Polynomial extension of Fleck’s congruence

by

ZHI-WEI SUN (Nanjing)

1. Introduction. As usual, we let \((\frac{x}{0}) = 1\) and
\[
\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!} \quad \text{for every } k = 1, 2, \ldots.
\]
For convenience, we also set \((\frac{x}{k}) = 0\) for any negative integer \(k\).

Let \(p\) be a prime and \(r\) be an integer. In 1913, A. Fleck (cf. Dickson [D, p. 274]) discovered that
\[
\sum_{k \equiv r \pmod{p}} \binom{n}{k} (-1)^k \equiv 0 \pmod{p^\lfloor (n-1)/(p-1) \rfloor}
\]
for all \(n \in \mathbb{Z}^+ = \{1, 2, \ldots\}\), where \(\lfloor \cdot \rfloor\) is the well-known floor function. Sums of the form \(\sum_{k \equiv r \pmod{m}} \binom{n}{k} (-1)^k\) (with \(m \in \mathbb{Z}^+\)) have various applications in number theory and combinatorics (see, e.g., [SS], [H] and [S02]).

In 1977, by a very complicated method, C. S. Weisman [W] extended Fleck’s congruence to prime power moduli in the following way:
\[
\sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k \equiv 0 \pmod{p^\lfloor (n-p^{\alpha-1})/\varphi(p^\alpha) \rfloor},
\]
where \(\alpha, n \in \mathbb{N} = \{0, 1, 2, \ldots\}\) and \(n \geq p^{\alpha-1}\), and \(\varphi\) denotes Euler’s totient function. Unaware of Fleck’s previous work, Weisman was motivated by studying the relation between two different ways (Mahler’s and van der Put’s) to express a \(p\)-adically continuous function.

Quite recently, in his lecture notes on Fontaine’s rings and \(p\)-adic \(L\)-functions given at Irvine (Spring, 2005), D. Wan got the following new ex-
tension of Fleck’s congruence:

\[(1.3) \sum_{k \equiv r \pmod{p}} \binom{n}{k} (-1)^k \left(\frac{k-r}{l}\right) \equiv 0 \pmod{p^{l(n-lp-1)/(p-1)}} ,\]

where \(l, n \in \mathbb{N}\) and \(n > lp\). Wan was led to this when trying to understand a sharp estimate for the \(\psi\)-operator in Fontaine’s theory of \((\phi, \Gamma)\)-modules.

For a prime \(p\), we let \(\mathbb{Q}_p\) and \(\mathbb{Z}_p\) denote the field of \(p\)-adic numbers and the ring of \(p\)-adic integers respectively; the \(p\)-adic order of \(\omega \in \mathbb{Q}_p\) is defined by \(\text{ord}_p(\omega) = \sup\{a \in \mathbb{Z} : \omega/p^a \in \mathbb{Z}_p\}\) (whence \(\text{ord}_p(0) = +\infty\)). Throughout this paper, the Kronecker symbol \(\delta_{m,n}\) with \(m,n \in \mathbb{N}\) equals 1 or 0 according as \(m = n\) or not.

Clearly both Weisman’s and Wan’s extensions of Fleck’s congruence follow from the special case \(\alpha = \beta\) of the following theorem, which we will establish by a combinatorial approach.

**Theorem 1.1.** Let \(p\) be a prime, and let \(f(x) \in \mathbb{Q}_p[x], \deg f \leq l \in \mathbb{N}\) and \(f(a) \in \mathbb{Z}_p\) for all \(a \in \mathbb{Z}\). Provided that \(\alpha, \beta \in \mathbb{N}\) and \(\alpha \geq \beta\), we have

\[(1.4) \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k f \left(\frac{k-r}{p^\alpha} \right) \in p^{l(n-p^{\alpha-1}-l)/\varphi(p^{\alpha})-(l-1)\alpha-\beta}\mathbb{Z}_p\]

for all integers \(n \geq p^{\alpha-1}\) and \(r\); moreover, we can substitute \(\delta_{\beta,0}\) for the first \(l\) in (1.4) if \(\alpha\) is greater than one.

