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On a conjecture of Yiming Long

by

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1. Introduction. In 2000 when studying the Maslov-type index theory for Hamiltonian systems, Yiming Long [1] put forward the following two conjectures.

Conjecture 1. *For any positive integer k, there are infinitely many pairs of prime numbers which are of the form $(kn-1, kn+1)$ with a positive integer n.*

Conjecture 2. *For any irrational number φ in the interval $(0,1)$, there are infinitely many prime numbers p which cannot be expressed as*

$$p = 2n + 2[n\varphi] + 1 \tag{1}$$

with a positive integer n.

In January 2004, Professor Yiming Long proposed his conjectures to me. From the viewpoint of Diophantine equations Conjecture 1 seems as difficult as the prime twins conjecture. In this paper, we shall give a positive answer to Conjecture 2. In fact, we can get more information.

In the following, we suppose that φ is an irrational number in the interval $(0,1)$ and that

$$\alpha = -\frac{1}{2(1+\varphi)}, \tag{2}$$

which is also irrational. By a well known result of Dirichlet (see page 9 of [4]), there are infinitely many rational numbers a/q ($(a,q)=1$, $q \to \infty$) such that $|\alpha - a/q| \le 1/q^2$. We suppose that q is sufficiently large and that ε is a sufficiently small positive constant, $\delta = \varepsilon^2$. Write $((x)) = x - [x] - 1/2$ and $e(x) = e^{2\pi i x}$. Let $\|y\|$ denote the smallest distance from y to integers, p a prime number and $\Lambda(n)$ the Mangoldt function.

On the above supposition, we have

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THEOREM. *Let $T(p)$ be the number of solutions of the equation* (1) *in positive integers n. If $q^{1+\varepsilon} < x \le q^{1/\varepsilon}$, then*

$$\sum_{\substack{x<p\le 2x\\ T(p)=0}} 1 \ge \frac{\varphi-\varepsilon}{1+\varphi}\cdot\frac{x}{\log x}.$$

We see that not only Conjecture 2 is true but also in a lot of intervals the relevant prime numbers have a positive density by the fact that

$$\sum_{x<p\le 2x} 1 \sim \frac{x}{\log x}.$$

2. Proof of the Theorem. For a prime number $p \ge 3$, the following equivalences hold:

$$2n+2[n\varphi]+1=p,\ n>0 \Leftrightarrow [n\varphi]=\frac{p-1}{2}-n,\ n>0$$
$$\Leftrightarrow n\varphi-1<\frac{p-1}{2}-n\le n\varphi,\ n>0 \Leftrightarrow \frac{p-1}{2(1+\varphi)}\le n<\frac{p+1}{2(1+\varphi)}$$
$$\Leftrightarrow -\frac{p+1}{2(1+\varphi)}<-n\le -\frac{p-1}{2(1+\varphi)}.$$

Now the number of $-n$, which is also the number of n, is equal to

$$\left[-\frac{p-1}{2(1+\varphi)}\right]-\left[-\frac{p+1}{2(1+\varphi)}\right].$$

Hence,

$$T(p)=\left[-\frac{p-1}{2(1+\varphi)}\right]-\left[-\frac{p+1}{2(1+\varphi)}\right]. \tag{3}$$

Since

$$0\le\left(-\frac{p-1}{2(1+\varphi)}\right)-\left(-\frac{p+1}{2(1+\varphi)}\right)=\frac{1}{1+\varphi}<1,$$

we have

$$T(p)=0 \text{ or } 1. \tag{4}$$

Now we study the sum

$$\sum_{x<p\le 2x} T(p)\log p. \tag{5}$$

We have

$$\sum_{x<p\le 2x} T(p)\log p=\sum_{x<p\le 2x}\left(\left[-\frac{p-1}{2(1+\varphi)}\right]-\left[-\frac{p+1}{2(1+\varphi)}\right]\right)\log p \tag{6}$$
$$\le\sum_{x<m\le 2x}\left(\left[-\frac{m-1}{2(1+\varphi)}\right]-\left[-\frac{m+1}{2(1+\varphi)}\right]\right)\Lambda(m)$$

$$= \sum_{x<m\le 2x}\left(\left(-\frac{m-1}{2(1+\varphi)}-\frac{1}{2}\right)-\left(\left(-\frac{m-1}{2(1+\varphi)}\right)\right)\right)\Lambda(m)$$

