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On the quantitative unit sum number problem—an application of the subspace theorem

by

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Dedicated to Wolfgang M. Schmidt on the occasion of his 75th birthday

1. Introduction. By Roth's famous theorem [12] for irrational algebraic numbers α and any $\delta > 0$ the inequality

$$|x_2 - \alpha x_1| < |x_1|^{-1-\delta}$$

has only finitely many integral solutions. Wolfgang Schmidt [16] extended this result to a simultaneous version which states that the inequality

$$|x_1 - \alpha_1 x_0| \cdots |x_n - \alpha_n x_0| < |x_0|^{-1-\delta} \qquad (\delta > 0)$$

has only finitely many solutions, where $1, \alpha_1, \ldots, \alpha_n$ are linearly independent algebraic numbers over $\mathbb{Q}$. These investigations cumulated in the celebrated subspace theorem of Schmidt [17]: Let L_i be linearly independent linear forms in n variables with algebraic coefficients; then all integral solutions of the inequality

$$\prod_{i=1}^{n} |L_i(x_1, \ldots, x_n)| \le \max\{|x_1|, \ldots, |x_n|\}^{-\delta} \qquad (\delta > 0)$$

lie in finitely many proper subspaces of $\mathbb{Q}^n$.

As an application of the subspace theorem Schmidt [18] described all norm form equations that have finitely many solutions for any non-zero constant term. The subspace theorem has been further developed by Schlickewei [13, 14] and is proved in its most general form by Evertse and Schlickewei [4] (see also [15]). These investigations led to many applications, e.g. to the

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finiteness of the number of solutions to S-unit equations (see e.g. [5]) or to estimates for the number of zeros of linear recurrence sequences (see e.g. [20]). In this paper we use these techniques to obtain results on a quantitative version of the so-called unit sum number problem. In particular, we solve a problem related to a recent paper of M. Jarden and W. Narkiewicz [11].

The investigation of the unit sum number of rings goes back to the 1950's, when Zelinsky [22] proved that every element of the endomorphism ring E of a vector space V over a division ring D can be written as the sum of two automorphisms (units in E) unless D is the field with two elements and the dimension of V is one. Following Goldsmith, Pabst and Scott [7] the *unit sum number* of a ring (with identity) is defined as the smallest number k such that every $r \in R$ can be represented as a sum of k units. If no such k exists and R is additively generated by its units, we say that R has unit sum number ω. Otherwise the unit sum number is ∞.

In 2005 Ashrafi and Vámos [1] showed that the rings of integers of quadratic fields, cubic fields and cyclotomic fields of the form $\mathbb{Q}(\zeta_{2^N})$, with ζ_{2^N} a 2^Nth primitive root of unity, do not have finite unit sum number. Moreover, they characterized all integers d for which the ring of integers of $\mathbb{Q}(\sqrt{d})$ is generated by its units. This has been shown by Belcher 30 years before by a similar method (see [2]). The result of Ashrafi and Vámos was completed by Jarden and Narkiewicz [11]. They proved that no ring of algebraic integers has finite unit sum number. However, the question remains which number fields have rings of integers that are generated by their units. This has been solved for quadratic fields by Belcher [2] and Ashrafi and Vámos [1], for pure cubic fields by Tichy and Ziegler [21], for complex biquadratic fields by Ziegler [23], and for complex pure quartic fields by Filipin, Tichy and Ziegler [6].

In this context the question arises how many integers can be represented as the sum of exactly m units. This question is one of the problems stated in the paper of Jarden and Narkiewicz [11, Problem C]. In order to be precise we use the following definition.

Definition 1. Let K be a number field. As usual, two integers α and β of K are said to be *associated* if there exists a unit ε such that $\alpha = \beta\varepsilon$; we then write $\alpha \sim \beta$.

We define the counting function $u_K(m;x)$ as the number of equivalence classes $[\alpha]_\sim$ such that

$$|\mathrm{N}_{K/\mathbb{Q}}(\alpha)| \le x, \qquad \alpha = \sum_{i=1}^{m} \varepsilon_i, \qquad \varepsilon_i \in \mathfrak{O}_K^*,$$

and no subsum vanishes. By $\mathfrak{O}_K$ we denote the ring of integers of K.

