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Short sums of restricted Möbius functions

by

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1. Introduction and result. In what follows, $10 \le y \le x$ are large real numbers, $e(t) = e^{2\pi i t}$, $[t]$ is the integer part of t and $\psi(t) = t - [t] - 1/2$. Finally, $\varepsilon > 0$ is an arbitrary small real number which does not need to be the same at each occurrence.

In 1976, Ramachandra [12] proved a general theorem for short sums of certain multiplicative functions from which he deduced that

$$\sum_{x<n\le x+y} \mu(n) = O(x^{1-1/B+\varepsilon} + y\exp(-(\log x)^{1/6}))$$

where $\mu(n)$ is the Möbius function and $B \ge 2$ is an admissible absolute constant occurring in zero-density estimates. From the work of Huxley [8], we know that $B = 12/5$ is admissible so that we have

$$\sum_{x<n\le x+y} \mu(n) = O(x^{7/12+\varepsilon} + y\exp(-(\log x)^{1/6})). \tag{1}$$

The density hypothesis states that $B = 2$ is admissible, so that

$$\sum_{x<n\le x+y} \mu(n) = O(x^{1/2+\varepsilon} + y\exp(-(\log x)^{1/6})) \tag{2}$$

if the density hypothesis is true.

It should be mentioned that (1) was also independently discovered by Motohashi [11], and that the paper of Ramachandra was later refined (see [13, 14]) and generalized to problems in number fields (see [5]).

From (1) we could easily infer that if $x^{7/12+\varepsilon} \le y \le x$ then

$$\sum_{x<n\le x+y} \mu(n) = o(y) \tag{3}$$

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unconditionally. Using the important identity $\sum_{d\mid n}\mu(d) = 0$ valid for any integer $n > 1$, we can write

$$\sum_{x<n\le x+y} \mu(n) = -\sum_{x<n\le x+y} M(n;x)$$

where we set

$$M(n;t) := \sum_{\substack{d\mid n\\ d\le t}} \mu(d)$$

so that (3) could be written as

$$\sum_{x<n\le x+y} M(n;x) = o(y) \tag{4}$$

for $x^{7/12+\varepsilon} \le y \le x$ unconditionally. With (2) and (4) in mind, this paper deals with the following slightly different version of this problem: we ask for the greatest exponent $\theta \in (0,1]$ so that the estimate

$$\sum_{x<n\le x+y} M(n;x^{\theta}) = o(y)$$

holds true for $x^{1/2+\varepsilon} \le y \le x$. If the density hypothesis is true, then $\theta = 1$ is admissible. Unconditionally, the answer depends on estimates of twisted exponential sums of types I and II. This leads to the following result:

Theorem 1.1. *Let $x^{1/2+6\varepsilon} \le y \le x$ be large real numbers. Then*

$$\sum_{x<n\le x+y} M(n;x^{4/7}) = y\sum_{d\le x^{4/7}} \frac{\mu(d)}{d} + O_{\varepsilon}(yx^{-\varepsilon}).$$

2. The sums $\sum_n \mu(n)\psi(x/n)$

2.1. Introduction of exponential sums. In this section,

$$\|t\| = \min\{1/2+\psi(t), 1/2-\psi(t)\}$$

is the distance of t to the nearest integer. We begin with the following result:

Proposition 2.1. *Let x be a sufficiently large real number, $\varepsilon > 0$ be a small real number and $4 \le H \le R \le x$ be integers. Then*

$$\sum_{R<n\le 2R} \mu(n)\psi\left(\frac{x}{n}\right) = -\sum_{0<|h|\le H} \frac{1}{2\pi i h} \sum_{R<n\le 2R} \mu(n)e\left(\frac{hx}{n}\right) + O_{\varepsilon}(RH^{-1}x^{\varepsilon}).$$

The proof needs the following two lemmata:

Lemma 2.2. *Let $N \ge 1$ and $H \ge 4$ be integers, and $f : [N,2N] \to \mathbb{R}$ be any map. For any real number $0 < \delta \le 1/4$ set*

$$\mathcal{R}(f,N,\delta) := |\{n \in (N,2N] \cap \mathbb{Z} : \|f(n)\| < \delta\}|$$

and let $K := [\log H/\log 2]$. *Then*

$$\sum_{N<n\le 2N} \min\Big(1, \frac{1}{H\|f(n)\|}\Big) < 24NH^{-1} + 2\sum_{k=0}^{K-2} 2^{-k}\mathcal{R}(f,N,2^kH^{-1}).$$

Proof. We have

$$\sum_{N<n\le 2N} \min\Big(1, \frac{1}{H\|f(n)\|}\Big) = \sum_{\substack{N<n\le 2N\\ \|f(n)\|<H^{-1}}} 1 + \frac{1}{H}\sum_{\substack{N<n\le 2N\\ \|f(n)\|\ge H^{-1}}} \frac{1}{\|f(n)\|}$$
$$= \mathcal{R}(f,N,H^{-1}) + \frac{1}{H}\sum_{\substack{N<n\le 2N\\ \|f(n)\|\ge H^{-1}}} \frac{1}{\|f(n)\|}.$$

Since

$$\{n\in(N,2N]\cap\mathbb{Z} : \|f(n)\|\ge H^{-1}\}$$
$$\subseteq \bigcup_{k=1}^{K}\{n\in(N,2N]\cap\mathbb{Z} : 2^{k-1}H^{-1}\le\|f(n)\|<2^kH^{-1}\}$$

we get

$$\sum_{\substack{N<n\le 2N\\ \|f(n)\|\ge H^{-1}}} \frac{1}{\|f(n)\|} \le \sum_{k=1}^{K}\sum_{\substack{N<n\le 2N\\ 2^{k-1}H^{-1}\le\|f(n)\|<2^kH^{-1}}} \frac{1}{\|f(n)\|}$$
$$\le (N+1)(2^{1-K}+2^{2-K})H + \sum_{k=1}^{K-2}\sum_{\substack{N<n\le 2N\\ 2^{k-1}H^{-1}\le\|f(n)\|<2^kH^{-1}}} \frac{1}{\|f(n)\|}$$
$$\le 6\cdot 2^{-K}(N+1)H + 2H\sum_{k=1}^{K-2} 2^{-k}\sum_{\substack{N<n\le 2N\\ \|f(n)\|<2^kH^{-1}}} 1$$
$$< 12(N+1) + 2H\sum_{k=1}^{K-2} 2^{-k}\mathcal{R}(f,N,2^kH^{-1})$$

since $2^{-K} < 2H^{-1}$. Thus we get

$$\sum_{N<n\le 2N} \min\Big(1, \frac{1}{H\|f(n)\|}\Big) < \mathcal{R}(f,N,H^{-1}) + 24NH^{-1}$$
$$+ 2\sum_{k=1}^{K-2} 2^{-k}\mathcal{R}(f,N,2^kH^{-1}),$$

which implies the desired result. ∎

Lemma 2.3. *Let* $1 \le y \le x$ *and* $0 < \varepsilon < 1/2$ *be real numbers. If* $\tau(n)$ *is the usual divisor function, then*

$$\sum_{x-y<n\le x+y} \tau(n) \ll_\varepsilon yx^\varepsilon.$$

Proof. If $1 \le y \le x^\varepsilon$ then

$$\sum_{x-y<n\le x+y} \tau(n) \le (2y+1) \max_{x-y<n\le x+y} \tau(n) \ll_\varepsilon yx^\varepsilon,$$

and if $x^\varepsilon < y \le x$ then the result is a consequence of Shiu's theorem [15]. ■

Now we turn to the proof of Proposition 2.1.

