ACTA ARITHMETICA
153.4 (2012)

Weak Néron models for cubic polynomial maps over a non-Archimedean field

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1. Introduction. Let V be smooth variety defined over a discretely valued non-Archimedean field $\mathbb{K}$ and let $\phi : V \to V$ be a morphism on V. Assume that there exists a divisor $E \in \mathrm{Div}(V) \otimes \mathbb{R}$ and a real number $\alpha > 1$ such that $\phi^*(E)$ is linear equivalent to αE. Call and Silverman [7] showed that there exists a Weil local height function $\hat{\lambda}_{V,E,\phi}$ that plays a role analogous to the Néron–Tate local height function on an Abelian variety. In the case of Abelian varieties, the Néron–Tate local height can be computed using intersection theory on the Néron model (see [13, 5]) of the Abelian variety in question. This motivates defining an analogous notion for a variety V to which a morphism $\phi : V \to V$ is attached. Such a generalization was proposed in the same paper [7] and gives rise to the notion of *weak Néron model* of the couple $(V/\mathbb{K}, \phi)$ over the ring of integers of $\mathbb{K}$ (see also [5, pp. 73 ff.] for an alternative definition which has nothing to do with the setting discussed here). It is shown in [7] that indeed if the pair $(V/\mathbb{K}, \phi)$ has a weak Néron model, then the local height $\hat{\lambda}_{V,E,\phi}$ can also be computed using intersection theory on the model.

However, in general a weak Néron model for a given pair $(V/\mathbb{K}, \phi)$ may not exist. In [10], Hsia showed that for a rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ over $\mathbb{K}$, the existence of a weak Néron model is closely related to dynamical properties of ϕ and more precisely to the presence of points of the Julia set of ϕ inside $\mathbb{P}^1(\mathbb{K})$. This leads to the question of whether or not one can effectively determine the existence of a weak Néron model for a given pair $(\mathbb{P}^1/\mathbb{K}, \phi)$. In the case of elliptic curves (one-dimensional Abelian variety), the Tate algorithm [14] computes, among other things, the reduction type of an el-

2010 *Mathematics Subject Classification*: Primary 37P05; Secondary 11G99, 14G20, 37P20.
Key words and phrases: weak Néron model, Julia set, fixed point, non-Archimedean dynamics.

DOI: 10.4064/aa153-4-5

liptic curves given by a Weierstrass equation. Analogously to the situation of elliptic curves, the question we raise here can be viewed as the search for an algorithm determining the existence of the weak Néron model for a given pair $(\mathbb{P}^1/\mathbb{K}, \phi)$ and computing the model when it exists. However, for general rational maps ϕ on $\mathbb{P}^1/\mathbb{K}$, it does not seem clear that such an effective algorithm exists. On the other hand, in the case of polynomial maps we think it might be plausible to have optimistic expectation (see Questions in Section 4) The aim of this paper is to give a positive answer to this question for cubic polynomial maps.

As mentioned above, whether or not a weak Néron model exists for a pair $(\mathbb{P}^1/\mathbb{K}, \phi)$ is closely related to the dynamics induced by the action of the given morphism ϕ on $\mathbb{P}^1(\mathbb{K})$. In this note, we consider the dynamics of cubic polynomial maps over the non-Archimedean field $\mathbb{K}$. In the classical theory of dynamical systems, dynamical properties of cubic polynomials over $\mathbb{C}$ have received a lot of attention since the pioneering work of B. Branner and J. H. Hubbard in [6]. To understand the parameter space of those polynomials one is naturally led to study the dynamics of cubic polynomials over the field of Puiseux series over $\mathbb{C}$ (or $\overline{\mathbb{Q}}$) and this has been carried out by J. Kiwi in [11].

Although the theory of non-Archimedean dynamical systems has its origin in the study of arithmetic problems, Kiwi's work shows that non-Archimedean dynamics can be of great value in understanding complex dynamics as well. On the other hand, over a discrete valued field it seems possible that one can describe every dynamics occurring for a cubic polynomial, even though examples show that they can be very complicated. We will be concerned with the determination of the $\mathbb{K}$-*rational Julia set* (see Section 2) associated to a given cubic polynomial $\phi(z) \in \mathbb{K}[z]$. As a consequence of our main result (Theorem 2.1), we show that the non-emptiness of the $\mathbb{K}$-rational Julia set of ϕ is closely related to the existence of a $\mathbb{K}$-rational repelling fixed point of ϕ (see Theorem 2.2).

The plan of the paper is as follows. In Section 2 we recall the definition of a weak Néron model for a given pair $(V/\mathbb{K}, \phi)$ where we restrict ourselves to the case $V = \mathbb{P}^1$. Then we state our main results (Theorems 2.1 and 2.2). Section 3 is devoted to the proof of Theorem 2.1. In Section 4 we give examples of polynomials maps with degree higher than three for which Theorem 2.2 does not hold.

2. Weak Néron models and Julia sets. Let us introduce some basic notation. $\mathbb{K}$ will denote a field endowed with a non-Archimedean discrete valuation v, which will be assumed to have $\mathbb{Z}$ as value group. We will furthermore assume that $(\mathbb{K}, v)$ is Henselian (see [8] for instance). It is a well known fact that for any algebraic extension $\mathbb{L}$ of $\mathbb{K}$ the valuation v has a unique extension to a valuation on $\mathbb{L}$ (and thus that the Galois group acts by isometries). We will use the same symbol v to denote the extension of the valuation to the algebraic closure of $\mathbb{K}$. We will denote by $\mathcal{O}_{\mathbb{K}}$ the valuation ring of $\mathbb{K}$ and by $\mathcal{M}_{\mathbb{K}}$ its unique maximal ideal. The residue field is then $\Bbbk = \mathcal{O}_{\mathbb{K}}/\mathcal{M}_{\mathbb{K}}$. We fix a uniformizer π of $\mathbb{K}$ so that $v(\pi) = 1$ and we endow $\mathbb{K}$ with an absolute value $|\cdot|$ associated to v so that $|\pi| < 1$.

