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The real $3x+1$ problem

by

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1. Introduction. First of all, let us fix some notations. Put

$$\mathbb{R}_1 = \{x \in \mathbb{R} : x \geq 1\}, \quad \mathbb{Q}_1 = \mathbb{Q} \cap \mathbb{R}_1, \quad \mathbb{N}_1 = \mathbb{Z} \cap \mathbb{R}_1 = \{1, 2, 3, \ldots\},$$
$$\mathbb{R}_0 = \{x \in \mathbb{R} : x \geq 0\}, \quad \mathbb{Q}_0 = \mathbb{Q} \cap \mathbb{R}_0, \quad \mathbb{N}_0 = \mathbb{Z} \cap \mathbb{R}_0 = \{0, 1, 2, \ldots\}.$$

For $x \in \mathbb{R}$, $\lfloor x \rfloor$ will denote the *floor* or *integer part* of x, that is to say, $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$.

The well known $3n+1$ *function* (see, e.g., [8] and [10]) is the function $T : \mathbb{N}_1 \to \mathbb{N}_1$ given by

$$T(n) = \begin{cases} T_0(n) = n/2 & \text{if } n \text{ is even}, \\ T_1(n) = (3n+1)/2 & \text{if } n \text{ is odd}. \end{cases} \tag{1}$$

In this work, we introduce an extension of T, namely the function $U : \mathbb{R}_1 \to \mathbb{R}_1$ defined by

$$U(x) = \begin{cases} U_0(x) = x/2 & \text{if } \lfloor x \rfloor \text{ is even}, \\ U_1(x) = (3x+1)/2 & \text{if } \lfloor x \rfloor \text{ is odd}. \end{cases} \tag{2}$$

Note that $U|_{\mathbb{N}_1}$ is indeed T. We shall call U the *real* $3x+1$ function (in contrast to the *integer* $3n+1$ function T). In Section 2, we propose a conjecture about the iterates of U that generalizes the famous $3n+1$ conjecture. We then prove our main result about the iterates of U (Theorem 2.1), which is directly related to both of these conjectures. We also introduce the *flipped* $3x+1$ function $\widetilde{U}$ and prove an analogous result for its iterates. In Section 3, we show a couple of simple propositions about the iterates of U and $\widetilde{U}$, introduce other related functions and propose some questions and conjectures about their iterates.

We hope that the results, conjectures and questions stated here will be not only relevant to the $3n+1$ conjecture itself, but also of interest in their own right. Some of the results presented here already appear in the literature. In those cases, we refer the reader to their proofs. However, for

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the reader's benefit, we recall some well known definitions (in a form slightly better suited to our purposes).

2. The conjecture and the main results. Given a (nonempty) set X and a function $f : X \to X$, the *iterates* of f will be denoted by f^i ($i \in \mathbb{N}_0$). They are defined by $f^0 = \mathrm{id}_X$ (the identity function on X) and $f^i = f \circ f^{i-1}$ for $i > 0$. For any given $x \in X$, the *f-trajectory of x* or *starting at x* is the sequence $\mathcal{T}_f(x) = (f^i(x))_{i=0}^{\infty}$. An *$f$-periodic trajectory* or, simply, an *f-cycle* is the f-trajectory of some $z \in X$ such that $f^n(z) = z$ for some $n \in \mathbb{N}_1$ (in this case, the f-cycles starting at $f^k(z)$, $k \in \mathbb{N}_0$, will sometimes be considered as being one and the same f-cycle). By an *f-cycle of length $l \in \mathbb{N}_1$* we mean any sequence in the set $\{(x, f(x), \ldots, f^l(x)) : x \in X,\ f^l(x) = x\}$.