Proof of Proposition 2.1. Since

$$\psi(t) = -\sum_{0<|h|\le H} \frac{e(ht)}{2\pi ih} + O\left(\min\left(1, \frac{1}{H\|t\|}\right)\right)$$

we easily see using Lemma 2.2 that

$$\begin{aligned}
\sum_{R<n\le 2R} \mu(n)\psi\left(\frac{x}{n}\right) &= -\sum_{0<|h|\le H} \frac{1}{2\pi ih} \sum_{R<n\le 2R} \mu(n)e\left(\frac{hx}{n}\right) \\
&\quad + O\left(\sum_{R<n\le 2R} \min\left(1, \frac{1}{H\|x/n\|}\right)\right) \\
&= -\sum_{0<|h|\le H} \frac{1}{2\pi ih} \sum_{R<n\le 2R} \mu(n)e\left(\frac{hx}{n}\right) \\
&\quad + O\left(RH^{-1} + \sum_{k=0}^{[\log H/\log 2]-2} 2^{-k}\mathcal{R}\left(\frac{x}{n}, R, \frac{2^k}{H}\right)\right).
\end{aligned}$$

Now interchanging the summations and using Lemma 2.3 we obtain

$$\begin{aligned}
\mathcal{R}\left(\frac{x}{n}, R, \frac{2^k}{H}\right) &\le \sum_{R<n\le 2R} \left(\left[\frac{x}{n} + \frac{2^k}{H}\right] - \left[\frac{x}{n} - \frac{2^k}{H}\right]\right) \\
&\le \sum_{x-2^{k+1}RH^{-1}<m\le x+2^{k+1}RH^{-1}} \sum_{\substack{d\mid m \\ R<d\le 2R}} 1 \\
&\le \sum_{x-2^{k+1}RH^{-1}<m\le x+2^{k+1}RH^{-1}} \tau(m) \ll_\varepsilon 2^k RH^{-1} x^\varepsilon,
\end{aligned}$$

which proves Proposition 2.1. ■

The following result improves slightly on Lemma 8 of [1].

COROLLARY 2.4. *Under the hypothesis of Proposition* 2.1 *with* $10 \le y \le x$ *we have*

$$\sum_{R<n\le 2R} \mu(n)\Big(\psi\Big(\frac{x+y}{n}\Big) - \psi\Big(\frac{x}{n}\Big)\Big)$$
$$\ll \frac{y}{R} \max_{R\le R'\le 2R} \max_{x\le z\le x+y} \max_{H_1\le H} \Big| \sum_{R<n\le R'} \mu(n) \sum_{H_1<h\le 2H_1} e\Big(\frac{hz}{n}\Big)\Big| \log H$$
$$+ RH^{-1}x^{\varepsilon}.$$

Proof. Using Proposition 2.1 we get

$$\sum_{R<n\le 2R} \mu(n)\Big(\psi\Big(\frac{x+y}{n}\Big) - \psi\Big(\frac{x}{n}\Big)\Big)$$
$$= -\sum_{0<|h|\le H} \frac{1}{2\pi i h} \sum_{R<n\le 2R} \mu(n)\Big\{e\Big(\frac{h(x+y)}{n}\Big) - e\Big(\frac{hx}{n}\Big)\Big\}$$
$$+ O_{\varepsilon}(RH^{-1}x^{\varepsilon}),$$

and the identity

$$e(a(x+y)) - e(ax) = 2\pi i a \int_x^{x+y} e(at)\,dt$$

and Abel summation give the asserted result. ∎

2.2. Sums of types I and II. Corollary 2.4 reduces the problem to finding bounds for sums

$$\sum_{n\sim R} \mu(n) \sum_{h\sim H} e\Big(\frac{hx}{n}\Big).$$

Such bounds are achieved by using clever identities discovered by Vaughan (see [10] for example) and generalized by Heath-Brown [6]. We sum up the process in Lemma 2.5 below (see also Lemma 2 of [3]). We consider integers $M, N, R, R' \ge 1$ such that $R < R' \le 2R$ and let $S > 0$ be any real number. If $f : (R, R'] \to \mathbb{C}$ is any function, it is convenient to define *sums of type I* (related to f) to be the sums

$$\mathcal{S}_I := \sum_{\substack{M<m\le 2M \\ R<mn\le R'}} \sum_{N<n\le 2N} a_m f(mn)$$

and *sums of type II* (related to f) to be the sums

$$\mathcal{S}_{II} := \sum_{\substack{M<m\le 2M \\ R<mn\le R'}} \sum_{N<n\le 2N} a_m b_n f(mn)$$

where a_m, b_n are complex numbers supported respectively on $(M, 2M]$ and $(N, 2N]$ and satisfying $a_m \ll_\varepsilon m^\varepsilon$ and $b_n \ll_\varepsilon n^\varepsilon$.