By Theorem 1.1 in the case \(\alpha = \beta = r = 0\), if \(f(x) \in \mathbb{Z}[x]\) and \(f(x) \neq 0\), then for any integer \(n > \deg f + 1\) we have \(\sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) = 0\) since the sum is divisible by all primes. In fact, a known identity due to L. Euler (cf. [LW, pp. 90–91]) states that

\[\sum_{k=0}^{n} \binom{n}{k} (-1)^k k^l = \begin{cases} (-1)^n n! & \text{if } l = n \in \mathbb{N}, \\ 0 & \text{if } 0 \leq l < n. \end{cases}\]

Now we derive more consequences of Theorem 1.1.

**Corollary 1.1.** Let \(p\) be a prime, \(m \in \mathbb{Z}^+\) and \(\alpha = \text{ord}_p(m)\). Let \(l, n \in \mathbb{N}\) and \(r \in \mathbb{Z}\). Then

\[(1.5) \text{ord}_p \left( \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k B_l \left(\frac{k-r}{m}\right) \right) \geq \left\lfloor \frac{n-p^{\alpha-1}-l(\delta_{\alpha,0} + \delta_{\alpha,1})}{\varphi(p^{\alpha})} \right\rfloor - l\alpha,\]

where \(B_l(x)\) is the Bernoulli polynomial of degree \(l\).
Proof. (1.5) holds trivially if \( n < p^{\alpha - 1} \). Below we suppose \( n \geq p^{\alpha - 1} \).

When \( l = 0 \), (1.5) reduces to Weisman’s congruence (1.2). In the case \( \alpha = 0 \), if the lower bound in (1.5) is nonnegative (i.e., \( l < n \)) then the summation in (1.5) vanishes by Euler’s identity.

Now we assume \( l \alpha \neq 0 \), and let \( B_l = B_l(0) \) be the \( l \)th Bernoulli number. Note that \( m_0 = m/p^{\alpha} \) is relatively prime to \( p \).

For any \( a \in \mathbb{Z} \) we have

\[
B_l \left( \frac{a}{m_0} \right) - B_l \equiv \left( m_0 B_l \left( \frac{a}{m_0} \right) - B_l \right) - (m_0 B_l(0) - B_l) \in \mathbb{Z}_p
\]

by [S03, Corollary 1.3]. Applying Theorem 1.1 with \( f(x) = B_l(x/m_0) - B_l \) and \( \beta = \alpha \), we get

\[
\text{ord}_p \left( \sum_{k \equiv r \pmod{p^{\alpha}}} \binom{n}{k} (-1)^k B_l \left( \frac{k - r}{m} \right) - B_l \Sigma \right) \geq \left[ \frac{n - p^{\alpha - 1} - l \delta_{\alpha,1}}{\varphi(p^{\alpha})} \right] - l \alpha,
\]

where \( \Sigma = \sum_{k \equiv r \pmod{p^{\alpha}}} \binom{n}{k} (-1)^k \). Recall that \( pB_l \in \mathbb{Z}_p \) by the von Staudt–Clausen theorem (cf. [IR, pp. 233–236]). This, together with (1.2), shows that

\[
\text{ord}_p(B_l \Sigma) \geq \text{ord}_p(\Sigma) - 1 \geq \left[ \frac{n - p^{\alpha - 1}}{\varphi(p^{\alpha})} \right] - 1 \geq \left[ \frac{n - p^{\alpha - 1} - l \delta_{\alpha,1}}{\varphi(p^{\alpha})} \right] - l \alpha.
\]

So the desired (1.5) follows.

**Corollary 1.2.** Let \( p \) be a prime, and let \( f(x) \in \mathbb{Q}_p[x] \), \( \deg f = l \geq 0 \) and \( f(a) \in \mathbb{Z}_p \) for all \( a \in \mathbb{Z} \). Let \( \alpha \in \mathbb{N} \) and \( r \in \mathbb{Z} \). Then, for any integer \( n \geq p^{\alpha - 1} \), we have

\[
(1.6) \quad \text{ord}_p \left( \sum_{k=0}^{n} \binom{n}{k} (-1)^k (k - r, p^{\alpha}) f \left( \left\lfloor \frac{k - r}{p^{\alpha}} \right\rfloor \right) \right) \geq \left[ \frac{n - p^{\alpha - 1} - l (\delta_{\alpha,0} + \delta_{\alpha,1})}{\varphi(p^{\alpha})} \right] - (l - 1) \alpha - 1,
\]

where \( (k - r, p^{\alpha}) \) is the greatest common divisor of \( k - r \) and \( p^{\alpha} \).