Let $\phi \in \mathbb{K}(z)$ be a rational map. As a map, it acts on $\mathbb{P}^1(\mathbb{K})$ as well as on $\mathbb{P}^1(\mathbb{C}_v)$, where $\mathbb{C}_v$ is the completion (with respect to the unique extension of v) of an algebraic closure of $\mathbb{K}$. On the latter, one can define the *Julia set* of ϕ, which we denote by J_ϕ, as the set of points around which the family of iterates of ϕ is not equicontinuous with respect to the chordal metric on $\mathbb{P}^1(\mathbb{C}_v)$ (see [10]). In contrast with the complex case, the Julia set may be empty, for instance if the polynomial ϕ has *good reduction* as defined in [12] (see §2.1 for the definition). Furthermore, as usually $\mathbb{K}$ is far from being algebraically closed, it is often the case that J_ϕ is non-empty (and even rather large) while the $\mathbb{K}$-rational Julia set $J_\phi(\mathbb{K}) := J_\phi \cap \mathbb{P}^1(\mathbb{K})$ is itself empty, meaning that all the complicated dynamical behavior of ϕ takes place outside of $\mathbb{K}$.

2.1. Reduction of a morphism and weak Néron model. One advantage of working over a non-Archimedean field is that one can reduce maps modulo the maximal ideal $\mathcal{M}_{\mathbb{K}}$. Fixing homogeneous coordinates $[x, y]$ on $\mathbb{P}^1$ over $\mathbb{K}$, we may write a rational map on $\mathbb{P}^1$ as $\phi([x, y]) = [f(x, y), g(x, y)]$ where $f, g \in \mathcal{O}_{\mathbb{K}}[x, y]$ are homogeneous polynomials in x, y without common divisor. Multiplying both homogeneous coordinates by an appropriate $\lambda \in \mathbb{K}^*$, we may further assume that some coefficient of f, g is a unit of $\mathcal{O}_{\mathbb{K}}$. Let $\widetilde{\phi} = [\widetilde{f}, \widetilde{g}]$, where $\widetilde{f}, \widetilde{g}$ denote the reductions of the polynomials f, g by reducing their coefficients modulo the maximal ideal $\mathcal{M}_{\mathbb{K}}$ respectively. We say that ϕ *has good reduction* (with respect to the coordinate $[x, y]$) if $\deg(\widetilde{\phi}) = \deg(\phi)$ when $\widetilde{\phi}$ is viewed as a morphism on $\mathbb{P}^1$ over the residue field $\Bbbk$ (cf. [12, §4]). A morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ over $\mathbb{K}$ will be said to have good reduction if there exists a possible change of coordinates over $\mathbb{K}$ such that ϕ has good reduction with respect to the new system of coordinates. Moreover, ϕ is said to have *potential good reduction* if there exists a finite extension $\mathbb{L}$ over $\mathbb{K}$ so that after a change of coordinates over $\mathbb{L}$ the given rational map ϕ has good reduction with respect to the new system of coordinates.

Even if ϕ does not have good reduction, it is possible to define the reduction of ϕ by considering a *weak Néron model* of ϕ which is a dynamical analogue of a Néron model for an Abelian variety over $\mathbb{K}$. As in the arithmetic theory of Abelian varieties, the notion of weak Néron model of a morphism (first introduced by Call and Silverman in [7]) is devised to study the canonical (local) height associated to a morphism $\phi : V \to V$ on a smooth, projective variety $V/\mathbb{K}$. In the following, we recall the definition of a weak Néron model by restricting ourselves to the case of rational maps on $\mathbb{P}^1$ over $\mathbb{K}$. We refer the readers to the paper [7] for a general definition.

Let $S = \mathrm{Spec}(\mathcal{O}_{\mathbb{K}})$. We say that $(\mathbb{P}^1/\mathbb{K}, \phi)$ *admits a weak Néron model over* S if there exists a smooth, separated scheme $\mathcal{V}$ of finite type over S together with a morphism $\Phi : \mathcal{V}/S \to \mathcal{V}/S$ such that the following conditions hold:

(i) the generic fiber of $\mathcal{V}/S$ is isomorphic to $\mathbb{P}^1$ over $\mathbb{K}$,
(ii) every point $P \in \mathbb{P}^1(\mathbb{K})$ extends to a section $\overline{P} : S \to \mathcal{V}$,
(iii) the restriction of Φ to the generic fiber of $\mathcal{V}$ is exactly ϕ.

To fix the terminology, we say that a separated scheme $\mathcal{V}/S$ is a *model* of $\mathbb{P}^1$ (over S) if it satisfies (i); if furthermore it satisfies (ii) we say that it has the *extension property for étale points*. The most basic example of a map admitting a weak Néron model is that of a rational map having good reduction, in which case we can take $\mathcal{V} = \mathbb{P}^1_S$.

