$$

Then for $n \geq 0$,

$$G_n = P_1(n)\alpha_1^n + P_2(n)\alpha_2^n + \ldots + P_t(n)\alpha_t^n,$$

where $P_i(n)$ is a polynomial with degree less than the multiplicity of α_i; the coefficients of $P_i(n)$ are elements of the field $\mathbb{Q}(G_0, \ldots, G_{k-1}, A_1, \ldots, A_k, \alpha_1, \ldots, \alpha_t)$.

The recurring sequence is called *simple* if all characteristic roots are simple. It is called *nondegenerate* if no quotient α_i/α_j for all $1 \leq i < j \leq t$ is equal to a root of unity, and *degenerate* otherwise. Observe that even if (G_n) is degenerate, there exists a positive integer d such that (G_{r+md}) is nondegenerate on each of the d arithmetic progressions with $0 \leq r < d$. Therefore, restricting to nondegenerate recurring sequences causes no substantial loss of generality.

In the present paper we deal with the Diophantine equation

(3) $$G_n = Ex^q, \qquad E \in \mathbb{Z}\backslash\{0\},$$

which was earlier investigated by several authors (e.g. cf. [21]).

For the Fibonacci sequence (F_n), Cohn [2] and Wyler [28] independently proved that F_n is a square only if $n = 0, 1, 2$ and 13. Cohn [3] and Steiner

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[24] solved the equations $F_n = 2x^2$ and $F_n = 3x^2$. They also proved the corresponding results for the Lucas sequence (L_n). London and Finkelstein [10] determined all cubes in the Fibonacci sequence; Lagarias and Weisser [9] gave another proof. Steiner [23] derived some partial results for higher powers. The proofs of these results do not depend on estimates for linear forms in logarithms. Pethő [14], [15] used the theory of linear forms in logarithms and computer calculations to determine all the cubes and fifth powers in the Fibonacci sequence.

For a nondegenerate recurring sequence (G_n) of order 2 induced by a (rational) integral recurrence, it has been proved independently by Pethő [13] and Shorey and Stewart [19] that for the solutions $x \in \mathbb{Z}, |x| > 1$ and $q \geq 2$ of (3), $\max(|x|, q, n)$ is bounded by an effectively computable constant depending only on E and on the sequence (G_n). In fact, Pethő proved that $\max(|x|, q, n)$ is bounded by an effectively computable number depending only on the greatest prime divisor of E and on the coefficients and initial values of (G_n) (provided that the coefficients are coprime integers). Pethő extended this result to the equation $G_n = bx^q$ with $b \in S$, where S is a set of integers composed solely of a finite number of primes.

Shorey and Stewart [19] proved the above finiteness result for certain recurring sequences of order > 2. Let (G_n) be a nondegenerate linear recurring sequence given by

$$G_n = \lambda_1 \alpha_1^n + P_2(n)\alpha_2^n + \ldots + P_t(n)\alpha_t^n, \tag{4}$$

where λ_1 is a nonzero constant, $|\alpha_1| > |\alpha_j|$ for $j = 2, \ldots, t$, and $G_n - \lambda_1\alpha_1^n \neq 0$. Then assuming $x, q > 1$ the solutions q of (3) can be bounded by an effectively computable constant which depends on the coefficients and initial values of the recurrence. Kiss [8] proved that, in fact, q is less than a number which is effectively computable in terms of the greatest prime divisor of E and the coefficients and the initial values of the sequence (G_n).

Shorey and Stewart [20] considered recurring sequences (G_n) satisfying (4). Assuming that $P_2(n)$ is a nonzero constant, $t = 3$ and $|\alpha_1| = |\alpha_2| = |\alpha_3|$, they showed that (3) has only finitely many integral solutions E, x, q and n with the greatest prime divisor of E bounded by a given positive integer P, and $|x| > 1$, $q > 2$ and $n \geq 0$.

Finkelstein [6], Williams [27] and Steiner [24] proved that $1, 2$ and 5 are the only Fibonacci numbers of the form $x^2 + 1$. Finkelstein [7] established a similar result for Lucas numbers. Stewart [25] and Shorey and Stewart [20] investigated the equation

$$G_n = x^q + c, \tag{5}$$

where $c \in \mathbb{Z}$ and (G_n) is a simple, nondegenerate second order recurring sequence of rational integers. Assuming $|A_2| = 1$, they showed for the integral solutions of (5) that the maximum of $n \geq 0, |x| > 1$ and $q \geq 3$ is less than an effectively computable constant depending on c, the coefficients and the initial values of the recurrence. In the case $q = 2$ they obtained a similar result under additional technical conditions. Furthermore, Shorey and Stewart [20] proved that if α_1 and α_2 are multiplicatively independent with one root inside the unit circle, then (5) has only finitely many solutions in integers n, x and q with $n \geq 0, |x| > 1$ and $q > 2$.