Lemma 2.5. *Suppose that the estimates*

$$\mathcal{S}_I \ll S \quad \text{for } N \gg R^{1/2},$$
$$\mathcal{S}_{II} \ll S \quad \text{for } R^{1/3} \ll N \ll R^{1/2}$$

hold true for all sums of type I and type II. Then

$$\sum_{R<n\le R'} \mu(n) f(n) \ll S(\log 3R)^5.$$

It is well-known that the multiplicative restrictions $R < mn \le R'$ could be removed from sums $\mathcal{S}_I$ and $\mathcal{S}_{II}$ at a cost of a factor $\log R$ (see [1, Lemma 15] for instance).

To treat sums of type I we appeal to the following result which is the estimate (5.9) of Corollary 8 from [9].

Lemma 2.6. *Let $X > 0$ be a real number, $H, M, N \ge 1$ be integers and $\alpha, \beta \in \mathbb{R}$ such that $\beta \ne -1, 0$ and $\alpha/(1+\beta) \ne 0, 1$. Let $I \subseteq (N, 2N]$ and let $(a_m), (c_h) \in \mathbb{C}$ satisfy $|a_m|, |c_h| \le 1$. Then for all $\varepsilon > 0$,*

$$\sum_{H\le h<2H} \sum_{M\le m<2M} \sum_{n\in I} a_m c_h e\left(X\left(\frac{m}{M}\right)^\alpha \left(\frac{hN}{nH}\right)^\beta\right)$$
$$\ll \{(X^3H^6M^6N^2)^{1/8} + H(XM)^{1/2} + HM + (XH^3N)^{1/4}M + X^{-1}HMN\}(HMN)^\varepsilon.$$

In the last two decades, many authors provided nontrivial bounds for sums of type II. Among these we pick up the following estimate with the exponent pair $(k, l) = (1/2, 1/2)$ ([4], see also [2]). The idea of the proof goes back to Heath-Brown [7].

Lemma 2.7. *Let $z > 0$ be a real number, $H, M, N \ge 1$ be integers and let $(a_m), (B_{h,n}) \in \mathbb{C}$ satisfy $|a_m|, |B_{h,n}| \le 1$. Set $L := \log(2HMN)$. Then*

$$\sum_{M\le m<2M} \sum_{H\le h<2H} \sum_{N\le n<2N} a_m B_{h,n} e\left(\frac{hz}{mn}\right)$$
$$\ll \{H(zM^3N^4)^{1/6} + M(HN)^{1/2} + M^{1/2}HN + (Hz^{-1})^{1/2}(MN)^{3/2}\}L^3.$$

3. Proof of Theorem 1.1

Proposition 3.1. *Let $x^{2/5} \le y \le x$ be real numbers, and $10 \le R \le x$ be a large integer. Then for every $\varepsilon > 0$ we have*

$$\sum_{R<n\le 2R}\mu(n)\left(\psi\left(\frac{x+y}{n}\right)-\psi\left(\frac{x}{n}\right)\right)$$
$$\ll\{x^{1/12}y^{1/2}R^{7/24}+x^{-1/24}y^{3/4}R^{3/16}+x^{-1/12}y^{1/2}R^{11/24}+x^{-13/24}y^{3/4}R^{41/48}$$
$$+x^{1/32}y^{7/16}R^{-5/64}+x^{3/8}y^{1/4}R^{-3/16}+x^{-1}yR\}x^{\varepsilon}.$$