**Proof.** Let \( g(1) = p \) and \( g(p^\beta) = p - 1 \) if \( 0 < \beta \leq \alpha \). By Theorem 1.1, the \( p \)-adic order of

\[
\sum_{\beta=0}^{\alpha} g(p^\beta) p^\beta \sum_{k \equiv r \pmod{p^{\beta}}} \binom{n}{k} (-1)^k f \left( \left\lfloor \frac{k - r}{p^{\alpha}} \right\rfloor \right) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f \left( \left\lfloor \frac{k - r}{p^{\alpha}} \right\rfloor \right) \sum_{d \mid (k - r, p^{\alpha})} g(d) d
\]
is at least
\[ \nu = \left\lfloor \frac{n - p^{\alpha - 1} - l(\delta_{\alpha,0} + \delta_{\alpha,1})}{\varphi(p^\alpha)} \right\rfloor - (l - 1)\alpha. \]

We note in passing that in the case \( \alpha > 1 \),
\[ \text{ord}_p(g(p^0)) + \left\lfloor \frac{n - p^{\alpha - 1} - \delta_{0,0}}{\varphi(p^\alpha)} \right\rfloor \geq \left\lfloor \frac{n - p^{\alpha - 1}}{\varphi(p^\alpha)} \right\rfloor. \]

Now, since
\[ \sum_{d | (k-r,p^\alpha)} g(d)d = p + \sum_{1 < d | (k-r,p^\alpha)} (p-1)d = \sum_{d | (k-r,p^\alpha)} \varphi(d)p = (k-r,p^\alpha)p, \]
by the above the sum in (1.6) has \( p \)-adic order at least \( \nu - 1 \).

**Corollary 1.3.** Let \( p \) be a prime, and let \( \alpha, \beta, a, n, r \) be integers for which
\[ \alpha > 1, \quad \alpha \geq \beta \geq 0, \quad a \equiv 1 \pmod{p^\alpha}, \quad n \geq p^{\alpha - 1}, \quad r < p^\beta. \]
Then
\[ (1.7) \sum_{k \equiv r \pmod{p^\beta}} \binom{n}{k} (-1)^k a^{[(k-r)/p^\alpha]} \equiv 0 \pmod{p^{[(n-p^{\alpha - 1}-\delta_{\beta,0})/\varphi(p^\alpha)] + \alpha - \beta}}. \]

**Proof.** When \( a = 1, (1.7) \) holds by Theorem 1.1 in the case \( l = 0 \). So it suffices to show that
\[ D := \sum_{k \equiv r \pmod{p^\beta}} \binom{n}{k} (-1)^k (a^{[(k-r)/p^\alpha]} - 1) \]
is divisible by \( p^\lambda \) where
\[ \lambda = \left\lfloor \frac{n - p^{\alpha - 1} - \delta_{\beta,0}}{\varphi(p^\alpha)} \right\rfloor + \alpha - \beta. \]

Write \( a = 1 + p^\alpha b \) with \( b \in \mathbb{Z} \). Then
\[ D = \sum_{k \equiv r \pmod{p^\beta}} \binom{n}{k} (-1)^k \sum_{0 < l \leq [(k-r)/p^\alpha]} \binom{[(k-r)/p^\alpha]}{l} (p^\alpha b)^l = \sum_{0 < l \leq [(n-r)/p^\alpha]} p^{l\alpha} b^l \sum_{k \equiv r \pmod{p^\beta}} \binom{n}{k} (-1)^k \binom{[(k-r)/p^\alpha]}{l}. \]
For each \( 0 < l \leq [(n-r)/p^\alpha] \), applying Theorem 1.1 with \( f(x) = \binom{x}{l} \) we find that
\[ p^{l\alpha} \sum_{k \equiv r \pmod{p^\beta}} \binom{n}{k} (-1)^k \binom{[(k-r)/p^\alpha]}{l} \equiv 0 \pmod{p^\lambda}. \]
Therefore \( D \equiv 0 \pmod{p^\lambda} \). This concludes the proof. \( \blacksquare \)
Let $a \in \mathbb{Z}$ be congruent to 1 modulo a prime $p$. By induction, $a^{p^\alpha} \equiv 1 \pmod{p^{\alpha+1}}$ for any $\alpha \in \mathbb{N}$. Let $n, r \in \mathbb{Z}$ and $n \geq p^{\alpha-1}$. If $\alpha \geq 2$, then by Corollary 1.3 in the case $\beta = \alpha$ we have

$$
\sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-a)^k \equiv 0 \pmod{p^{n-p^{\alpha-1} / \varphi(p^\alpha)}}.
$$