Now, let $\mathbb{Q}[(2)]$ denote the set of all rational numbers having an odd denominator when written in lowest terms (see [5]). A number $a/b \in \mathbb{Q}[(2)]$ (with b odd) is *even* (resp. *odd*) if its numerator a is even (resp. odd). The *rational Collatz sequence generated* by $r_0 \in \mathbb{Q}[(2)]$ is the g-trajectory of r_0, where $g : \mathbb{Q}[(2)] \to \mathbb{Q}[(2)]$ is given by $g(r) = g_0(r) = r/2$ if r is even, and $g(r) = g_1(r) = (3r+1)/2$ if r is odd. A *rational Collatz cycle* (*of length l*) is simply a g-cycle (of length l). Given $l \in \mathbb{N}_1$ and $n \in \mathbb{N}_0$, let $S_{l,n}$ be the set of all 0-1 sequences of length l containing exactly n 1's, and put $S_l = \bigcup_{n=0}^{l} S_{l,n}$ and $S = \bigcup_{l=1}^{\infty} S_l$. If $s \in S$, we denote the number of 1's in s by $n(s)$ and the length of s by $l(s)$. Given $s = (s_1, \ldots, s_l) \in S$, define $\phi_s : \mathbb{R} \to \mathbb{R}$ by $\phi_s = g_{s_l} \circ \cdots \circ g_{s_1}$. A sequence $(x_0, x_1, \ldots, x_l)$ of real numbers is called a *pseudo-cycle of length l* if there exists $s = (s_1, \ldots, s_l) \in S$ such that $x_l = x_0$ and $x_i = g_{s_i}(x_{i-1})$ for $i = 1, \ldots, l$ (note that $x_l = \phi_s(x_0)$). Finally, define $\varphi : S \to \mathbb{N}_0$ by

$$\varphi(s) = \sum_{j=1}^{l(s)} s_j 2^{j-1} 3^{s_{j+1}+s_{j+2}+\cdots+s_{l(s)}}.$$

Let $x_0 \in \mathbb{R}_1$ be given. If

$$\{\lim_{k\to\infty} U^{2k}(x_0), \lim_{k\to\infty} U^{2k+1}(x_0)\} = \{1, 2\},$$

then we say that the U-trajectory $\mathcal{T}_U(x_0)$ *tends to* $\{1,2\}$ and write $\mathcal{T}_U(x_0) \to \{1,2\}$. Our *real* $3x+1$ *conjecture* is

RU: For all $x \in \mathbb{R}_1$, $\mathcal{T}_U(x) \to \{1,2\}$.

Note that, for all $n \in \mathbb{N}_1$, $\mathcal{T}_T(n) = \mathcal{T}_U(n)$. The famous (integer) $3n+1$ conjecture may then be stated as

NU: For all $n \in \mathbb{N}_1$, $\mathcal{T}_U(n) \to \{1,2\}$.

One could also state both of these conjectures in terms of the *U-parity sequence* associated with $x \in \mathbb{R}_1$, which is simply the infinite 0-1 sequence

$$\mathcal{P}_U(x) = (\lfloor U^i(x) \rfloor \bmod 2)_{i=0}^{\infty}.$$

Note that this sequence encodes which branch of U (U_0 or U_1) is used in each step of $\mathcal{T}_U(x)$. Now, an infinite 0-1 sequence $(p_i)_{i=0}^{\infty}$ will be called *eventually periodic with period* $(0,1)$ if there exists $j \in \mathbb{N}_0$ such that $(p_i, p_{i+1}) = (0,1)$ for all $i = j + 2m$, $m \in \mathbb{N}_0$. It is a simple matter (see Proposition 3.1) to show that, for each $x \in \mathbb{R}_1$, $\mathcal{P}_U(x)$ is eventually periodic with period $(0,1)$ if, and only if, $\mathcal{T}_U(x) \to \{1,2\}$. In other words, the conjectures **RU** and **NU** above can be stated in the following alternative, equivalent forms:

RU′: For all $x \in \mathbb{R}_1$, $\mathcal{P}_U(x)$ is eventually periodic with period $(0,1)$.
NU′: For all $n \in \mathbb{N}_1$, $\mathcal{P}_U(n)$ is eventually periodic with period $(0,1)$.

Now, we observe that our **RU** conjecture clearly implies both of the following two conjectures.

OU: The only U-cycle is the T-cycle $(1,2,1,2,1,\ldots)$.
BU: Every U-trajectory is bounded.

Of course, all T-cycles are U-cycles, and one would naturally expect to find (many) more U-cycles than T-cycles. However, our main result, which is directly related to the conjectures **RU** and **OU** above, tells us that in fact quite the opposite happens.

THEOREM 2.1. *The only U-cycles are the T-cycles.*

Proof. Let us first state two lemmas that will be used in this and subsequent proofs. The reader may find their proofs in [5] and [7] (the basic idea of most of Lemma 2.2 below is due originally to Böhm and Sontacchi [1]).