$$-\sum_{x<m\le 2x}\left(\left(-\frac{m+1}{2(1+\varphi)}-\frac{1}{2}\right)-\left(\left(-\frac{m+1}{2(1+\varphi)}\right)\right)\right)\Lambda(m)$$

$$=\frac{1}{1+\varphi}\sum_{x<m\le 2x}\Lambda(m)+\sum_{x<m\le 2x}((\alpha m+\alpha))\Lambda(m)$$

$$-\sum_{x<m\le 2x}((\alpha m-\alpha))\Lambda(m).$$

The prime number theorem yields

$$\frac{1}{1+\varphi}\sum_{x<m\le 2x}\Lambda(m)\sim\frac{x}{1+\varphi}. \tag{7}$$

By the formula on page 254 of [2],

$$((t)) = -\sum_{1\le|h|\le x}\frac{e(ht)}{2\pi ih}+O\left(\min\left(1,\frac{1}{x\|t\|}\right)\right).$$

Hence,

$$\begin{aligned}\sum_{x<m\le 2x}((\alpha m+\alpha))\Lambda(m) &= -\sum_{x<m\le 2x}\left(\sum_{1\le|h|\le x}\frac{e(\alpha hm)}{2\pi ih}\,e(\alpha h)\right)\Lambda(m)\\ &\quad+O\left(\sum_{x<m\le 2x}\min\left(1,\ \frac{1}{x\|\alpha m+\alpha\|}\right)\Lambda(m)\right)\\ &\ll \sum_{1\le h\le x}\frac{1}{h}\Big|\sum_{x<m\le 2x}\Lambda(m)e(\alpha hm)\Big|\\ &\quad+O\left(\log x\sum_{x<m\le 2x}\min\left(1,\ \frac{1}{x\|\alpha m+\alpha\|}\right)\right).\end{aligned} \tag{8}$$

When $J\le 2x$, Theorem 1 in [3] states

$$\sum_{1\le h\le J}\Big|\sum_{x<m\le 2x}\Lambda(m)e(\alpha hm)\Big|\ll x^{\delta}(Jx/\sqrt{q}+Jx^{3/4}+(Jqx)^{1/2}+J^{3/5}x^{4/5}).$$

It follows that

$$\begin{aligned}\sum_{1\le h\le x}\frac{1}{h}\Big|\sum_{x<m\le 2x}\Lambda(m)e(\alpha hm)\Big| &\ll \log x\max_{J\le x}\sum_{J\le h\le 2J}\frac{1}{h}\Big|\sum_{x<m\le 2x}\Lambda(m)e(\alpha hm)\Big|\\ &\ll x^{1-2\delta}.\end{aligned} \tag{9}$$

By Lemma 1 in [3],

$$\sum_{x<m\le 2x}\min\Big(1,\frac{1}{x\|\alpha m+\alpha\|}\Big)=\frac{1}{x}\sum_{x<m\le 2x}\min\Big(x,\frac{1}{\|\alpha m+\alpha\|}\Big) \ll \frac{1}{x}\Big(\frac{x^2}{q}+x+(x+q)\log q\Big) \ll x^{1-2\delta}. \tag{10}$$

The combination of (8), (9) and (10) produces

$$\sum_{x<m\le 2x}((\alpha m+\alpha))\Lambda(m)=O(x^{1-\delta}). \tag{11}$$

In the same way,

$$\sum_{x<m\le 2x}((\alpha m-\alpha))\Lambda(m)=O(x^{1-\delta}). \tag{12}$$

It follows from (6), (7), (11) and (12) that

$$\sum_{x<p\le 2x}T(p)\log p\le\frac{1+\delta}{1+\varphi}\cdot x. \tag{13}$$

Hence,

$$\sum_{x<p\le 2x}T(p)\le\frac{1+\delta}{1+\varphi}\cdot\frac{x}{\log x}.$$

Thus we have

$$\sum_{\substack{x<p\le 2x\\ T(p)=0}}1=\sum_{x<p\le 2x}1-\sum_{x<p\le 2x}T(p)\ge\frac{\varphi-\varepsilon}{1+\varphi}\cdot\frac{x}{\log x},$$

so the Theorem is proved.

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