Nemes and Pethő [11], [12] studied the more general equation

$$G_n = Ex^q + T(x), \tag{6}$$

where $T(x)$ is a polynomial of degree r and of height H with integral coefficients. For fixed $E \in \mathbb{Z}$ and T they established bounds for the integral solutions n, q, x with $|x|, q > 1$. Let (G_n) be defined as in (4) and assume

$$|\alpha_1| > |\alpha_2| > |\alpha_j| \quad \text{for } j = 3, \ldots, t, \tag{7}$$

with $\alpha_2 \neq \pm 1$. Nemes and Pethő showed that $q < C_1$ provided that $n > C_2$ and $r < C_3 q$, where C_1, C_2 and C_3 are suitable positive numbers which are effectively computable in terms of E, H and the coefficients and initial values of the recurrence. Nemes and Pethő were also able to show that if q is a fixed integer larger than one and (6) has infinitely many integral solutions n and x, then $T(x)$ can be characterized in terms of the Chebyshev polynomials. Kiss [8] and Shorey and Stewart [20] dealt with equation (6) for nondegenerate linear recurring sequences (G_n) of arbitrary order, under condition (7) and the additional assumptions that $E = 1$ and d is the degree of α_1 over $\mathbb{Q}$, α_1 and α_2 are multiplicatively independent and $\alpha_2 \neq \pm 1$. They showed that there are then only finitely many integers n, x and q with $n \geq 0, |x| > 1$ and

$$q > \max\left(\frac{d \log|\alpha_1|}{\log(|\alpha_1|/\max(1, |\alpha_2|))}, d + r\right)$$

for which (6) holds.

Recently Corvaja and Zannier [4] considered linear recurrences defined by

$$G_n = a_1\alpha_1^n + \ldots + a_t\alpha_t^n,$$

where $t \geq 2, a_1, \ldots, a_t$ are nonzero rational numbers, $\alpha_1 > \ldots > \alpha_t > 0$ are integers. They used Schmidt's Subspace Theorem [16], [17] to show that for every integer $q \geq 2$ the equation

$$G_n = x^q \tag{8}$$

has only finitely many solutions $(n, x) \in \mathbb{N}^2$ assuming that G_n is not identically a perfect qth power for any n in a suitable arithmetic progression.

2. Results. Our first main result gives a quantitative version of the above result of Corvaja and Zannier [4].

THEOREM 1. *Let (G_n) be a linear recurring sequence defined by*

$$G_n = a_1\alpha_1^n + \ldots + a_t\alpha_t^n, \tag{9}$$

where $t \geq 2, a_1, \ldots, a_t$ are nonzero rational numbers, and $\alpha_1 > \ldots > \alpha_t > 0$ are integers such that for given $q \geq 2$ there is no $r \in \{0, \ldots, q-1\}$ with G_{mq+r} a perfect qth power for all $m \in \mathbb{N}$. Then the number of solutions $(n, x) \in \mathbb{N}^2$ of the equation

$$G_n = x^q$$

is finite and can be bounded above by an explicitly computable number depending on $q, a_1, \ldots, a_t, \alpha_1, \ldots, \alpha_t$.

REMARK 1. Corvaja and Zannier [4] showed that (9) is the qth power of an integer for infinitely many $n \in \mathbb{N}$ if and only if there exist integers $r \in \{0, \ldots, q-1\}$, $b \geq 1$ and

$$H_n = c_1\beta_1^n + \ldots + c_s\beta_s^n,$$

where $c_1, \ldots, c_s$ are nonzero rational numbers and $\beta_1 > \ldots > \beta_s > 0$ are integers as above, such that

$$G_n = b^{n-r}H_n^q.$$

In particular, at least one of the functions $m \mapsto G_{mq+r}$ $(r = 0, \ldots, q-1)$ is a qth power in the ring of complex functions (with pointwise multiplication) of the form (9), or G_n is a perfect qth power for any n in a suitable arithmetic progression.

REMARK 2. Observe that one can effectively determine whether G_{mq+r} is a perfect qth power or not (see again [4]). A sufficient condition is that α_1, α_2 are coprime.

REMARK 3. The example

$$G_n = 18^n + 2 \cdot 6^n + 2^n$$

shows that the condition in Theorem 1 that G_{mq+r} not be a perfect qth power for every m cannot be removed. Indeed, in this example the coefficients and roots of G_n satisfy the conditions of Theorem 1, but

$$G_{2m} = (18^m + 2^m)^2,$$

so G_{2m} is a perfect square for all $m \in \mathbb{N}$.

REMARK 4. The assumption $\alpha_1 > \ldots > \alpha_t > 0$ guarantees that the recurring sequence (G_n) is nondegenerate.

REMARK 5. We mention that the proof of Theorem 1 should also work in the case when (G_n) is a linear recurring sequence with algebraic characteristic roots $\alpha_1, \ldots, \alpha_t$, which are multiplicatively independent and satisfy

$$|\alpha_1| > |\alpha_i| \quad \forall i = 2, \ldots, t,$$

and with $a_i \in \mathbb{Q}(\alpha_1, \ldots, \alpha_t)$ for all $i = 1, \ldots, t$.

Our second main result extends Theorem 1 to the situation when also q is considered to be variable.

THEOREM 2. *Let (G_n) be a linear recurring sequence defined by*

$$G_n = a_1\alpha_1^n + \ldots + a_t\alpha_t^n,$$

where $a_1, \ldots, a_t$ $(t \geq 3)$ are nonzero rational numbers and $\alpha_1 > \ldots > \alpha_t > 0$ are integers such that (for fixed $q \geq 2$) there is no $r \in \{0, \ldots, q-1\}$ with G_{mq+r} a perfect qth power for all $m \in \mathbb{N}$. Then the equation

$$G_n = x^q$$

has only finitely many integral solutions $n, x > 1, q$. The number of solutions can be bounded by an explicitly computable constant C depending only on the recurrence.