LEMMA 2.2 (Böhm and Sontacchi, Lagarias, Halbeisen and Hungerbühler). *A sequence $(x_0, x_1, \ldots, x_l)$ is a rational Collatz cycle of length l if, and only if, it is a pseudo-cycle of length l. Moreover, if a rational Collatz cycle is not the cycle $(0,0,\ldots)$, then its elements are either all strictly positive or all strictly negative.* ■

LEMMA 2.3 (Lagarias). *For any $s \in S$ and any $x \in \mathbb{R}$, we have*

$$\phi_s(x) = \frac{3^{n(s)}x + \varphi(s)}{2^{l(s)}}. \tag{3}$$

Therefore, given $s \in S$,

$$x_0(s) = \frac{\varphi(s)}{2^{l(s)} - 3^{n(s)}} \in \mathbb{Q}[(2)] \tag{4}$$

is the **unique** *number that generates the rational Collatz cycle of length $l(s)$ that is also the pseudo-cycle of length $l(s)$ determined by s.* ■

To begin with, we note that all U-cycles start at numbers in $\mathbb{Q}_1$, since, for each $k \in \mathbb{N}_1$, every solution of $x = U^k(x)$ is rational. Suppose then that there exist $x_0 \in \mathbb{Q}_1 \setminus \mathbb{N}_1$ and $l \in \mathbb{N}_1$ such that there is a U-cycle of length l starting at x_0, namely $\Omega(x_0) = (x_0, U(x_0), \ldots, U^l(x_0) = x_0)$. If we derive a contradiction from this hypothesis, then we will be done. Note that it is immediate (by inspection) that the only U-cycle of length less than 4 is the T-cycle $(1,2,1)$. Hence, without loss of generality, we may assume that $l \geq 4$, which avoids our having to consider some trivial cases separately in what follows. Now, since $U_\iota \equiv g_\iota$ $(\iota = 0,1)$, $\Omega(x_0)$ is a pseudo-cycle of length l. Thus, by Lemma 2.2, $\Omega(x_0)$ is a rational Collatz cycle of length l as well. Therefore, by using Lemma 2.3 and the fact that $U|_{\mathbb{N}_1} = T$, one deduces both that all $U^i(x_0)$ are in $\mathbb{Q}[(2)] \cap \mathbb{Q}_1 \setminus \mathbb{N}_1$ and that

$$x_0 = x_0(s) = \frac{\varphi(s)}{2^{l(s)} - 3^{n(s)}}, \tag{5}$$

where $s = (s_1, \ldots, s_l) \in S$ is the 0-1 sequence associated with (the pseudo-cycle) $\Omega(x_0)$, i.e., s consists of the first $l = l(s)$ terms in $\mathcal{P}_U(x_0)$. For convenience, put $n = n(s)$ and $d = 2^l - 3^n$. Now, given any a/b in $\mathbb{Q}[(2)]$ (with b odd), it is clear that every term in the rational Collatz sequence generated by a/b may be written with denominator b. As d happens to be odd, one may, for $i = 0,1,\ldots,l$, write

$$x_i = U^i(x_0) = \frac{c_i}{d} = \frac{q_i d + r_i}{d} = q_i + \frac{r_i}{d}, \tag{6}$$

where q_i is the quotient and r_i the remainder in the Euclidean division of c_i by d. Note that all c_i, q_i and r_i lie in $\mathbb{N}_1$ and that $d \geq 5$ (for all $x_i = c_i/d$ are in $\mathbb{Q}_1 \setminus \mathbb{N}_1$, $\varphi(s) > 0$ and 3 does not divide d). In particular, no r_i is 0, and so all r_i satisfy $0 < r_i < d$. Moreover, because $d = 2^l - 3^n > 0$, one has

$$n < l \log_3 2. \tag{7}$$

Now, since $\Omega(x_0) = (x_0, x_1, \ldots, x_l = x_0)$ is both a U-cycle and a rational Collatz cycle (of length l), we infer, for $i = 0,1,\ldots,l$, that $q_i = \lfloor x_i \rfloor$ is even (resp. odd) if, and only if, c_i is even (resp. odd). Thus, all $r_i = c_i - dq_i$ are even. Write $r_i = 2^{e_i} o_i$, where $e_i \geq 1$ and o_i is odd, think of $r_0, r_1, \ldots, r_l = r_0$ as being arranged (in this order) in a circular manner and observe that, for $i = 0,1,\ldots,l$,

$$r_i = \begin{cases} \frac{1}{2} r_{i-1} & \text{if } q_{i-1} \text{ is even,} \\ \frac{3}{2} r_{i-1} & \text{if } q_{i-1} \text{ is odd and } r_{i-1} < \frac{2}{3} d, \\ \frac{3}{2} r_{i-1} - d & \text{if } q_{i-1} \text{ is odd and } r_{i-1} > \frac{2}{3} d. \end{cases} \tag{8}$$