Note that the function $u_K(m;x)$ is well defined, since $|\mathrm{N}_{K/\mathbb{Q}}(\alpha)| = |\mathrm{N}_{K/\mathbb{Q}}(\beta)|$ if $\alpha \sim \beta$.

In the case of imaginary quadratic integers this problem is equivalent to the circle problem (see [24, Section 2]). Filipin et al. [6] investigated the case $K = \mathbb{Q}(\sqrt[4]{-d^2})$ and found asymptotic expansions for $u_K(m;x)$, where m is small with respect to the fundamental unit of K. In this paper we investigate the function $u_K(m;x)$ for K a real quadratic field and m arbitrary. In particular, we prove an asymptotic expansion, where the remainder term will be specified later.

THEOREM 1. *Let $K = \mathbb{Q}(\sqrt{d})$ with $1 < d \in \mathbb{Z}$ squarefree. Then the asymptotic formula*

$$u_K(m;x) = \frac{1}{(m-1)!}\left(\frac{2\log x}{\log \varepsilon}\right)^{m-1}(1+o(1))$$

holds, where $\varepsilon > 1$ is the fundamental unit of K.

More accurate results can be found in the sections below. We shall treat the cases $m = 2$, $m \le \varepsilon/2$, and m arbitrary separately.

Note that this result is closely related to a result due to Everest [3] who counts the number of weighted sums $c_0x_0 + \cdots + c_mx_m$ of S-units $x_0, \ldots, x_m$ of a number field K with fixed $c_0, \ldots, c_m \in K$ that have norm less than x. Everest uses a combination of analytic and Diophantine methods to obtain precise error terms. Unfortunately, these precise error terms cannot be obtained for $|S| = 1$. Moreover Everest's theorem does not explicitly give the coefficient for the main term, in particular this coefficient is quoted as a combinatorial constant depending only on m. However, in a subsequent paper we plan to use Everest's method in order to obtain more general results on the quantitative unit sum number problem. In this context we also mention a result due to Győry, Mignotte and Shorey [8] who counted the number of weighted sums of S-units with the additional property that the greatest prime divisor of the norm is less than P.

2. Plan of the paper. In this section we introduce the main ideas of the proofs. First, we note that by Dirichlet's unit theorem the unit rank of K is one. Therefore any unit of $\mathfrak{O}_K$ is of the form $\pm\varepsilon^k$, where $\varepsilon > 1$ is the fundamental unit. Assume $\alpha = (-1)^{l_1}\varepsilon^{k_1} + \cdots + (-1)^{l_m}\varepsilon^{k_m}$ can be written as a sum of m units; then we may assume $\alpha = 1 + (-1)^{l_2}\varepsilon^{k_2} + \cdots + (-1)^{l_m}\varepsilon^{k_m}$ with $0 = k_1 \le \cdots \le k_m$, since we are interested only in equivalence classes of associated integers. Now let us compute the norm of α:

$$\begin{aligned}|\mathrm{N}_{K/\mathbb{Q}}(\alpha)| &= |(1 + (-1)^{l_2}\varepsilon^{k_2} + \cdots + (-1)^{l_m}\varepsilon^{k_m}) \\ &\qquad \times (1 + (-1)^{l'_2}\varepsilon^{-k_2} + \cdots + (-1)^{l'_m}\varepsilon^{-k_m})| \\ &= \varepsilon^{k_m} + O(m\varepsilon^{k_m-1}).\end{aligned}$$

Using this estimate we will solve the following two problems:

- Find N_1 such that for any $k_m \leq N_1$ we have $\mathrm{N}_{K/\mathbb{Q}}(\alpha) \leq x$.
- Find N_2 such that for any $k_m > N_2$ we have $\mathrm{N}_{K/\mathbb{Q}}(\alpha) > x$.

Counting all α such that $k_m \leq N_1$ we deduce a lower bound for $u_K(m;x)$ and similarly an upper bound. These bounds yield asymptotic expansions for $u_K(m;x)$. It is easy to find N_1 and N_2 if m is small with respect to ε, but if m is large the O-term might absorb the dominant term ε^{k_m}. To overcome this problem we use the subspace theorem to prove that this absorption occurs only in very few cases, i.e. we are able to compute N_1 and N_2.