REMARK 6. The assumption $t \geq 3$ means no loss of generality, because for $t = 2$ this theorem is already well known (see [13], [19]).

REMARK 7. Our proof of Theorem 2 depends on an application of the result of Nemes and Pethő [11] mentioned in the introduction.

REMARK 8. By Remark 5 and the theorem in [11] it should also be possible to obtain the following finiteness result: Let G_n be the nth term of a linear recurrence sequence defined by (9), where $t \geq 3, \alpha_1, \ldots, \alpha_t$ are multiplicatively independent algebraic numbers with

$$|\alpha_1| > |\alpha_2| > |\alpha_i|, \qquad \forall i = 3, \ldots, t,$$

and $a_i \in \mathbb{Q}(\alpha_1, \ldots, \alpha_t)$ for all $i = 1, \ldots, t$. Assuming that for fixed $q \geq 2$ there is no $r \in \{0, \ldots, q-1\}$ with G_{mq+r} a perfect qth power for all $m \in \mathbb{N}$, the equation

$$G_n = x^q$$

has only finitely many integral solutions $n, x > 1, q$.

3. Auxiliary results. Our proof of Theorem 1 depends on a quantitative version of the Subspace Theorem due to J.-H. Evertse [5].

Let K be an algebraic number field. Denote its ring of integers by O_K and its collection of places by M_K. For $v \in M_K$, $x \in K$, we define the absolute value $|x|_v$ by

(i) $|x|_v = |\sigma(x)|^{1/[K:\mathbb{Q}]}$ if v corresponds to the embedding $\sigma : K \hookrightarrow \mathbb{R}$;

(ii) $|x|_v = |\sigma(x)|^{2/[K:\mathbb{Q}]} = |\overline{\sigma}(x)|^{2/[K:\mathbb{Q}]}$ if v corresponds to the pair of conjugate complex embeddings $\sigma, \overline{\sigma} : K \hookrightarrow \mathbb{C}$;

(iii) $|x|_v = (N\wp)^{-\mathrm{ord}_\wp(x)/[K:\mathbb{Q}]}$ if v corresponds to the prime ideal $\wp$ of O_K.

Here $N\wp = \#(O_K/\wp)$ is the norm of $\wp$ and $\mathrm{ord}_\wp(x)$ the exponent of $\wp$ in the prime ideal composition of (x), with $\mathrm{ord}_\wp(0) := \infty$. In cases (i) or (ii) we call v *real infinite* or *complex infinite*, respectively; in case (iii) we call v *finite*. These absolute values satisfy the *product formula*

$$\prod_{v\in M_K} |x|_v = 1 \quad \text{for } x \in K^*. \tag{10}$$

The *height* of $\mathbf{x} = (x_1, \ldots, x_n) \in K^n$ with $\mathbf{x} \neq \mathbf{0}$ is defined as follows: for $v \in M_K$ put

$$|\mathbf{x}|_v = \begin{cases} \Big(\sum_{i=1}^{n} |x_i|_v^{2[K:\mathbb{Q}]}\Big)^{1/(2[K:\mathbb{Q}])} & \text{if } v \text{ is real infinite,} \\ \Big(\sum_{i=1}^{n} |x_i|_v^{[K:\mathbb{Q}]}\Big)^{1/[K:\mathbb{Q}]} & \text{if } v \text{ is complex infinite,} \\ \max(|x_1|_v, \ldots, |x_n|_v) & \text{if } v \text{ is finite} \end{cases}$$

(note that for infinite places v, $|\cdot|_v$ is a power of the Euclidean norm). Now define

$$\mathcal{H}(\mathbf{x}) = \mathcal{H}(x_1, \ldots, x_n) = \prod_v |\mathbf{x}|_v.$$

For a linear form $l(\mathbf{X}) = a_1X_1 + \ldots + a_nX_n$ with algebraic coefficients we define $\mathcal{H}(l) := \mathcal{H}(\mathbf{a})$, where $\mathbf{a} = (a_1, \ldots, a_n)$, and if $\mathbf{a} \in K^n$ then we put $|l|_v = |\mathbf{a}|_v$ for $v \in M_K$. Further we define the number field $K(l) := K(a_1/a_j, \ldots, a_n/a_j)$ for any j with $a_j \neq 0$; this is independent of the choice of j.

We are now ready to state Evertse's result [5]. The following notation is used:

- S is a finite set of places on K of cardinality s containing all infinite places;
- $\{l_{1v}, \ldots, l_{nv}\}$, $v \in S$, are linearly independent sets of linear forms in n variables with algebraic coefficients such that

$$\mathcal{H}(l_{iv}) \le H, \quad [K(l_{iv}) : K] \le D \quad \text{for } v \in S,\ i = 1, \ldots, n.$$

For every place $v \in M_K$ we choose a continuation of $|\cdot|_v$ to the algebraic closure of K and denote it also by $|\cdot|_v$.