Note that, since 3 does not divide d, it is never the case that $r_{i-1} = 2d/3$ in (8). As usual, indices are to be considered modulo l whenever necessary. Now, if r_i is such that $r_i = 3r_{i-1}/2 - d$, then we say that this r_i is *new*. Note that if r_{j+1} is not new, then either $r_{j+1} = 2^{e_j - 1} o_j$ or $r_{j+1} = 2^{e_j - 1}(3o_j)$. This clearly means that at least one of $r_0, r_1, \ldots, r_{l-1}$ is new. By renaming the x_i's if necessary, we can assume that r_0 $(= r_l)$ is new. Now, let $0 \leq p < q \leq l$ be such that r_p and r_q are *consecutive new*, that is, both r_p and r_q are new and, for all $p < k < q$, r_k is not new (if r_0 is the only new one, then put $p = 0$ and $q = l$). Because there are no new r_k's strictly between r_p and r_q, one has

$$r_{q-1} = 2(3^{n(p,q-1)} o_p) \quad \text{and} \quad e_p = q - p, \tag{9}$$

where, for any $0 \le i \le j \le l$, $n(i,j)$ is the number of times U_1 is used from x_i to x_j, i.e., $n(i,j)$ is the number of 1's in $\{s_{i+1}, s_{i+2}, \dots, s_j\}$. Since r_q is new, we have both $n(p,q) = n(p,q-1)+1$ and $d < 3r_{q-1}/2$. From this and (9), it follows that

$$d < 3^{n(p,\,q-1)+1} o_p = \frac{3^{n(p,\,q-1)+1}}{2^{e_p}}\, r_p = \frac{3^{n(p,\,q)}}{2^{q-p}}\, r_p. \tag{10}$$

Now, because r_p is new, $r_p = 3r_{p-1}/2 - d$, and so, since $0 < r_{p-1} < d$, we obtain $r_p < d/2 < 2d/3$. From this and (10), one gets

$$d < \frac{3^{n(p,\,q)-1}}{2^{q-p-1}}\, d \;\Rightarrow\; 3^{n(p,\,q)-1} > 2^{q-p-1} \;\Rightarrow\; n(p,q) > \log_3 2^{q-p-1} + 1. \tag{11}$$

Therefore, $n(p,q) > \log_3 2^{q-p-1} + \log_3 2 = \log_3 2^{q-p}$, and so

$$n(p,q) > (q-p)\log_3 2. \tag{12}$$

Now, let $0 = i_0 < i_1 < \dots < i_m = l$, $m \ge 1$, be such that $r_{i_0}, r_{i_1}, \dots, r_{i_m}$ are all the new r_i's in $\{r_0, r_1, \dots, r_l\}$. Then we have $n = \sum_{k=1}^{m} n(i_{k-1}, i_k)$, $l = \sum_{k=1}^{m} (i_k - i_{k-1})$ and $r_{i_{k-1}}$ and r_{i_k} are consecutive new for all $k = 1, \dots, m$. Consequently, inequality (12) gives us $n > l \log_3 2$, contrary to (7). ∎

We note that some authors have already investigated a variety of interesting *smooth* extensions of T to the real (and even complex) numbers (see, e.g., [2]–[4], [6] and [9]). Unlike the *conjectured* case of U, however, the dynamics of these extensions outside the integers are always extraneous to the $3x+1$ conjecture (i.e., there exist periodic and divergent trajectories).