(1.8)

By the binomial theorem, (1.8) is also valid with $\alpha = 0$. We remark that (1.8) also holds when $\alpha = 1$, as pointed out by Fleck (cf. [D, p. 274]).

In the next section we will provide some lemmas. Section 3 is devoted to the proof of Theorem 1.1.

### 2. Some lemmas.

Let us recall the following well-known convolution identity of Chu and Vandermonde (see, e.g., [GKP, (5.27)]):

$$
\sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}
$$

for all $n = 0, 1, 2, \ldots$.

This can be seen by comparing the power series expansions of $(1+t)^x(1+t)^y$ and $(1+t)^{x+y}$.

**Lemma 2.1.** Let $f(x)$ be a function from $\mathbb{Z}$ to a field, and let $m, n \in \mathbb{Z}^+$. Then for any $r \in \mathbb{Z}$ we have

$$
\sum_{k=0}^{n} \binom{n}{k} (-1)^k f\left(\left\lfloor \frac{k-r}{m} \right\rfloor \right) = \sum_{k \equiv r \pmod{m}} \binom{n-1}{k} (-1)^{k-1} \Delta f\left(\left\lfloor \frac{k-r}{m} \right\rfloor \right),
$$

where $\overline{r} = r + m - 1$ and $\Delta f(x) = f(x+1) - f(x)$.

**Proof.** By the Chu–Vandermonde identity, for any $h \in \mathbb{N}$ we have

$$
\sum_{k=0}^{h} \binom{n}{k} (-1)^k = (-1)^h \sum_{k=0}^{h} \binom{n}{h-k} (-1)^{h-k} = (-1)^h \binom{n-1}{h}.
$$

Therefore

$$
\sum_{k=0}^{n} \binom{n}{k} (-1)^k f\left(\left\lfloor \frac{k-r}{m} \right\rfloor \right) = \sum_{j \in \mathbb{Z}} c_j f(j),
$$

where

$$
c_j = \sum_{\left\lfloor \frac{k-r}{m} \right\rfloor = j} \binom{n}{k} (-1)^k
$$

$$
= \sum_{0 \leq k < (j+1)m+r} \binom{n}{k} (-1)^k - \sum_{0 \leq k < jm+r} \binom{n}{k} (-1)^k
$$

$$
= (-1)^{(j+1)m+r-1} \binom{n-1}{(j+1)m+r-1} - (-1)^{jm+r-1} \binom{n-1}{jm+r-1}.
$$
(Note that \( \binom{n-1}{i} \neq 0 \) only for \( i \in \{0, \ldots, n-1\} \).) So we have
\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k f\left( \left\lfloor \frac{k-r}{m} \right\rfloor \right)
\]
\[
= \sum_{j \in \mathbb{Z}} (-1)^{(j+1)m+r-1} \binom{n-1}{(j+1)m + r - 1} f(j)
\]
\[
- \sum_{j \in \mathbb{Z}} (-1)^{jm+r-1} \binom{n-1}{jm + r - 1} f(j)
\]
\[
= \sum_{k \equiv r \pmod{m}} \binom{n-1}{k} (-1)^k f\left( \frac{k-r}{m} \right) - f\left( \frac{k-r+1}{m} \right)
\]
\[
= \sum_{k \equiv r \pmod{m}} \binom{n-1}{k} (-1)^{k-1} \Delta f\left( \frac{k-r}{m} \right).
\]
This proves the desired identity. ■

It is interesting to compare the identity in Lemma 2.1 with the following observation:
\[
\sum_{\substack{0 \leq k \leq n \\ k \equiv r \pmod{m}}} \Delta f\left( \frac{k-r}{m} \right)
= f\left( \left\lfloor \frac{n-r}{m} \right\rfloor + 1 \right) - f\left( \left\lfloor \frac{-r-1}{m} \right\rfloor + 1 \right),
\]
which appeared in the author’s proof of [S03, Lemma 3.1].