Theorem 3 (Quantitative Subspace Theorem, Evertse). *Let* $0 < \delta < 1$ *and for* $\mathbf{x} \in K^n$ *consider the inequality*

$$\prod_{v\in S}\prod_{i=1}^{n} \frac{|l_{iv}(\mathbf{x})|_v}{|\mathbf{x}|_v} < \Big(\prod_{v\in S} |\det(l_{1v}, \ldots, l_{nv})|_v\Big) \cdot \mathcal{H}(\mathbf{x})^{-n-\delta}. \tag{11}$$

Then the following assertions hold:

(i) *There are proper linear subspaces* $T_1, \ldots, T_{t_1}$ *of* K^n *with*

$$t_1 \le (2^{60n^2}\delta^{-7n})^s \log 4D \cdot \log\log 4D$$

such that every solution $\mathbf{x} \in K^n$ *of* (11) *satisfying* $\mathcal{H}(\mathbf{x}) \geq H$ *belongs to* $T_1 \cup \ldots \cup T_{t_1}$.

(ii) *There are proper linear subspaces* $S_1, \ldots, S_{t_2}$ *of* K^n *with*

$$t_2 \leq (150n^4\delta^{-1})^{ns+1}(2 + \log\log 2H)$$

such that every solution $\mathbf{x} \in K^n$ *of* (11) *satisfying* $\mathcal{H}(\mathbf{x}) < H$ *belongs to* $S_1 \cup \ldots \cup S_{t_2}$.

We also need the following theorem of W. M. Schmidt [18] concerning the zero multiplicity of a nondegenerate recurring sequence.

THEOREM 4 (W. M. Schmidt). *Suppose that* $(G_n)_{n\in\mathbb{Z}}$ *is a nondegenerate linear recurring sequence of complex numbers whose characteristic polynomial has* k *distinct roots of multiplicity* $\leq a$. *Then the number of solutions* $n \in \mathbb{Z}$ *of the equation*

$$G_n = 0$$

can be bounded above by

$$c(k,a) = e^{(7k^a)^{8k^a}}.$$

(*This number of solutions is called the zero multiplicity of the recurrence.*)

In the proof of Theorem 2 we also apply the following result due to A. Baker [1]. Let us first recall the definition of the absolute logarithmic Weil height: For an algebraic number β let $P_\beta(x) = x^k + a_{k-1}x^{k-1} + \ldots + a_0 \in \mathbb{Z}[x]$ denote the minimal polynomial of β. Further let $\beta_1 = \beta, \beta_2, \ldots, \beta_k$ denote the conjugates of β. Then we call

$$h(\beta) = \frac{1}{k}\log\Big(\prod_{i=1}^{k}\max\{1, |\beta_i|\}\Big)$$

the *absolute logarithmic Weil height* of β.

THEOREM 5 (A. Baker). *Let* $\alpha_1, \ldots, \alpha_k$ *be algebraic numbers, different from* 0 *or* 1, $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_k)$, *and let* d *be the degree* $[K : \mathbb{Q}]$. *For* $i = 1, \ldots, k$ *set*

$$h_i = \max\left\{h(\alpha_i), \frac{e|\log\alpha_i|}{d}, \frac{1}{d}\right\}.$$

Let $b_1, \ldots, b_k \in \mathbb{Z}$, $b_k > 0$, $\Lambda = b_1\log\alpha_1 + \ldots + b_k\log\alpha_k \neq 0$ *and* $B = \max\{2, |b_1|, \ldots, |b_{k-1}|\}$. *Then*

$$\log|\Lambda| > -C(k)d^{k+2}h_1\ldots h_k\log(C(k)d^{k+2}h_1\ldots h_{k-1})\log b_k - \frac{B}{b_k}, \tag{12}$$

where

$$C(k) = 2^{26k}k^{3k}.$$

A proof of this theorem with the given explicit constants can be found in the monograph of Waldschmidt [26, p. 309, Corollary 9.24].

Below we have collected some simple lemmas which are needed in our proofs.

Lemma 1. *Let $N_{j,k}$ denote the number of formal summands of $(a_1+\ldots+a_k)^j$, where $a_1,\ldots,a_k$ denote formal commuting variables. Then*

$$N_{j,k}=\binom{k+j-1}{j}.$$

This is well known from combinatorics.

Lemma 2. *Let d be a positive integer. Then for the complex function $f(z)=(1+z)^{1/d}$ we have*

$$\left|f(z)-\sum_{k=0}^{n}\binom{1/d}{k}z^k\right|\leq\frac{1}{d(n+1)(1-|z|)}\cdot|z|^{n+1}$$

for $z\in\mathbb{C}$, $|z|<1$, where we have chosen the branch of $(1+z)^{1/d}$ which is holomorphic on $\mathbb{C}\setminus(-\infty,-1]$ and equal to the positive dth root of $1+z$ for $z\in\mathbb{R}$, $z>-1$.

Proof. It is well known that for $z\in\mathbb{C}$, $|z|<1$ we have

$$f(z)=\sum_{k=0}^{\infty}\binom{1/d}{k}z^k.$$

From

$$(n+1)\left|\binom{1/d}{n+1}\right|=\frac{\frac{1}{d}\cdot\left(1-\frac{1}{d}\right)\cdot\ldots\cdot\left(n-\frac{1}{d}\right)}{1\cdot\ldots\cdot n}<\frac{1}{d}<1$$

we obtain

$$\begin{aligned}\left|f(z)-\sum_{k=0}^{n}\binom{1/d}{k}z^k\right|&\leq\sum_{k=n+1}^{\infty}\left|\binom{1/d}{k}\right|\cdot|z|^k\\&\leq\frac{1}{d(n+1)}\sum_{k=n+1}^{\infty}|z|^k\\&=\frac{1}{d(n+1)(1-|z|)}\cdot|z|^{n+1},\end{aligned}$$

and therefore the proof is complete. ∎

Lemma 3. *Let $a,b\geq 0$ and let $x\in\mathbb{R}$ be the largest solution of $x=a+b\log x$. If $b>e^2$ then*

$$x<2(a+b\log b).$$

This lemma is due to A. Pethő and B. M. M. de Weger [22].