Now, the previous theorem illustrated the relative ease one has in obtaining some results if one is allowed the freedom to work in $\mathbb{R}_1$ (instead of having to concentrate on $\mathbb{N}_1$). For another example along these lines, consider the *flipped* $3x+1$ function $\widetilde{U} : \mathbb{R}_0 \to \mathbb{R}_0$ defined by

$$\widetilde{U}(x) = \begin{cases} \widetilde{U}_0(x) = U_1(x) & \text{if } \lfloor x \rfloor \text{ is even,} \\ \widetilde{U}_1(x) = U_0(x) & \text{if } \lfloor x \rfloor \text{ is odd.} \end{cases} \tag{13}$$

Clearly, $\widetilde{U}|_{\mathbb{N}_0}$ is not a function from $\mathbb{N}_0$ to $\mathbb{N}_0$. Naturally, one would like to know what happens to the $\widetilde{U}$-trajectories. In particular, one could try to obtain all $\widetilde{U}$-cycles. This is in fact done in our next theorem, which is a bonus result we obtain from the method used to prove Theorem 2.1.

THEOREM 2.4. *There are no $\widetilde{U}$-cycles.*

Proof. The proof is almost entirely analogous to the proof of Theorem 2.1 above, and so we will be brief and point out only the required modifications. Clearly, no $\widetilde{U}$-cycles start at numbers in $\mathbb{N}_0$. Suppose then that there exist $x_0 \in \mathbb{Q}_0 \setminus \mathbb{N}_0$ and $l \in \mathbb{N}_1$ such that there is a $\widetilde{U}$-cycle of length l starting at x_0, namely $\widetilde{\Omega}(x_0) = (x_0, \widetilde{U}(x_0), \ldots, \widetilde{U}^l(x_0) = x_0)$. By inspection, there are no $\widetilde{U}$-cycles of length less than 4, and so we may assume that $l \geq 4$ (again, this assumption is made to avoid trivialities). Now, with similar notations and the same arguments from the proof of Theorem 2.1, one finds, for $i = 0, 1, \ldots, l$, that

$$x_i = \widetilde{U}^i(x_0) = \frac{c_i}{d} = \frac{q_i d + r_i}{d} = q_i + \frac{r_i}{d} = (q_i + 1) - \frac{d - r_i}{d}, \tag{14}$$

where q_i is the quotient and r_i the remainder in the Euclidean division of c_i by d. Since no x_i's belong to $\mathbb{N}_0$, all r_i satisfy $0 < d - r_i < d$. Moreover, because $d = 2^l - 3^n > 0$, we have, as before,

$$n < l \log_3 2. \tag{15}$$

Since $\widetilde{\Omega}(x_0) = (x_0, x_1, \ldots, x_l = x_0)$ is both a $\widetilde{U}$-cycle and a rational Collatz cycle (of length l), it follows, for $i = 0, 1, \ldots, l$, that $q_i = \lfloor x_i \rfloor$ is even (resp. odd) if, and only if, c_i is odd (resp. even). Thus, all $r_i = c_i - dq_i$ are odd, i.e., all $d - r_i$ are even. Now, think of $d - r_0, d - r_1, \ldots, d - r_l = d - r_0$ as being arranged (in this order) in a circular fashion and note that, for $i = 0, 1, \ldots, l$,

$$d - r_i = \begin{cases} \frac{1}{2}(d - r_{i-1}) & \text{if } q_{i-1} \text{ is odd,} \\ \frac{3}{2}(d - r_{i-1}) & \text{if } q_{i-1} \text{ is even and } d - r_{i-1} < \frac{2}{3}d, \\ \frac{3}{2}(d - r_{i-1}) - d & \text{if } q_{i-1} \text{ is even and } d - r_{i-1} > \frac{2}{3}d. \end{cases} \tag{16}$$

Arguing exactly in the same way as in the proof of Theorem 2.1, we conclude that $n > l \log_3 2$, which contradicts (15). ■

Note that yet another equivalent way of phrasing the conjecture **RU** is to say that, for every $x \in \mathbb{R}_1$, there exists $k \in \mathbb{N}_0$ such that $U^k(x) \in [1, 3)$. Our corresponding conjecture for the iterates of $\widetilde{U}$ is

$\mathbf{R\widetilde{U}}$: For every $x \in \mathbb{R}_0$ there exists $k \in \mathbb{N}_0$ such that $\widetilde{U}^k(x) \in [0, 2)$.

Of course, Theorem 2.4 is directly related to the conjecture $\mathbf{R\widetilde{U}}$ above. Let us conclude this section by observing that our $\mathbf{R\widetilde{U}}$ conjecture clearly implies the following conjecture.