In general, one cannot expect that a weak Néron model always exists for an arbitrary rational map ϕ. The link between the notions of Julia set and weak Néron model was made by Hsia in [10] who proved that if the $\mathbb{K}$-rational Julia set is non-empty, $(\mathbb{P}^1/\mathbb{K}, \phi)$ does not admit any weak Néron model. This provides an easy way of constructing maps without weak Néron model, by choosing one with a sufficiently complicated dynamics, or even just one with a $\mathbb{K}$-rational repelling fixed point. On the other hand, some interesting families of rational maps, like Lattès maps arising from isogenies on elliptic curves, have been proven to admit weak Néron models (see [2, 3]). If ϕ is a polynomial, by [10, Theorems 4.3 and 4.8] we have a better result: $(\mathbb{P}^1/\mathbb{K}, \phi)$ admits a weak Néron model if and only if the $\mathbb{K}$-rational Julia set is empty. Thus, in practice, one can determine whether the $\mathbb{K}$-rational Julia set of a given polynomial map ϕ is empty or not, although it can be very complicated.

2.2. Statement of the main result. Before we state our main result, let us recall that a *fixed point* $z \in \mathbb{C}_v$ is a point such that $\phi(z) = z$ and that z is *repelling* if $v(\phi'(z)) < 0$ (equivalently $|\phi'(z)| > 1$).

THEOREM 2.1. *Let* $\phi \in \mathbb{K}[z]$ *be a polynomial of degree three. Then* $(\mathbb{P}^1/\mathbb{K}, \phi)$ *admits a weak Néron model if and only if* ϕ *admits no* $\mathbb{K}$*-rational repelling fixed point.*

REMARK 1. It follows from [10, Theorem 3.1] that $(\mathbb{P}^1/\mathbb{K}, \phi)$ does not admit a weak Néron model if there exists a repelling periodic point for ϕ. A repelling periodic point is necessarily in the Julia set of ϕ, so that the $\mathbb{K}$-rational Julia set of ϕ is non-empty if ϕ admits a $\mathbb{K}$-rational repelling fixed point. The converse is not a priori clear as there could well exist some high period repelling points in $J_\phi(\mathbb{K})$ that could be hard to detect due to the high degree of the equation defining them.

Also, notice that computing or even determining non-emptiness of the $\mathbb{K}$-rational Julia set is often difficult in general (although it has been carried out completely in the quadratic case in [1]) but for cubic polynomials we prove the following theorem giving an easily verifiable criterion for determining the existence of $\mathbb{K}$-rational Julia set.

THEOREM 2.2. *With notation as above, let $\phi \in \mathbb{K}[z]$ be a polynomial of degree three. Then the $\mathbb{K}$-rational Julia set $J_\phi(\mathbb{K})$ of ϕ is non-empty if and only if it contains a repelling fixed point of ϕ.*

As a direct consequence of Theorem 2.2, we have the following.

COROLLARY 2.1. *If all $\mathbb{K}$-rational fixed points are non-repelling, then all periodic points in $\mathbb{K}$ are non-repelling.*

We first show that Theorem 2.1 implies Theorem 2.2.

Proof of Theorem 2.2. As a repelling fixed point is necessarily in the Julia set, we see that one direction of the implication is clear. So, let us assume that there is no $\mathbb{K}$-rational repelling fixed point of ϕ. By Theorem 2.1, $(\mathbb{P}^1/\mathbb{K}, \phi)$ admits a weak Néron model over S. We know that if $(\mathbb{P}^1/\mathbb{K}, \phi)$ admits a weak Néron model over S, then by Theorem 3.3 of [10], the family of morphisms $\{\phi^i\}_{i=0}^{\infty}$ is equicontinuous on $\mathbb{P}^1(\mathbb{K})$. It follows that the rational Julia set $J_\phi(K)$ is empty as desired. ∎

The proof of Theorem 2.1 will have the following simple corollary:

COROLLARY 2.2. *Let $\phi \in \mathbb{K}[z]$ be a polynomial of degree three. If all the fixed points of ϕ (in $\mathbb{C}_v$) are non-repelling, then J_ϕ is empty.*

REMARK 2.

(1) Corollary 2.2 is true only for non-Archimedean dynamics as the example $\phi(z) = z^3 + z$ shows: its complex Julia set is non-empty and it admits a lot of repelling periodic points, although its only fixed point 0 is non-repelling.
(2) Theorem 3 of Bézivin's article [4] implies that for a degree 3 polynomial with coefficients in $\mathbb{C}_p$ whose $\mathbb{C}_p$-Julia set is non-empty, the Julia set is equal to the closure of the set of repelling periodic points.

The remaining part of this note is devoted to proving Theorem 2.1. We present our proof in the next section. In the final section, we give a counterexample to the conclusion of Theorem 2.2 in higher degrees and pose some questions.

3. Proof of the main result

3.1. Preliminaries. As the theorem is stated in terms of fixed points, we let $\phi(z) = g(z) + z$ so $g(z) = 0$ is the fixed point equation. We will split the proof into two parts, according to whether ϕ admits a fixed point in $\mathbb{K}$ or not, in which case the polynomial g is irreducible over $\mathbb{K}$. Before dealing with the proof proper, we make two remarks:

(1) it is not hard to see, from the definition of a weak Néron model, that if $(\mathbb{P}^1/\mathbb{K}, \phi)$ admits a weak Néron model, then so does $(\mathbb{P}^1/\mathbb{L}, \phi)$, where $\mathbb{L}$ is any unramified algebraic extension of $\mathbb{K}$;
(2) the property of having a weak Néron model over $\mathbb{K}$ or not is invariant under conjugacy by a Möbius map with coefficients in $\mathbb{K}$: $(\mathbb{P}^1/\mathbb{K}, \phi)$ admits a weak Néron model if and only if $(\mathbb{P}^1/\mathbb{K}, f \circ \phi \circ f^{-1})$ does, for one and hence all $f \in \mathrm{PGL}(2, \mathbb{K})$.