4. Proof of Theorem 1. According to Theorem 4 the number of solutions of (8) of the form $(n,0)$, $n\in\mathbb{N}$, can be estimated by

$$c_1(t)\le e^{(7t)^{8t}}.$$

Therefore we can restrict ourselves to solutions of the form $(n,x)\in\mathbb{N}^2$ with $x\neq 0$. These solutions are denoted by $(n,x_n)\in\mathbb{N}^2$ with $n\in\Sigma$, where Σ is a set of positive integers.

Let us now consider the expansion of the function $f(z)=(1+z)^{1/q}$ around the origin,

$$(1+z)^{1/q}=\sum_{j=0}^{\infty}\binom{1/q}{j}z^j \qquad \text{with } |z|\le 1, z\neq -1.$$

We approximate $G_n^{1/q}$ by defining

$$H_m := (a_1\alpha_1^r)^{1/q}\alpha_1^m\left[1+\sum_{j=1}^{R}\binom{1/q}{j}\cdot\left(\sum_{i=2}^{t}\frac{a_i\alpha_i^{mq+r}}{a_1\alpha_1^{mq+r}}\right)^j\right],$$

where $R\ge 1$ is an integer to be chosen later and where we have set $n=mq+r$ with $n\in\mathbb{N}$, $r\in\{0,\dots,q-1\}$. We write

$$H_m=\sum_{i=1}^{h}d_i\left(\frac{e_i}{b}\right)^m,$$

where $d_i\in\mathbb{Q}((a_1\alpha_1^r)^{1/q})^*$, e_i, b are integers, $b>0$, and the e_i/b are nonzero distinct rational numbers. Clearly, H_m is nondegenerate (the roots are all positive) and we have

$$[\mathbb{Q}((a_1\alpha_1^r)^{1/q}):\mathbb{Q}]\le q.$$

By Lemma 1 we obtain

$$h\le\binom{R+t-1}{R}.$$

On the other hand, we have

$$\left|\sum_{i=2}^{t}\frac{a_i\alpha_i^{mq+r}}{a_1\alpha_1^{mq+r}}\right|\le (t-1)c\left(\frac{\alpha_2}{\alpha_1}\right)^{mq+r}\le\frac{1}{2}<1$$

where

$$c:=\max\{|a_i/a_1|\mid i=2,\dots,t\},$$

if

$$m\ge\frac{\log 2(t-1)c}{q\log\frac{\alpha_1}{\alpha_2}}. \tag{13}$$

Therefore, by Lemma 2 we get, for m large enough,

$$
\begin{aligned}
|H_m - x_{mq+r}| &= |G_{mq+r}^{1/q} - H_m| \\
&\le |a_1^{1/q}| \cdot \alpha_1^{r/q} \cdot \alpha_1^m \cdot \frac{1}{q(R+1)\left(1 - \left|\sum_{i=2}^{t} \frac{a_i \alpha_i^{mq+r}}{a_1 \alpha_1^{mq+r}}\right|\right)} \left|\sum_{i=2}^{t} \frac{a_i \alpha_i^{mq+r}}{a_1 \alpha_1^{mq+r}}\right|^{R+1} \\
&\le |a_1^{1/q}| \cdot \alpha_1 \cdot \alpha_1^m \cdot \frac{2}{q(R+1)} \left[(t-1)c\left(\frac{\alpha_2}{\alpha_1}\right)^r \left(\frac{\alpha_2}{\alpha_1}\right)^{mq}\right]^{R+1} \\
&\le |a_1^{1/q}| \cdot \alpha_1 \cdot [(t-1)c]^{R+1} \cdot \alpha_1^m \cdot \left(\frac{\alpha_2}{\alpha_1}\right)^{mq(R+1)}.
\end{aligned}
$$

Thus we derive

$$
|H_m - x_{mq+r}| \le c_2(R) l_1^m \tag{14}
$$

where we have set

$$
l_1 := \alpha_1 \left(\frac{\alpha_2}{\alpha_1}\right)^{q(R+1)} \quad \text{and} \quad c_2(R) := |a_1^{1/q}| \alpha_1 [(t-1)c]^{R+1}.
$$

Now choose

$$
R > \max\left\{1, \frac{1}{q} \cdot \frac{\log \alpha_1}{\log \frac{\alpha_1}{\alpha_2}} - 1\right\}. \tag{15}
$$

Then $0 < l_1 < 1$. Put

$$
l := \frac{l_1 + 1}{2}.
$$

Then for m large enough, to be more precise, for

$$
m > \frac{\log c_2(R)}{\log \frac{l}{l_1}} \tag{16}
$$

we have

$$
c_2(R) l_1^m = c_2(R) \left(\frac{l_1}{l}\right)^m l^m < l^m.
$$

Consequently, we obtain

$$
|H_m - x_{mq+r}| < l^m \quad \text{with } 0 < l < 1, \tag{17}
$$

provided that R satisfies (15) and m satisfies (13) and (16).