**Lemma 2.2.** Let \( p \) be a prime and \( \alpha \) be a positive integer. Then, for any \( k = 0, 1, \ldots, \varphi(p^{\alpha}) \), we have
\[
\binom{\varphi(p^{\alpha})}{k} \equiv \begin{cases} 
(-1)^k \pmod{p} & \text{if } p^{\alpha-1} \mid k, \\
0 \pmod{p} & \text{otherwise}.
\end{cases}
\]

**Proof.** Let \( k = k_0 + k_1 p + \cdots + k_{\alpha-1} p^{\alpha-1} \) be the \( p \)-adic expansion of \( k \), where \( k_0, k_1, \ldots, k_{\alpha-1} \in \{0, \ldots, p-1\} \). By a well-known theorem of E. Lucas (see, e.g., [HS]),
\[
\binom{\varphi(p^{\alpha})}{k} = \binom{\sum_{0 \leq j < \alpha-1} 0p^j + (p-1)p^{\alpha-1}}{\sum_{0 \leq j < \alpha-1} k_j p^j + k_{\alpha-1} p^{\alpha-1}}
\]
\[
\equiv \binom{p-1}{k_{\alpha-1}} \prod_{0 \leq j < \alpha-1} \binom{0}{k_j} \pmod{p}.
\]
If \( p^{\alpha-1} \mid k \), then \( k_j > 0 \) for some \( j < \alpha - 1 \), and hence \( \left( \frac{\varphi(p^\alpha)}{k} \right) \equiv 0 \pmod{p} \). When \( p^{\alpha-1} \mid k \), we have \( k_j = 0 \) for all \( j < \alpha - 1 \), and thus

\[
\left( \frac{\varphi(p^\alpha)}{k} \right) \equiv \left( \frac{p - 1}{k_{\alpha-1}} \right) = \prod_{0 < s \leq k_{\alpha-1}} \frac{p - s}{s} \pmod{p}
\]

\[
\equiv (-1)^{k_{\alpha-1}} \equiv (-1)^{p^{\alpha-1}k_{\alpha-1}} = (-1)^k \pmod{p}.
\]

This completes the proof. ■

3. Proof of Theorem 1.1. We use induction on \( w_l(\alpha, \beta) := l(\alpha+1) + \beta \).

In the case \( w_l(\alpha, \beta) = 0 \) (i.e., \( l = \beta = 0 \)), the desired result is trivial because \( \sum_{k=0}^{n} \binom{n}{k} (-1)^k = (1 - 1)^n = 0 \) for all \( n \in \mathbb{Z}^+ \).

Let \( w \) be a positive integer, and assume that the desired result holds whenever \( w_l(\alpha, \beta) < w \). Now we deal with the case \( w_l(\alpha, \beta) = w \).

Case 1: \( \beta = 0 \). In this case, \( l \) is positive. Let \( n \in \mathbb{N} \), \( n \geq p^{\alpha-1} \), \( r \in \mathbb{Z} \) and \( r = r + p^{\alpha-1} - 1 \). By Lemma 2.1,

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k f\left( \left\lfloor \frac{k - r}{p^\alpha} \right\rfloor \right) = \sum_{k \equiv r \pmod{p^\alpha}} \binom{n-1}{k} (-1)^{k-1} \Delta f\left( \frac{k - r}{p^\alpha} \right).
\]

Clearly \( \Delta f(x) \) is a polynomial of degree at most \( l-1 \), and \( \Delta f(a) \in \mathbb{Z}_p \) for all \( a \in \mathbb{Z} \). Also, \( w_{l-1}(\alpha, \alpha) < w_l(\alpha, 0) = w \). In view of (3.1) and the induction hypothesis,

\[
\text{ord}_p \left( \sum_{k=0}^{n} \binom{n}{k} (-1)^k f\left( \left\lfloor \frac{k - r}{p^\alpha} \right\rfloor \right) \right) \geq \left\lfloor \frac{(n - 1) - p^{\alpha-1} - (l - 1)}{\varphi(p^\alpha)} \right\rfloor - (l - 2)\alpha - \alpha
\]

\[
= \left\lfloor \frac{n - p^{\alpha-1} - l}{\varphi(p^\alpha)} \right\rfloor - (l - 1)\alpha - 0.
\]