Proof. Note that if $10\le R\le (x^2y^{12})^{1/17}$, then $x^{1/12}y^{1/2}R^{7/24}\ge R$ so that we may suppose $(x^2y^{12})^{1/17}<R\le x$. To treat the sum of Corollary 2.4, we apply Lemma 2.5 with

$$f(n)=\sum_{H_1<h\le 2H_1}e\left(\frac{hz}{n}\right)$$

where $R<n\le R'$, $1\le H_1\le H$ and $x\le z\le x+y$. Using Lemma 2.6 with $-\alpha=c_h=\beta=1$, $H=H_1$, $z=XMNH_1^{-1}$ and supposing that $MN\asymp R$ with $N\gg R^{1/2}$, we get

$$\mathcal{S}_I\ll\{(zH_1^9R)^{1/8}+(z^2H_1^6R^{-1})^{1/4}+H_1R^{1/2}+H_1(z^2R^3)^{1/8}+z^{-1}R^2\}(H_1R)^{\varepsilon}$$

and, similarly, using Lemma 2.7 with $B_{h,n}=b_n$, $H=H_1$ and supposing that $MN\asymp R$ with $R^{1/3}\ll N\ll R^{1/2}$, we obtain

$$\mathcal{S}_{II}\ll\{H_1(z^2R^7)^{1/12}+H_1^{1/2}R^{5/6}+H_1R^{3/4}+(H_1z^{-1})^{1/2}R^{3/2}\}(\log 2H_1R)^4$$

so that for every integer $4\le H\le R$, we get, using Corollary 2.4 and Lemma 2.5,

$$\sum_{R<n\le 2R}\mu(n)\left(\psi\left(\frac{x+y}{n}\right)-\psi\left(\frac{x}{n}\right)\right)$$
$$\ll\{yH(x^2R^{-5})^{1/12}+yH^{1/2}R^{-1/6}+yHR^{-1/4}+y(x^{-1}H^9R^{-7})^{1/8}$$
$$+y(x^2H^6R^{-5})^{1/4}+yH(x^2R^{-5})^{1/8}+x^{-1}yR\}(HR)^{\varepsilon}+RH^{-1}x^{\varepsilon}.$$

Since $y\ge x^{2/5}$, we have $R>(x^2y^{12})^{1/17}\ge x^{2/5}$, so that $yH(x^2R^{-5})^{1/8}$ is dominated by the first term, and the choice of $H=[4x^{-1/12}y^{-1/2}R^{17/24}]$ gives the desired result. ∎

The following result is an easy consequence of Proposition 3.1.

Corollary 3.2. *If $x^{1/2+6\varepsilon}\le y\le x$ then*

$$\max_{x^{1/2}<R\le x^{4/7}}\sum_{R<n\le 2R}\mu(n)\left(\psi\left(\frac{x+y}{n}\right)-\psi\left(\frac{x}{n}\right)\right)\ll yx^{-2\varepsilon}.$$

Proof. Indeed, we get

$$\max_{x^{1/2}<R\le x^{4/7}} \sum_{R<n\le 2R} \mu(n)\left(\psi\left(\frac{x+y}{n}\right)-\psi\left(\frac{x}{n}\right)\right)$$
$$\ll x^{1/4+\varepsilon}y^{1/2}+x^{11/168+\varepsilon}y^{3/4}+x^{9/32+\varepsilon}y^{1/4}+yx^{-3/7+\varepsilon}\ll yx^{-2\varepsilon}$$

since $x^{1/2+6\varepsilon}\le y\le x$. ∎

Now we are able to prove Theorem 1.1. Interchanging the summations we obtain

$$\begin{aligned}\sum_{x<n\le x+y} M(n;x^{4/7}) &= \sum_{d\le x^{4/7}} \mu(d)\left(\left[\frac{x+y}{d}\right]-\left[\frac{x}{d}\right]\right)\\ &= y\sum_{d\le x^{4/7}}\frac{\mu(d)}{d}-\sum_{d\le x^{4/7}}\mu(d)\left(\psi\left(\frac{x+y}{d}\right)-\psi\left(\frac{x}{d}\right)\right)\\ &= y\sum_{d\le x^{4/7}}\frac{\mu(d)}{d}-\sum_{x^{1/2}<d\le x^{4/7}}\mu(d)\left(\psi\left(\frac{x+y}{d}\right)-\psi\left(\frac{x}{d}\right)\right)\\ &\quad + O(x^{1/2})\end{aligned}$$

and using Corollary 3.2 along with a splitting argument gives the result. ∎

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