$\mathbf{B\widetilde{U}}$: Every $\widetilde{U}$-trajectory is bounded.

3. Other results, conjectures and questions. One way to find out if studying what happens to the iterates of U can shed some new light on the $3n + 1$ conjecture or not would be to try to answer our first question.

Q1: Does the $3n+1$ conjecture imply our real $3x+1$ conjecture **RU**?

On one hand, if the answer to this question is *yes*, then this would show that looking at the iterates of U amounts to essentially the same thing as looking at those of T (as far as the $3n+1$ conjecture is concerned). On the other hand, we note that if the $3n+1$ conjecture is true, then the answer to the question **Q1** above could very well be *no*. To see why, suppose that, instead of T, one considered the original Collatz function, i.e., $f:\mathbb{N}_1\to\mathbb{N}_1$ given by

$$f(n)=\begin{cases} f_0(n)=n/2 & \text{if } n \text{ is even,}\\ f_1(n)=3n+1 & \text{if } n \text{ is odd.}\end{cases} \tag{17}$$

Its extension to $\mathbb{R}_1$ (in our sense) is the function $F:\mathbb{R}_1\to\mathbb{R}_1$ given by

$$F(x)=\begin{cases} F_0(x)=x/2 & \text{if } \lfloor x\rfloor \text{ is even,}\\ F_1(x)=3x+1 & \text{if } \lfloor x\rfloor \text{ is odd.}\end{cases} \tag{18}$$

The statement for the F-trajectories which corresponds to the conjecture **RU** would be that $\mathcal{T}_F(x)\to\{1,2\}$ for all $x\in\mathbb{R}_1$. However, this is readily seen to be *false*, since, for example, all F-trajectories starting at $2m+3/2$, $m\in\mathbb{N}_0$, diverge (monotonically) to $+\infty$. Now, the $3n+1$ conjecture for the iterates of T is equivalent to the (same) one for the iterates of f. Thus, if the $3n+1$ conjecture turns out to be true, then the question for the F-trajectories that is the counterpart to question **Q1** will have a negative answer. Moreover, if our real $3x+1$ conjecture **RU** is true, then the U-trajectories and the F-trajectories will be seen to have quite different behaviors in $\mathbb{R}_1$ (as opposed to what happens in $\mathbb{N}_1$). In our view, comparisons between the U-trajectories and the F-trajectories may play an important rôle in some future $3x+1$-type investigations. Let our next question emphasize this point.

Q2: Are the F-trajectories starting at $2m+3/2$, $m=0,1,2,\ldots$, the only F-trajectories that do not tend to $\{1,2\}$?

Of course, analogous questions on similar notions regarding the iterates of $\widetilde{U}$ could be posed as well. We now show a simple result about the iterates of U. Its proof will suggest a new approach one might consider in trying to prove the conjecture **OU** (see Remark 3.3). A corresponding result for the iterates of $\widetilde{U}$ will then be obtained as a corollary. Before we can state these results, a couple of definitions are needed.

Given $x_0\in\mathbb{R}_1$, we say that $\mathcal{P}_U(x_0)=(p_i)_{i=0}^{\infty}$ is *eventually periodic with period* $s=(s_0,s_1,\ldots,s_{l(s)-1})\in S$ if there exists $j\in\mathbb{N}_0$ such that $(p_i,p_{i+1},\ldots,p_{i+l(s)-1})=(s_0,s_1,\ldots,s_{l(s)-1})$ for all $i=j+ml(s)$, $m\in\mathbb{N}_0$. Moreover, if $a\in\mathbb{N}_1$ is such that there is a U-cycle of length l starting at a, then we say that $\mathcal{T}_U(x_0)$ *tends to* $\{U^t(a)\}$ *from above* (in symbols, $\mathcal{T}_U(x_0) \xrightarrow{+} \{U^t(a)\}$) whenever there is $j_0 \in \mathbb{N}_0$ such that, for all $j \in \{0,1,\dots,l-1\}$, $U^{kl}(U^{j+j_0}(x_0)) \to U^j(a)^+$ as $k \to +\infty$.