Now let S be the set of places of $\mathbb{Q}$ consisting of ∞ and all primes dividing some of the e_i or b. Extend each place in S to $K := \mathbb{Q}((a_1\alpha_1^r)^{1/q})$ in some way, the infinite place being extended so as to coincide with the complex absolute value in the given embedding of K in $\mathbb{C}$. Define the linear forms $L_{i,v}$ for $v \in S$ and $i = 1, \ldots, h$ as follows: $L_{0,\infty} := L := X_0 - \sum_{i=1}^h d_i X_i$, $L_{i,\infty} := X_i$ for $i = 1, \ldots, h$, while for $v \in S$, $v \neq \infty$, put $L_{i,v} := X_i$ for $i = 1, \ldots, h$. Then $\{L_{0,v}, \ldots, L_{h,v}\}$, $v \in S$, are linearly independent sets of linear forms in $h+1$ variables with coefficients in K. Furthermore we have

$$\mathcal{H}(L_{i,v}) = \begin{cases} \mathcal{H}(1,-d_1,\dots,-d_h) =: \widetilde{H} & \text{for } i=0,\ v=\infty, \\ \mathcal{H}(0,\dots,0,1,0,\dots,0) = 1 & \text{else.} \end{cases}$$

We set $H := \max\{1,\widetilde{H}\}$. Then it follows $\mathcal{H}(L_{i,v}) \le H$, for $v \in S$, $i = 0,\dots,h$. Hence $\mathbb{Q}(L_{i,v}) = \mathbb{Q}$ for $v \neq \infty$, $[\mathbb{Q}(L_{0,\infty}):\mathbb{Q}] \le q$ and therefore

$$[\mathbb{Q}(L_{i,v}):\mathbb{Q}] \le q \qquad \forall v \in S,\ i = 0,\dots,h.$$

For $n \in \Sigma$ define the vector $\mathbf{x}_m = (b^m x_{mq+r}, e_1^m, \dots, e_h^m) \in \mathbb{Z}^{h+1}$.

From (17) we obtain $|L_{0,\infty}(\mathbf{x}_m)| \le (bl)^m$. Recall that S includes all primes dividing b and that the x_{mq+r} are integers. Thus by the product formula (10),

$$\prod_{v\in S\setminus\{\infty\}} |L_{0,v}(\mathbf{x}_m)|_v = \prod_{v\in S\setminus\{\infty\}} |b^m x_{mq+r}|_v \le \prod_{v\in S\setminus\{\infty\}} |b^m|_v = b^{-m}.$$

Moreover, since S also includes the primes dividing the numbers e_i, the product formula (10) gives

$$\prod_{v\in S}\prod_{i=1}^{h} |L_{i,v}(\mathbf{x}_m)|_v = \prod_{v\in S}\prod_{i=1}^{h} |e_i^m|_v = 1.$$

Thus we obtain

$$\prod_{v\in S}\prod_{i=0}^{h} \frac{|L_{i,v}(\mathbf{x}_m)|_v}{|\mathbf{x}_m|_v} \le \Big(\prod_{v\in S} |\mathbf{x}_m|_v\Big)^{-h-1} \cdot l^m.$$

Since the coordinates of the vectors $\mathbf{x}_m$ are integers we have $|\mathbf{x}_m|_v \le 1$ for $v \in M_{\mathbb{Q}}\setminus\{\infty\}$. Further, we have

$$|\mathbf{x}_m|_\infty \le A^m$$

for some real A independent of m. Indeed, we have

$$|x_{mq+r}| \le |x_{mq+r} - H_m| + |H_m| \le l^m + h\widetilde{c}a^m \le 1 + h\widetilde{c}a^m \le \widetilde{a}^m$$

with

$$\widetilde{c} := \max\{|d_i| \mid i = 1,\dots,h\}, \qquad a := \max\{|e_i/b| \mid i = 1,\dots,h\},$$

and $\widetilde{a} := (1+h\widetilde{c})(1+a)$. Hence

$$|\mathbf{x}_m|_\infty = \Big(|b^m x_{mq+r}|^2 + \sum_{i=1}^{h} |e_i|^2\Big)^{1/2} \le ((b\widetilde{a})^{2m} + h(ba)^{2m})^{1/2} \le A^m$$

with $A := (h+1)\widetilde{a}b$. It follows that

$$\mathcal{H}(\mathbf{x}_m) = \prod_{v\in M_{\mathbb{Q}}} |\mathbf{x}_m|_v \le \prod_{v\in S} |\mathbf{x}_m|_v \le |\mathbf{x}_m|_\infty \le A^m.$$