We can thus assume that $\mathbb{K}$ is strictly Henselian and in particular that the residue field $\Bbbk$ is algebraically closed.

We will use the method setup in [10] to construct a weak Néron model for cubic polynomials. For the convenience of the reader, we sketch it briefly. Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a given morphism over $\mathbb{K}$ and take $X_0 := \mathbb{P}^1_S$. Then X_0 is a proper (and smooth) model of $\mathbb{P}^1$ over S and X_0 has the extension property for étale points by the valuative criterion for properness (see [9, pp. 95–105]). We know that ϕ extends at least to an S-rational map $\Phi_0 : \mathbb{P}^1_S \dashrightarrow \mathbb{P}^1_S$. Now, we proceed inductively. Suppose that we have a separated and smooth S-model X_i of $\mathbb{P}^1$ having the extension property for étale points and an S-rational map $\Phi_i : X_i \dashrightarrow X_i$ extending ϕ for integer $i \geq 0$. If Φ_i is an S-morphism, then (X_i, Φ_i) is a weak Néron model for $(\mathbb{P}^1/K, \phi)$. Otherwise, the set of points where Φ_i is not defined is of codimension 2 in X_i. Hence, there are only finitely many closed points on the special fiber of X_i where Φ_i is not defined. We eliminate the indeterminacies by blowing up the closed points where Φ_i is not defined and let Y_{i+1} be the resulting scheme. Then Y_{i+1} is a separated S-model of $\mathbb{P}^1$ and still has the extension property for étale points. Removing the singular points of Y_{i+1} yields a new scheme denoted by X_{i+1} which is a smooth, separated S-model of $\mathbb{P}^1$ having the extension property for étale points. We consider again the extension map $\Phi_{i+1} : X_{i+1} \dashrightarrow X_{i+1}$ and test if any new indeterminacies occur. In the case of polynomial maps, either the process continues indefinitely, in which case the $\mathbb{K}$-rational Julia set is not empty, or there is an integer $n \geq 0$ such that the extension Φ_n is an S-morphism and (X_n, Φ_n) is a weak Néron model for $(\mathbb{P}^1/\mathbb{K}, \phi)$.

It is a standard fact that one can eliminate the points of indeterminacy of a rational map by blowing up a coherent sheaf of ideals (see for instance [9, Example 7.17.3]). In the proof of Theorem 2.1, we shall perform explicit blowups. For that purpose, we now describe more precisely how one performs blowups to eliminate indeterminacies of a rational map ϕ on $\mathbb{P}^1$. Specifically, let X be a smooth S-model of $\mathbb{P}^1$ having the extension property for étale points and let the S-rational map $\Phi : X \dashrightarrow X$ be the extension of ϕ. Note that as X is a smooth model of $\mathbb{P}^1$, the irreducible components of its special fiber $\widetilde{X}$ are isomorphic to the projective line $\mathbb{P}^1_{\Bbbk}$ over $\Bbbk$ with at most finitely many closed points removed. Each point of $\widetilde{X}(\Bbbk)$ can be lifted to a point $P \in \mathbb{P}^1(\mathbb{K})$. Hence, we will write a closed point of $\widetilde{X}$ as $\widetilde{P}$ with $P \in \mathbb{P}^1(\mathbb{K})$. Suppose that there is a closed point $\widetilde{P}$ on some irreducible component Z of $\widetilde{X}$ where Φ is not defined. The closed point $\widetilde{P}$ as a reduced closed subscheme of X is locally defined by the ideal $\mathcal{I} \subset \mathcal{O}_{\mathbb{K}}[z]$ generated by π and z. This means that $\widetilde{P}$ in $\widetilde{X}$ has local coordinate $\widetilde{z} = 0$. Let $X' \to X$ be the blowup of $\widetilde{P}$ in X. The exceptional divisor in X' is thus isomorphic to a projective line $\mathbb{P}^1$ over $\Bbbk$. Let X'_π denote the subset of X' defined by the equation $z = \pi z'$ where z' is a local coordinate in X'. Following [5, §3.2], we call X'_π the *dilatation* of $\widetilde{P}$ in X (not to be confused with the term *dilation* which we shall use below to denote homotheties $z \mapsto \lambda z$).

Note that Φ is defined at the generic point η_Z of Z. It follows that its image $\Phi(\eta_Z)$ is either a generic point or a closed point of some irreducible component W of $\widetilde{X}$. Let w be a local coordinate in a neighborhood of $\Phi(\eta_Z)$. Then we may represent the rational map $\Phi : X \dashrightarrow X$ locally in terms of the coordinates z and w so that

$$\phi_w(z) := w \circ \phi(z) = \frac{f(z)}{g(z)} \quad \text{with } f(z), g(z) \in \mathcal{O}_{\mathbb{K}}[z].$$

Note that $\widetilde{\phi_w}(\widetilde{z}) = \widetilde{f}(\widetilde{z})/\widetilde{g}(\widetilde{z})$ represents the rational map from Z to W over $\Bbbk$. By assumption $\widetilde{\phi_w}$ is not defined at $\widetilde{P}$. It follows that either $\widetilde{f}$ and $\widetilde{g}$ have the common zero $\widetilde{z} = 0$ or $\widetilde{\phi_w}(0)$ is equal to a point $\alpha \in W$ where W meets another component of $\widetilde{X}$. Notice that the dilatation X'_π of $\widetilde{P}$ has integral points $X'_\pi(\mathcal{O}_{\mathbb{K}})$ corresponding bijectively to points of $\mathbb{P}^1(\mathbb{K})$ with coordinate $|z| \le |\pi|$. The extension of Φ to X'_π amounts to replacing z by $\pi z'$ on X'_π. Then we examine whether or not the extension $\Phi : X' \dashrightarrow X'$ is well defined on the dilatation X'_π until we attain a model $\mathcal{X}$ such that the extension $\Phi : \mathcal{X} \dashrightarrow \mathcal{X}$, which we denote by Φ again, is an S-morphism.