Proposition 3.1. *If $a \in \mathbb{N}_1$ is such that there is a U-cycle of length l starting at a, then, for all $x \in \mathbb{R}_1$, $\mathcal{P}_U(x)$ is eventually periodic with period* $(a \bmod 2, U(a) \bmod 2, \dots, U^{l-1}(a) \bmod 2)$ *if, and only if, $\mathcal{T}_U(x)$ tends to $\{U^t(a)\}$ from above.*

Proof. Suppose first that $x \in \mathbb{R}_1$ is such that $\mathcal{T}_U(x) \xrightarrow{+} \{U^t(a)\}$. Since there is a U-cycle of length l starting at a, it clearly follows that there is some $0 < \theta \in \mathbb{R}$ such that, for all $y \in [a, a+\theta)$ and all $m \in \mathbb{N}_0$,

$$(\lfloor U^{ml}(y)\rfloor, \lfloor U^{ml+1}(y)\rfloor, \dots, \lfloor U^{ml+l-1}(y)\rfloor) = (a, U(a), \dots, U^{l-1}(a)). \tag{19}$$

For instance, any $0 < \theta < (2/3)^l$ will do. Since $\mathcal{T}_U(x) \xrightarrow{+} \{U^t(a)\}$, there is some $k_0 \in \mathbb{N}_0$ such that $U^{k_0}(x) \in [a, a+\theta)$. Hence, $\mathcal{P}_U(x)$ is eventually periodic with period $(a \bmod 2, U(a) \bmod 2, \dots, U^{l-1}(a) \bmod 2)$.

For the other direction, suppose now that $x \in \mathbb{R}_1$ is such that $\mathcal{P}_U(x)$ is eventually periodic with period $s = (a \bmod 2, U(a) \bmod 2, \dots, U^{l-1}(a) \bmod 2) \in S$. By using Lemma 2.3, one sees that there is some $j_0 \in \mathbb{N}_0$ such that

$$\begin{aligned} U^{l(s)}(U^{j_0}(x)) &= \frac{3^{n(s)}U^{j_0}(x) + \varphi(s)}{2^{l(s)}} = \frac{3^{n(s)}(a + U^{j_0}(x) - a) + \varphi(s)}{2^{l(s)}} \\ &= \frac{3^{n(s)}a + \varphi(s)}{2^{l(s)}} + \frac{3^{n(s)}(U^{j_0}(x) - a)}{2^{l(s)}} = a + \frac{3^{n(s)}}{2^{l(s)}}\,(U^{j_0}(x) - a). \end{aligned}$$

Analogously, for $j = 0, 1, \dots, l-1$ we have

$$U^{l(s)}(U^{j+j_0}(x)) = U^j(a) + \frac{3^{n(s)}}{2^{l(s)}}\,(U^{j+j_0}(x) - U^j(a)).$$

Therefore, for all $m \in \mathbb{N}_0$ and all $j \in \{0,1,\dots,l-1\}$,

$$U^{ml(s)}(U^{j+j_0}(x)) = U^j(a) + \left(\frac{3^{n(s)}}{2^{l(s)}}\right)^m (U^{j+j_0}(x) - U^j(a)). \tag{20}$$

Since $3^{n(s)} < 2^{l(s)}$, it is not hard to conclude now that $\mathcal{T}_U(x) \xrightarrow{+} \{U^t(a)\}$. ∎

Now, with the appropriate analogous definitions for the iterates of $\widetilde{U}$, the same argument presented in the proof of Proposition 3.1 above gives us the following result as well.

Proposition 3.2. *If $a \in \mathbb{N}_1$ is such that there is a U-cycle of length l starting at a, then, for all $x \in \mathbb{R}_0$, $\mathcal{P}_{\widetilde{U}}(x)$ is eventually periodic with period* $(1 - a \bmod 2, 1 - (U(a) \bmod 2), \dots, 1 - (U^{l-1}(a) \bmod 2))$ *if, and only if, $\mathcal{T}_{\widetilde{U}}(x)$ tends to $\{U^t(a)\}$ from below.* ∎

Note that if the $3n+1$ conjecture is true and $x_0 \in \mathbb{R}_1$ is such that $\mathcal{T}_U(x_0) \to a_0^+$ for some $a_0 \in \mathbb{N}_1$, then $\mathcal{T}_U(x_0) \to \{1,2\}$. This indicates one way in which one may try to give a positive answer to question **Q1**.