Lastly we have

$$\det(L_{0,v},\dots,L_{h,v}) = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ * & 1 & 0 & \dots & 0 \\ * & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ * & 0 & 0 & \dots & 1 \end{vmatrix} = 1,$$

which yields

$$|\det(L_{0,v},\dots,L_{h,v})|_v = 1 \qquad \forall v \in S.$$

Combining our estimates we get

$$\prod_{v\in S}\prod_{i=0}^{h}\frac{|L_{i,v}(\mathbf{x}_m)|_v}{|\mathbf{x}_m|_v} < \Big(\prod_{v\in S}|\det(L_{0,v},\dots,L_{h,v})|_v\Big)\mathcal{H}(\mathbf{x}_m)^{-h-1-\delta}$$

for all m with (16), provided that $\delta < \log(1/l)/\log A$. By Theorem 3 there exist finitely many nonzero rational linear forms $\Lambda_1(X_0,\dots,X_h),\dots$ $\dots,\Lambda_g(X_0,\dots,X_h)$ with

$$g \le (2^{60(h+1)^2}\delta^{-7(h+1)})^s \log 4q \cdot \log\log 4q \\ + (150(h+1)^4\delta^{-1})^{(h+1)s+1}(2+\log\log 2H),$$

such that each vector $\mathbf{x}_m$ is a zero of some Λ_j.

Suppose first Λ_j does not depend on X_0. Then, if $\Lambda_j(\mathbf{x}_m)=0$, we have a nontrivial relation

$$\sum_{i=1}^{h} u_i\left(\frac{e_i}{b}\right)^m = 0, \qquad u_i\in\mathbb{Q},\ i=1,\dots,h.$$

By Theorem 4 this can hold for at most a finite number of m. More precisely, the number of solutions m can be estimated by

$$c_1(h) = e^{(7h)^{8h}},$$

since (H_m) is nondegenerate.

Suppose that Λ_j depends on X_0 and that $\Lambda_j(\mathbf{x}_m)=0$. Then we have

$$\sum_{i=1}^{h} v_i\left(\frac{e_i}{b}\right)^m = x_{mq+r}, \qquad v_i\in\mathbb{Q},\ i=1,\dots,h. \tag{18}$$

Set

$$U_m := \sum_{i=1}^{h} v_i\left(\frac{e_i}{b}\right)^m;$$

then U_m is a nondegenerate, simple recurring sequence and we obtain

$$U_m^q = x_{mq+r}^q = G_{mq+r}.$$

Hence

$$V_m := \left(\sum_{i=1}^{h} v_i \left(\frac{e_i}{b}\right)^m\right)^q - \sum_{i=1}^{t} a_i \alpha_i^r (\alpha_i^q)^m$$

has the form

$$V_m = \sum_{i=1}^{p} b_i \beta_i^m$$

with $b_i \in \mathbb{Q}$, $\beta_i \in \mathbb{Q}^+$, $i = 1, \dots, p$. Therefore V_m is a nondegenerate, simple recurring sequence, and we conclude by Lemma 1 that

$$p \le t + \binom{h+q-1}{q} \le t + \binom{\binom{R+t-1}{t} + q - 1}{q}.$$

Observe that by our assumptions $V_m = 0$ does not hold for every m hence an i with $b_i \neq 0$ exists. Again by Theorem 4 we can bound the number of solutions of (18) by

$$c_1(p) = e^{(7p)^{8p}}.$$

Therefore the number of solutions of (8) can be estimated by

$$\begin{aligned}\widetilde{C}(q) := e^{(7t)^{8t}} + \frac{\log 2(t-1)c}{\log \frac{\alpha_1}{\alpha_2}} + q\Bigg[&\{(2^{60(\widetilde{h}+1)^2}\delta^{-7(\widetilde{h}+1)})^s \\ &\times \log 4q \cdot \log\log 4q + (150(\widetilde{h}+1)^4\delta^{-1})^{(\widetilde{h}+1)s+1}(2+\log\log 2H)\} \\ &\times \{e^{(7\widetilde{h})^{8\widetilde{h}}} + e^{(7\widetilde{p})^{8\widetilde{p}}}\} + \frac{\log c_2(R)}{\log \frac{l}{l_1}}\Bigg],\end{aligned}$$

where

$$\begin{aligned}&\widetilde{h} = \binom{R+t-1}{R}, \qquad \widetilde{p} = \binom{\widetilde{h}+q-1}{q} + t, \\ &H = \max\{1, \mathcal{H}(1, -d_1, \dots, -d_h)\}, \qquad s = |S|, \\ &c_2(R) = |a_1^{1/q}|\alpha_1[(t-1)c]^{R+1}, \qquad \delta < \log(1/l)/\log A,\end{aligned}$$

and l is as in (17). ∎

5. Proof of Theorem 2. Assume that $n, x > 1, q \ge 2$ is a solution of (8) and write

$$x^q = G_n = a_1\alpha_1^n + B(n).$$

We distinguish two cases:

CASE 1: $B(n) = 0$. Here we get

$$|B(n)| = \left|a_2\alpha_2^n\left(1 + \sum_{i=3}^{t} \frac{a_i}{a_2}\left(\frac{\alpha_i}{\alpha_2}\right)^n\right)\right| \ge |a_2|\alpha_2^n\left|1 - \left|\sum_{i=3}^{t} \frac{a_i}{a_2}\left(\frac{\alpha_i}{\alpha_2}\right)^n\right|\right| > 0,$$

since

$$\left|\sum_{i=3}^{t}\frac{a_i}{a_2}\left(\frac{\alpha_i}{\alpha_2}\right)^n\right| \le \max\left\{\left|\frac{a_i}{a_2}\right| \mid i=3,\dots,t\right\}\cdot(t-2)\left(\frac{\alpha_3}{\alpha_2}\right)^n$$
$$\le tc|a_2|^{-1}\left(\frac{\alpha_3}{\alpha_2}\right)^n < 1,$$

where $c=\max\{|a_i| \mid i=1,\dots,t\}$, whenever

$$n > \frac{\log(tc|a_2|^{-1})}{\log\frac{\alpha_2}{\alpha_3}} =: n_1.$$

Therefore $n \le n_1$ must hold and we deduce from $a_1\alpha_1^n = x^q$ and $x \ge 2$ that

$$q = \frac{\log(|a_1|\alpha_1^n)}{\log x} \le \frac{\log(c\alpha_1^{n_1})}{\log 2}.$$