After these preliminaries, we are ready to give a proof of Theorem 2.1. We split our arguments into two parts according to whether or not g is irreducible over $\mathbb{K}$. We fix an affine coordinate z on $\mathbb{P}^1$ so that $\phi(z)$ is a polynomial of degree 3. We first deal with the irreducible case in §3.2 below, then in §3.3 we treat the remaining case and finish the proof.

3.2. The irreducible case. Let us begin with the case where g is irreducible over $\mathbb{K}$, that is, when ϕ admits no $\mathbb{K}$-rational fixed point. In this case, we need to show that $(\mathbb{P}^1/K, \phi)$ has a weak Néron model. Conjugating ϕ by the dilation $z \mapsto z/\pi^s$ and by taking s large enough we may assume that ϕ is of the following form (without changing notation for the conjugated map):

$$\phi(z) = \frac{1}{\pi^n} f(z),$$

with $f(z) = uz^3 + a_1 z^2 + a_2 z + a_3 \in \mathcal{O}_{\mathbb{K}}[z]$, $|u| = 1$. If $n = 0$ then the polynomial has good reduction and thus it has a weak Néron model. Let us assume $n \geq 1$ from now on.

The reduction $\widetilde{f}$ of f must split over $\mathbb{k}$ since $\mathbb{k}$ is algebraically closed. Recall that $g(z) = \phi(z) - z$. From this we see that $g(z) = \pi^{-n} f^*(z)$ where $f^*(z) = f(z) - \pi^n z$ satisfying $\tilde{f}^* = \tilde{f}$. If $\tilde{f}$ has a simple root in $\mathbb{k}$ then so does $\tilde{f}^*$, which, by Hensel's lemma, implies that f has a fixed point in $\mathbb{K}$, contrary to hypothesis. We thus have $\tilde{f}(z) = \tilde{u}(z - \tilde{\alpha})^3$. Let $\alpha \in \mathcal{O}_{\mathbb{K}}$ be any lift of $\tilde{\alpha}$: $f(\alpha) \equiv 0 \pmod{\pi}$. Conjugating ϕ by the translation $z \mapsto z - \alpha$, we may assume that $\tilde{f}(z) = \tilde{u}z^3$ and hence $v(a_i) > 0$ for $i = 1, 2, 3$. Let $f^*(z) = uz^3 + a_1^* z^2 + a_2^* z + a_3^*$ and $n_i = v(a_i^*) > 0$ for $i = 1, 2, 3$.

As a consequence of our normalizations, the following are true:

(i) $n_i = v(a_i)$ for $i \neq 2$ and $n_2 = v(a_2 - \pi^n)$;
(ii) $n_3 = 3l + r$ with $r = 1$ or 2; $n_2 \geq 2l + 2r/3$ and $n_1 \geq l + r/3$.

As (i) is clear, we explain (ii). Notice that as g is irreducible over $\mathbb{K}$, so is f^*. It follows that every root of f^* has the same valuation. Let α_i, $i = 1, 2, 3$, be the roots of f^*. Then

$$n_3 = \sum_{i=1}^{3} v(\alpha_i) = 3v(\alpha) \quad \text{with } \alpha = \alpha_1.$$

By assumption $\mathbb{K}$ is strictly Henselian. It follows that $v(\alpha) \notin \mathbb{Z}$. Hence, n_i is of the form as claimed in (ii) and $v(\alpha_i) = l + r/3$ for $i = 1, 2, 3$. Now, observe that

$$a_1^* = -u \sum_{i=1}^{3} \alpha_i, \qquad a_2^* = u \sum_{1 \leq i < j \leq 3} \alpha_i \alpha_j.$$

Then the inequalities satisfied by n_1 and n_2 follow by applying the valuation v on both sides and the strong triangle inequality for v.

Before we proceed further, we observe that if $n \geq 2l + 2r/3 > 2l$ then we may conjugate ϕ by the dilation $z \mapsto \pi^l z$ and get (without changing notation for ϕ conjugated)

$$\phi(z) = \frac{1}{\pi^{n-2l}} (uz^3 + \pi^{-l} a_1 z^2 + \pi^{-2l} a_2 z + \pi^{-3l} a_3).$$

Notice that the polynomial $\pi^{n-2l}\phi(z)$ has all the coefficients in $\mathcal{O}_{\mathbb{K}}$ and $v(\pi^{-3l} a_3) = r$. So, after conjugation we may assume that $l = 0$ and $n_3 =$

$r = 1$ or 2. Since $n_2 \geq 2r/3$ ($l = 0$), we easily check that $n_2 \geq n_3$ in this case.