(Note that this is trivial if \( n - 1 < p^{\alpha-1} \).) Similarly, when \( \alpha > 1 \), by (3.1) and the induction hypothesis we have

\[
\text{ord}_p \left( \sum_{k=0}^{n} \binom{n}{k} (-1)^k f\left( \left\lfloor \frac{k - r}{p^\alpha} \right\rfloor \right) \right) \geq \left\lfloor \frac{(n - 1) - p^{\alpha-1} - \delta_0,0}{\varphi(p^\alpha)} \right\rfloor - (l - 2)\alpha - \alpha
\]

\[
= \left\lfloor \frac{n - p^{\alpha-1} - \delta_0,0}{\varphi(p^\alpha)} \right\rfloor - (l - 1)\alpha - 0.
\]
Case 2: $0 < \beta \leq \alpha$. If $l = 0$ (i.e., $f(x)$ is constant), then $w_l(\beta, \beta) = w_l(\alpha, \beta) = w$ and it suffices to handle the case $\alpha = \beta$. In fact, when $l = 0$, $n \geq p^{\alpha - 1}$ and $r \in \mathbb{Z}$, provided that

$$\sum_{k \equiv r \pmod{p^\beta}} \binom{n}{k} (-1)^k f \left( \frac{k - r}{p^\beta} \right) \in p^{l(n-p^{\beta-1}/\varphi(p^\beta))} \mathbb{Z}_p,$$

we have

$$\sum_{k \equiv r \pmod{p^\beta}} \binom{n}{k} (-1)^k f \left( \left\lfloor \frac{k - r}{p^\alpha} \right\rfloor \right) \in p^{l(n-p^{\alpha-1}/\varphi(p^\alpha))-(0-1)\alpha-\beta} \mathbb{Z}_p,$$

because

$$\frac{n-p^{\beta-1}}{\varphi(p^\beta)} - \frac{n-p^{\alpha-1}}{\varphi(p^\alpha)} = \frac{n}{p^{\alpha-1}} \sum_{0 \leq s < \alpha - \beta} p^s \geq \alpha - \beta.$$ 

Below we simply let $(l-1)\alpha + \beta \geq 0$ (i.e., $\alpha = \beta$ if $l = 0$).

Let us use induction on $n \geq p^{\alpha - 1}$. The desired result is trivial when $n - p^{\alpha-1} < \varphi(p^\alpha) = p^\alpha - p^{\alpha-1}$.

Below we let $n \geq p^\alpha$ and assume that the desired result holds for smaller values of $n$ not less than $p^{\alpha-1}$. Note that $n' = n - \varphi(p^\beta) < n$ and also $n' \geq n - \varphi(p^\alpha) \geq p^{\alpha-1}$.

Let $r$ be any integer, and set

$$(3.2) \quad S = \sum_{k \equiv r \pmod{p^\beta}} \binom{n}{k} (-1)^k f \left( \left\lfloor \frac{k - r}{p^\alpha} \right\rfloor \right).$$

By the Chu–Vandermonde identity,

$$S = \sum_{k \equiv r \pmod{p^\beta}} \sum_{j=0}^{\varphi(p^\beta)} \binom{\varphi(p^\beta)}{j} \binom{n'}{k-j} (-1)^k f \left( \left\lfloor \frac{k - j}{p^\alpha} \right\rfloor \right)$$

$$= \sum_{j=0}^{\varphi(p^\beta)} \binom{\varphi(p^\beta)}{j} \sum_{k \equiv r \pmod{p^\beta}} \binom{n'}{k-j} (-1)^k f \left( \left\lfloor \frac{k - j - (r-j)}{p^\alpha} \right\rfloor \right)$$

$$= \sum_{j=0}^{\varphi(p^\beta)} \binom{\varphi(p^\beta)}{j} (-1)^j S_j,$$ 

where

$$(3.3) \quad S_j = \sum_{k \equiv r-j \pmod{p^\beta}} \binom{n'}{k} (-1)^k f \left( \left\lfloor \frac{k - (r-j)}{p^\alpha} \right\rfloor \right).$$