Now, consider $\mathcal{I}_U(\mathbb{N}_1) = \{x \in \mathbb{R}_1 : \exists\, k \in \mathbb{N}_0 \text{ with } U^k(x) \in \mathbb{N}_1\}$ and $\mathcal{N}_U(\mathbb{N}_1) = \mathbb{R}_1 \setminus \mathcal{I}_U(\mathbb{N}_1)$. Of course, our **RU** conjecture implies the following conjecture.

$\mathcal{N}$**U**: For all $x \in \mathcal{N}_U(\mathbb{N}_1)$, $\mathcal{T}_U(x) \to \{1,2\}$.

We may now pose our next question, which can also be thought of as being one of the possible (nontrivial) ways of turning question **Q1** around.

Q3: Does the $\mathcal{N}$**U** conjecture above imply the $3n+1$ conjecture?

REMARK 3.3. Let us just note here an interesting corollary of the proof of Proposition 3.1: if one proves that, for all $n \in \mathbb{N}_1$ and all $0 < \varrho \in \mathbb{R}$, there exists some $z \in (n, n+\varrho) \cap \mathcal{N}_U(\mathbb{N}_1)$ such that $\mathcal{T}_U(z) \to \{1,2\}$, then it will follow that the **OU** conjecture (which, in light of Theorem 2.1, states in fact that "there are no nontrivial T-cycles") is true.

To try to answer the question **Q3** above might be an even better way of seeing whether there are some real advantages in shifting one's attention from T to U. Let us end this line of inquiry now by registering the following very broad (but also potentially very productive) question.

Q4: What kind of results for the iterates of U does one get by attempting to translate known results for the iterates of T?

In conclusion, let us just remark that the apparent general project would be for one to study the dynamical system in $\mathbb{R}$ generated by the iterates of (discontinuous) piecewise linear functions of the following "simple" kind.

Let $\alpha, \beta, \gamma, \delta, \tau \in \mathbb{R}$ be fixed, with $\tau \in [0,2)$, and consider the function $\Phi = \Phi(\alpha, \beta, \gamma, \delta, \tau) : \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi(x) = \begin{cases} \Phi_0(x) = \alpha x + \beta & \text{if } \lfloor x+\tau \rfloor \text{ is even,} \\ \Phi_1(x) = \gamma x + \delta & \text{if } \lfloor x+\tau \rfloor \text{ is odd.} \end{cases} \tag{21}$$

Naturally, the crux of the matter here is to find out how the parameters $\alpha, \beta, \gamma, \delta$ and τ affect the behavior of the $\Phi = \Phi(\alpha, \beta, \gamma, \delta, \tau)$-trajectories. This brings us to our final (albeit seemingly intractable as of yet!) question.

Q5: How do the general properties of the dynamical system in $\mathbb{R}$ generated by the iterates of the function $\Phi = \Phi(\alpha, \beta, \gamma, \delta, \tau)$ defined as in (21) depend on the values of the real parameters $\alpha, \beta, \gamma, \delta$ and τ?

We have, e.g., $U = \Phi(1/2, 0, 3/2, 1/2, 0)|_{\mathbb{R}_1}$ and $\widetilde{U} = \Phi(1/2, 0, 3/2, 1/2, 1)|_{\mathbb{R}_0}$. Note also that the functions $\Phi(1/2, 0, 3/2, 1/2, \tau_0)|_{\mathbb{R}_1}$, with $0 \le \tau_0 < 1$, are all extensions of T. Finally, we bring into attention $V = \Phi(1/2, 0, 3/2, 0, 0)|_{\mathbb{R}_1}$, i.e., the function $V : \mathbb{R}_1 \to \mathbb{R}_1$ given by

$$V(x) = \begin{cases} V_0(x) = \frac{1}{2}x & \text{if } \lfloor x \rfloor \text{ is even,} \\ V_1(x) = \frac{3}{2}x & \text{if } \lfloor x \rfloor \text{ is odd.} \end{cases} \tag{22}$$

Of course, there are no V-cycles. It might be worthwhile for one to try to find out the status of the following two final conjectures, as well as their possible connections to the $3n+1$ and **RU** conjectures, if any:

RV: For every $x \in \mathbb{R}_1$ there exists $k \in \mathbb{N}_0$ such that $V^k(x) \in [1, 3)$.
BV: Every V-trajectory is bounded.

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