If $n < 2l + 2r/3$, then conjugating ϕ by $z \mapsto \pi^k z$ with $k = [n/2]$ gives

$$\phi(z) = \frac{1}{\pi^{n-2k}}(uz^3 + \pi^{-k}a_1z^2 + \pi^{-2k}a_2z + \pi^{-3k}a_3).$$

Similarly, $\pi^{n-2k}\phi(z)$ is a polynomial with coefficients in $\mathcal{O}_{\mathbb{K}}$. If $n = 2k$ is even, then $\deg\widetilde{\phi} = \deg\phi$. Thus ϕ has good reduction in this case and it admits a weak Néron model. On the other hand, if n is odd then $n - 2k = 1$. Hence, after conjugation, we may assume $n = 1$ in this case. In our discussion for the remaining case below, we further assume that either (1) $l = 0$ and $n \geq 2r/3$, or (2) $n = 1 < 2l + 2r/3$.

Notice that $\widetilde{z} = 0$ is the only place where the extension of ϕ on $\mathbb{P}^1_S$ has indeterminacy. As explained in §3.1, we perform a blowup of the special fiber at $\widetilde{z} = 0$ in $\mathbb{P}^1_S$. Let $X_1 \to \mathbb{P}^1_S$ be the blowup and let $\mathcal{X}$ be the smooth locus of X_1.

Proposition 3.1. *Let $\phi(z)$ be a cubic polynomial as above. Then ϕ extends to a morphism $\Phi : \mathcal{X} \to \mathcal{X}$ over S so that $(\mathcal{X}, \Phi)$ is a weak Néron model for $(\mathbb{P}^1/\mathbb{K}, \phi)$.*

Proof. Let $X_{1,\pi}$ be the dilatation of $\widetilde{z} = 0$. Then on $X_{1,\pi}$ we may use an affine coordinate z_1 so that $z = \pi z_1$ and on $X_{1,\pi}$ the polynomial map ϕ can be represented by

$$\psi(z_1) = \phi(\pi z_1) = \frac{1}{\pi^n}(u\pi^3z_1^3 + a_1\pi^2z_1^2 + a_2\pi z_1 + \pi^{n_3}u'),$$

where u' is a unit such that $a_3 = \pi^{n_3}u'$.

By our assumption above, we have either (1) $l = 0$ and $n \geq 2r/3$, or (2) $n = 1 < 2l + 2r/3$. For case (1), we have $n_3 = 1$ or 2; $n \geq n_2 \geq n_3$; and $v(a_2) \geq n_2 \geq n_3$. Then

$$\psi(z_1) = \frac{1}{\pi^{n-n_3}}(\pi^{3-n_3}uz_1^3 + \pi^{2-n_3}a_1z_1^2 + \pi^{1-n_3}a_2z_1 + u') = \frac{1}{\pi^{n-n_3}}\psi_1(z_1).$$

Note that $\psi_1(z_1) \equiv \widetilde{u'} \pmod{\pi}$. Thus ψ sends the component $\widetilde{X_{1,\pi}}$ on which it is defined to $\widetilde{\infty}$ if $n > n_3$ or to $\widetilde{u'} \neq \tilde{0}$ in the special fiber of $\mathbb{P}^1_S$ if $n = n_3$. From this, we conclude that the extension $\Phi : \mathcal{X} \to \mathcal{X}$ is an S-morphism. Thus, $(\mathcal{X}, \Phi)$ is a Néron model for $(\mathbb{P}^1/\mathbb{K}, \phi)$ and this completes the proof of the first case.

Now consider case (2). As $2r/3 > 1$ in this case, we have $n_3 = 3l + r \geq 2$. Therefore,

$$\psi(z_1) = u\pi^2z_1^3 + a_1\pi z_2^2 + a_1z_2 + u'\pi^{n_3-1} = \pi\psi_1(z_1)$$

where $\psi_1(z_1) \in \mathcal{O}_{\mathbb{K}}[z_1]$. We see that ϕ extends to an S-morphism that maps the component $\widetilde{X_{1,\pi}}$ to itself. In this case, we may also conclude that ϕ

extends to an S-morphism $\Phi : \mathcal{X} \to \mathcal{X}$ and $(\mathcal{X}, \Phi)$ is a weak Néron model of $(\mathbb{P}^1/\mathbb{K}, \phi)$. ■

This concludes the case when the fixed point equation is irreducible over $\mathbb{K}$.

3.3. The reducible case. We assume in this section that ϕ admits a $\mathbb{K}$-rational fixed point, which, after conjugating by a translation, is equal to 0. So ϕ has the form

$$\phi(z) = \lambda z + a_2 z^2 + a_3 z^3, \qquad \lambda, a_2, a_3 \in \mathbb{K}.$$

As λ is the multiplier of the fixed point 0, if $v(\lambda) < 0$ then 0 is a repelling fixed point and ϕ does not admit a weak Néron model. We thus assume now that $v(\lambda) \geq 0$. We let $\nu = v(a_3)/2 - v(a_2)$ and notice that this quantity is invariant under conjugacy of ϕ by a dilation centered at 0. Moreover, by a suitable choice of dilation $z \mapsto \pi^l z$, we may also assume that $a_2, a_3 \in \mathcal{O}_{\mathbb{K}}$. As before, we let $n_i = v(a_i)$ and distinguish two cases according to the sign of ν.