For any $j = 0, 1, \ldots, \varphi(p^\beta)$, by the induction hypothesis we have

$$\text{ord}_p(S_j) \geq \gamma = \left\lfloor \frac{n' - p^{\alpha-1} - l\delta_{\alpha,1}}{\varphi(p^\alpha)} \right\rfloor - (l-1)\alpha - \beta,$$
and Lemma 2.2 yields
\[
\left( \frac{\varphi(p^\beta)}{j} \right) \equiv \begin{cases} 
(-1)^j \pmod{p} & \text{if } p^\beta - 1 \mid j, \\
0 \pmod{p} & \text{if } p^\beta - 1 \nmid j.
\end{cases}
\]

Thus, if \( \gamma \geq 0 \) then
\[
S \equiv \sum_{j=0}^{p-1} \left( \frac{\varphi(p^\beta)}{p^{\beta-1}j} \right) \cdot (-1)^{p^\beta - 1}j \cdot S_{p^{\beta-1}j} \equiv \sum_{j=0}^{p-1} S_{p^{\beta-1}j} \pmod{p^{\gamma+1}}.
\]

Observe that
\[
\sum_{j=0}^{p-1} S_{p^{\beta-1}j} = \sum_{k \equiv r \pmod{p^{\beta-1}}} \left( \frac{n'}{k} \right)(-1)^{k} \left( \frac{k - (r - p^{\beta - 1}j_k)}{p^\alpha} \right),
\]
where \( j_k \) is the unique integer in \( \{0, \ldots, p - 1\} \) with \( p^\beta \mid k - (r - p^{\beta - 1}j_k) \).

For \( k \equiv r \pmod{p^{\beta-1}} \), clearly
\[
\frac{k - r + p^{\beta - 1}j_k}{p^\beta} = \frac{k - r' - p^{\beta - 1}(p - 1 - j_k)}{p^\beta} = \left\lfloor \frac{k - r'}{p^\beta} \right\rfloor,
\]
where \( r' = r - \varphi(p^\beta) \). Therefore \( \sum_{j=0}^{p-1} S_{p^{\beta-1}j} = S' \), where
\[(3.4) \quad S' = \sum_{k \equiv r' \pmod{p^{\beta-1}}} \left( \frac{n'}{k} \right)(-1)^{k} \left( \frac{k - r'}{p^\alpha} \right)
\]

From the above it follows that
\[
\text{ord}_p(S - S') \geq \gamma + 1 \geq \left\lfloor \frac{n - p^{\alpha-1} - l\delta_{\alpha,1}}{\varphi(p^\alpha)} \right\rfloor - (l - 1)\alpha - \beta.
\]

Let \( l_0 = l \) if \( \alpha = 1 \), and \( l_0 = \min\{l, \delta_{\beta - 1,0}\} \) if \( \alpha > 1 \). As \( w_l(\alpha, \beta - 1) < w_l(\alpha, \beta) = w \), by the induction hypothesis we have
\[
\text{ord}_p(S') \geq \left\lfloor \frac{n' - p^{\alpha-1} - l_0}{\varphi(p^\alpha)} \right\rfloor - (l - 1)\alpha - (\beta - 1) \geq \left\lfloor \frac{n' - p^{\alpha-1} - l\delta_{\alpha,1}}{\varphi(p^\alpha)} \right\rfloor - (l - 1)\alpha - \beta.
\]

(Note that if \( \alpha > 1 = \delta_{\beta - 1,0} \) then \( \beta = 1 < \alpha \) and hence \( n' - 1 + \varphi(p^\alpha) \geq n' + \varphi(p^\beta) = n \).)

Combining the above we finally obtain
\[
\text{ord}_p(S) = \text{ord}_p((S - S') + S') \geq \left\lfloor \frac{n - p^{\alpha-1} - l\delta_{\alpha,1}}{\varphi(p^\alpha)} \right\rfloor - (l - 1)\alpha - \beta.
\]

Since \( \delta_{\beta,0} = 0 \), this concludes the induction step in Case 2.

The proof of Theorem 1.1 is now complete. \( \blacksquare \)
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Department of Mathematics (and Institute of Mathematical Science)
Nanjing University
Nanjing 210093, People’s Republic of China
E-mail: zwsun@nju.edu.cn
http://pweb.nju.edu.cn/zwsun

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