Case 2: $B(n) \neq 0$. In this case we first set

$$\delta := \frac{1}{2}\left(1-\frac{\log\alpha_2}{\log\alpha_1}\right).$$

Then we get

$$|B(n)| \le tc\alpha_2^n < \frac{1}{2}\alpha_1^{n(1-\delta)}$$

if

$$n > \frac{2\log(2ct)}{\log\frac{\alpha_1}{\alpha_2}} =: n_2.$$

Further

$$\frac{x^q}{a_1\alpha_1^n} = 1+\frac{B(n)}{a_1\alpha_1^n}, \tag{19}$$

so

$$1-(|a_1|\alpha_1^{\delta n})^{-1} \le |a_1|^{-1}\alpha_1^{-n}x^q \le 1+(|a_1|\alpha_1^{\delta n})^{-1}, \tag{20}$$

where we have used the fact that $(|a_1|\alpha_1^{\delta n})^{-1} < 1/2$ if

$$n > \frac{\log(2|a_1|^{-1})}{\delta\log\alpha_1} =: n_3.$$

Taking logarithms and using the inequalities

$$|\log(1+x)| \le x \quad\text{ and }\quad |\log(1-x)| \le 2x \quad\text{ for } 0 \le x < 1/2,$$

from (20) we derive

$$-2|a_1|^{-1}\alpha_1^{-\delta n} \le -\log|a_1| - n\log\alpha_1 + q\log x \le 2|a_1|^{-1}\alpha_1^{-\delta n}.$$

Thus

$$|-\log|a_1| - n\log\alpha_1 + q\log x| \le 2|a_1|^{-1}\alpha_1^{-\delta n}. \tag{21}$$

Put $\Lambda = -\log|a_1| - n\log\alpha_1 + q\log x$. From (19) and the fact that $B(n) \neq 0$, we get $\Lambda \neq 0$. Thus we can employ Theorem 5 to obtain, for $n \geq 2$,

$$|\Lambda| > \exp\left\{-C(3)h_1h_2\log\left(C(3)h_1h_2\right)e\log x\log q - \frac{n}{q}\right\}, \tag{22}$$

where $C(3) = 2^{78}3^9$ and

$$h_1 = \max\{h(|a_1|^{-1}), e\,|\log|a_1||, 1\},$$
$$h_2 = \max\{h(\alpha_1), e\log\alpha_1, 1\} = e\log\alpha_1.$$

Set

$$c_3 := C(3)h_1h_2\log(C(3)h_1h_2)e.$$

A comparison of (21) and (22) reveals that

$$-c_3\log q\log x - \frac{n}{q} < \log(2|a_1|^{-1}) - n\delta\log\alpha_1. \tag{23}$$

However, for $n > \max\{n_1, n_2\}$,

$$\frac{1}{2}|a_1|\alpha_1^n \leq |a_1|\alpha_1^n - |B(n)| \leq x^q \leq |a_1|\alpha_1^n + |B(n)| \leq ct\alpha_1^n.$$

Thus, for

$$n > \max\left\{\frac{\log ct}{\log\alpha_1}, \frac{2\log(2|a_1|^{-1})}{\log\alpha_1}\right\} =: n_4$$

we obtain

$$\frac{\log\alpha_1}{2}n < q\log x < 2\log\alpha_1 n.$$

If we write this as

$$\frac{n}{q} < \frac{2\log x}{\log\alpha_1}, \qquad \frac{\log x}{2\log\alpha_1}q < n,$$

then (23) can be reformulated as

$$q < \frac{2\log\left(\frac{1}{2}|a_1|^{-1}\right)}{\delta\log 2} + \frac{4}{\delta\log\alpha_1} + \frac{2c_3}{\delta}\log q.$$

Thus by Lemma 3,

$$q < 2\left(\frac{2\log\left(\frac{1}{2}|a_1|^{-1}\right)}{\delta\log 2} + \frac{4}{\delta\log\alpha_1} + \frac{2c_3}{\delta}\log\left(\frac{2c_3}{\delta}\right)\right) =: C_1$$

if $n > \max\{2, n_2, n_3, n_4\} =: n_5$. Otherwise, we have

$$q \leq \frac{\log(ct\alpha_1^{n_5})}{\log 2}.$$

Altogether we derive

$$q \leq \max\left\{\frac{\log(ct\alpha_1^{\bar{c}})}{\log 2}, C_1(q)\right\} =: C,$$

where $\bar{c} := \max\{2, n_1, n_2, n_3, n_4\}$. For the number of solutions $n, x > 1, q \geq 2$ of (8) we finally obtain the upper bound

$$\sum_{q=2}^{C} \widetilde{C}(q),$$

and therefore the proof is finished. ∎

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