Let us assume first that $\nu \leq 0$, i.e. $n_2 \geq n_3/2$. By conjugating by the dilation $z \mapsto \pi^k z$ with $k = [n_3/2]$, we may assume that $n_3 = 0$ or 1. If $n_3 = 0$ then ϕ has good reduction and we are done. So, let us assume that $n_3 = 1$ and take $X_0 = \mathbb{P}^1_S$. In this case, we have $n_2 \geq 1$. Thus in homogeneous coordinates $\widetilde{\phi}([z : t]) \equiv [\widetilde{\lambda} z t^2 : t^3] \pmod{\pi}$ and it has indeterminacy at the point $[1 : 0] = \widetilde{\infty}$. Let $X_1 \to X_0$ be the blowup of the point $\widetilde{\infty}$ and $\mathcal{X}$ be the smooth locus of X_1. Let $X_{1,\pi}$ be the dilation of $\widetilde{\infty}$. We may use the affine coordinate $z_1 = \pi z$ on $X_{1,\pi}$ and represent ϕ by

$$\psi(z_1) = \phi(z_1/\pi) = \frac{\pi\lambda z_1 + a_2 z_1^2 + u z_1^3}{\pi^2}$$

where $a_3 = u\pi$ for some unit $u \in \mathcal{O}^*$. We see that on $X_{1,\pi}$, ψ has indeterminacy at $\widetilde{z_1} = 0$, which is the point where the special fiber $\widetilde{X_{1,\pi}}$ of $X_{1,\pi}$ meets the special fiber of X_0. Therefore, we conclude that ϕ extends to an S-morphism Φ on $\mathcal{X}$ and $(\mathcal{X}, \Phi)$ is a weak Néron model for $(\mathbb{P}^1/\mathbb{K}, \phi)$, which proves the case for $\nu \leq 0$.

If on the contrary we have $\nu > 0$ then, by conjugating ϕ by $z \mapsto \pi^{-n_2} z$, we can assume that $n_2 = 0$ and $n_3 = 2\nu > 0$, so ϕ can be written as

$$\phi(z) = \lambda z + u_2 z^2 + u_3 \pi^{2\nu} z^3$$

with $u_i \in \mathcal{O}_{\mathbb{K}}^*$ and $v(\lambda) \geq 0$. The equation for the non-zero fixed points is

$$u_3 \pi^{2\nu} z^2 + u_2 z + (\lambda - 1) = 0,$$

which is equivalent, upon letting $x = \pi^{2\nu} z$, to the equation

$$h(x) = u_3 x^2 + u_2 x + (\lambda - 1)\pi^{2\nu} = 0.$$

Observe that $h(x)$ has two simple roots modulo $\mathcal{M}_{\mathbb{K}}$ so by Hensel's lemma h splits over $\mathbb{K}$. We conclude that all fixed points of ϕ are $\mathbb{K}$-rational. One of the two roots of $h(x)$ is a unit $\epsilon \in \mathcal{O}_{\mathbb{K}}^*$ such that $\epsilon \equiv -u_2/u_3 \pmod{\pi}$. Let $\zeta = \epsilon\pi^{-2\nu}$ be the fixed point with largest absolute value. The multiplier of this fixed point is given by

$$\phi'(\zeta) = \frac{1}{\pi^{2\nu}}(u_3\epsilon^2 + 2\epsilon(u_3\epsilon + u_2) + \lambda\pi^{2\nu}) = \frac{g(\epsilon)}{\pi^{2\nu}}$$

and

$$\tilde{g}(\tilde{\epsilon}) = \tilde{g}\left(-\frac{\widetilde{u_2}}{\widetilde{u_3}}\right) = \frac{\widetilde{u_2}^2}{\widetilde{u_3}} \neq 0.$$

Thus $v(\phi'(\zeta)) = -2\nu < 0$ and ζ is a repelling fixed point. So $(\mathbb{P}^1/\mathbb{K}, \phi)$ does not admit a weak Néron model in this case, and this concludes the proof of Theorem 2.1. ∎

Remark 3.

(1) Let $\mathbb{L}$ be the quadratic ramified extension $\mathbb{K}[\sqrt{\pi}]$ of $\mathbb{K}$ and denote by $v_{\mathbb{L}}$ the extension of the valuation v on $\mathbb{L}$ such that $v_{\mathbb{L}} = 2v$ on $\mathbb{K}$ and $v_{\mathbb{L}}(\sqrt{\pi}) = 1$. Equivalently, $v_{\mathbb{L}}(\mathbb{L}^*) = \mathbb{Z}$. In the case where $\nu \leq 0$, we note that $v_{\mathbb{L}}(a_3) = 2v(a_3)$, which is an even integer. Then the same argument for the case where n_3 is even in the proof applies to this situation. In conclusion, $(\mathbb{P}^1/\mathbb{K}, \phi)$ has potential good reduction if $\nu \leq 0$.

(2) Let $\phi(z) \in \mathbb{K}[z]$ be a cubic polynomial as above. It follows from the proof of Theorem 2.1 that if $(\mathbb{P}^1/K, \phi)$ has a weak Néron model $(\mathcal{X}, \Phi)$, then the special fiber $\widetilde{\mathcal{X}}$ of $\mathcal{X}$ is either $\mathbb{P}^1_{\Bbbk}$, in which case ϕ has good reduction, or two components isomorphic to $\mathbb{P}^1_{\Bbbk}$ that intersect at one closed point transversally. In fact, the proof actually gives an algorithm to compute the reduction type of ϕ.

Proof of Corollary 2.2. As periodic points of ϕ are algebraic over $\mathbb{K}$, we may assume $\mathbb{K}$ contains the fixed points of ϕ. We are thus in the situation where the fixed point equation is reducible. As the fixed points are non-repelling, with the notation above we are necessarily in the case where $\nu \leq 0$, for which we showed that ϕ has potential good reduction (by Remark 3) and thus empty Julia set. ∎

4. Counterexamples in degree 4 and higher. The purpose of this section is to show that Theorem 2.2 proved above is not true any more in degrees higher than three. Let $d = \deg\phi \geq 4$ and let p be the characteristic of the residue class field $\Bbbk$. Then, except for the four cases $(p, d) = (2, 4), (2, 5), (2, 7), (3, 5)$, we can write $d = e_0 + e_1$ or $d = e_0 + e_1 + e_2$ such that $e_i \geq 2$ and $p \nmid e_i$ for $i = 0, 1, 2$. Let $n = \operatorname{lcm}(e_0, e_1, e_2)$ and

$n = e_0e'_0 = e_1e'_1 = e_2e'_2$ where by abuse of notation we assume that $e_2 = 0$ if $d = e_0 + e_1$. With the same notation as in the previous sections we let

$$\phi(z) = \frac{1}{\pi^n} z^{e_0}(z-1)^{e_1}(z-\alpha)^{e_2} + z, \qquad \alpha \not\equiv 0, 1 \pmod{\pi},$$

be a degree d polynomial with $0, 1$ and possibly α if $e_2 \neq 0$ as fixed points, which are not repelling. In the following, for simplicity we only consider the case where $e_2 = 0$, i.e. $d = e_0 + e_1$ with $e_i \geq 2$ and $p \nmid e_i$ for $i = 0, 1$. For the other case ($d = e_0 + e_1 + e_2$ with $e_2 \neq 0$), the arguments are similar. We leave it to the interested readers.

So, the polynomial map in question is of the form as above with $e_2 = 0$. The (finite) fixed points are 0 and 1, which are non-repelling. For ease of notation, we let $(x)_n = \{y \in \mathbb{K} \nmid v(x-y) \geq n\}$ be the ball centered at x with radius $|\pi|^n$. Let $r_i = \lceil e'_i/(e_i - 1) \rceil$, $i = 0, 1$, where $\lceil x \rceil$ is the ceiling of the real number x (i.e. the smallest integer not less than x). Put $s_i = e'_i + r_i$, $i = 0, 1$. Then the two balls $(0)_{s_0}$ and $(1)_{s_1}$ are both forward invariant (under the action of ϕ) and are thus in the Fatou set of ϕ. A simple computation shows that if z is not in $(0)_{e'_0} \cup (1)_{e'_1}$ then its orbit escapes to infinity. The Julia set is thus included in the union of the two annuli $(0)_{e'_0} \setminus (0)_{s_0}$ and $(1)_{e'_1} \setminus (1)_{s_1}$. Notice that ϕ takes the balls $(0)_{e'_0}$ and $(1)_{e'_1}$ onto the unit ball $(0)_0$. By symmetry, we may just consider $\phi : (0)_{e'_0} \to (0)_0$ only. Let $z = \pi^{e'_0} w$ so that w is a local coordinate on $(0)_{e'_0}$. Recall that $n = e_0e'_0$; we get

$$\phi(\pi^{e'_0} w) = w^{e_0}(\pi^{e'_0} w - 1)^{e_1} + \pi^{e'_0} w \equiv (-1)^{e_1} w^{e_0} \pmod{\pi}.$$

Since $p \nmid e_0$, there are $\zeta_i \in \Bbbk$, $i = 1, \ldots, e_0$, such that

$$(-1)^{e_1} \zeta_i^{e_0} \equiv 1 \pmod{\pi}.$$

As a simple consequence of Hensel's lemma, we find that there are points $a_i \in (0)_{e'_0}$ such that $w(a_i) \equiv \zeta_i \pmod{\pi}$ and that $\phi : (a_i)_{e'_0+1} \to (1)_1$ are bijectively expanding the distances by a factor of $|\pi|^{-e'_0}$ for all $i = 1, \ldots, e_0$. For the same reason, there are points $b_j \in (1)_{e'_1}$ such that $\phi : (b_j)_{e'_1+1} \to (0)_1$ are bijectively expanding by a factor of $|\pi|^{-e'_1}$ for $j = 1, \ldots, e_1$. We see that ϕ induces a subdynamics on

$$J_\phi \cap \left(\bigcup_{i=1}^{e_0} (a_i)_{e'_0+1} \cup \bigcup_{j=1}^{e_1} (b_j)_{e'_1+1} \right)$$

which can be conjugated to the subshift of finite type on $e_0 + e_1 = d$ symbols whose incidence graph is given by the complete bipartite graph with $e_0 + e_1$ vertices. In particular ϕ admits no repelling fixed point while on the other hand it admits period two repelling points: ϕ does not have a weak Néron model over $\mathcal{O}_{\mathbb{K}}$.

QUESTIONS.

(1) Is Theorem 2.2 true for the four exceptional cases $(p, d) = (2, 4)$, $(2, 5), (2, 7)$ and $(3, 5)$.
(2) For a polynomial map of degree d, is there a positive integer r depending on d and the characteristic p of the residue field $\Bbbk$ so that if all $\mathbb{K}$-rational periodic points with period less than r are non-repelling then there is no $\mathbb{K}$-rational Julia set of the polynomial map in question?

Acknowledgments. The authors have benefited from support from the project "Berko" of the French Agence Nationale pour la Recherche, of the Université de Provence in France and the National Center for Theoretical Sciences and National Central University in Taiwan. The first author would like to thank the Academia Sinica in Taipei for its hospitality. The second author's research is supported by NSC-95-2115-M-008-003 of the National Science Council of Taiwan.

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Received on 29.6.2011
and in revised form on 27.11.2011 (6747)