We divide our investigations into three parts. First, we investigate the cases $m = 2$, where we obtain very sharp estimates. For $m = 2$ we use only elementary methods (see Section 3). Next, we consider the case $m \leq \varepsilon/2$. In this case our estimations are less sharp. But m is still small with respect to ε and no absorption occurs (see Section 4). In the last section we consider m arbitrary. In this case absorption may occur. As mentioned above, we utilize the subspace theorem in order to obtain results for this case. In particular, we apply the following variant of the subspace theorem (cf. [19]).

Theorem 2 (Subspace Theorem). *Let K be an algebraic number field and let $S \subset M(K) = \{$canonical absolute values of $K\}$ be a finite set of absolute values which contains all the Archimedian ones. For each $\nu \in S$ let $L_{\nu,1}, \ldots, L_{\nu,n}$ be n linearly independent linear forms in n variables with coefficients in K. Then for given $\delta > 0$, the solutions of the inequality*

$$\prod_{\nu \in S} \prod_{i=1}^{n} |L_{\nu,i}(\mathbf{x})|_{\nu}^{n_\nu} < \lceil \mathbf{x} \rceil^{-\delta}$$

with $\mathbf{x} \in \mathfrak{O}_K^n$ and $\mathbf{x} \neq \mathbf{0}$, where

$$\lceil \mathbf{x} \rceil = \max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq \deg K}} |x_i^{(j)}|,$$

$|\cdot|_\nu$ denotes the valuation corresponding to ν, n_ν is the local degree and $\mathfrak{O}_K$ is the maximal order of K, lie in finitely many proper subspaces of K^n.

In order to get precise error terms we sometimes use the so-called Λ-notation instead of the O-notation. Let c be a real number, and assume $f(x)$, $g(x)$ and $h(x)$ are real functions and $h(x) > 0$ for $x > c$. We will write

$$f(x) = g(x) + \Lambda_c(h(x))$$

for

$$g(x) - h(x) \leq f(x) \leq g(x) + h(x).$$

This notation turned out to be useful in several papers, e.g. [9] or [10].

3. The case $m = 2$. We prove the following theorem:

THEOREM 3. *Let $K = \mathbb{Q}(\sqrt{d})$ with $1 < d \in \mathbb{Z}$ squarefree. Then*

$$u_K(2;x) = \frac{2\log x}{\log\varepsilon} - \frac{3}{2x\log\varepsilon} + \Lambda_{12}\left(1 + \frac{11}{2x\log\varepsilon}\right).$$

Proof. First we note that two sums $1+\eta_1$ and $1+\eta_2$ do not represent the same integer unless $\eta_1 = \eta_2$. As described in Section 2, we compute N_1 and N_2. Therefore we write $\alpha = 1 \pm \varepsilon^k$ and compute

$$|\mathrm{N}_{K/\mathbb{Q}}(1 \pm \varepsilon^k)| \le |1+\varepsilon^k|\,|1+\varepsilon^{-k}| \le \varepsilon^k + 2 + \varepsilon^{-k} \le \varepsilon^k + 3.$$

This yields $N_1 = \log(x-3)/\log\varepsilon$. Similarly we obtain

$$|\mathrm{N}_{K/\mathbb{Q}}(1 \pm \varepsilon^k)| \ge |\varepsilon^k - 1|\,|1-\varepsilon^{-k}| \ge \varepsilon^k - 2 + \varepsilon^{-k} \ge \varepsilon^k - 2,$$

hence $N_2 = \log(x+2)/\log\varepsilon$. Since there are exactly $2\lfloor N\rfloor + 1$ representations with $k \le N$ we find

$$2\lfloor N_1\rfloor + 1 \le u_K(2;x) \le 2\lfloor N_2\rfloor + 1.$$

In order to estimate the logarithms we use the following lemma:

LEMMA 1. *Let $0 \le a < x$. Then*

$$\log x - \frac{a(2x-a)}{2x(x-a)} \le \log(x-a) \le \log x \le \log(x+a) \le \log x + \frac{a}{x},$$

where equality holds if and only if $a = 0$.

Proof of Lemma 1. The estimates follow immediately from the Taylor expansions of $\log(x+a) = \log(x) + \log(1+x/a)$ and $\log(x-a) = \log x + \log(1-x/a)$ respectively. ∎

If we assume $x \ge 12$, we obtain, by Lemma 1,

$$(1)\qquad u_K(2;x) \le 2\left\lfloor\frac{\log(x+2)}{\log\varepsilon}\right\rfloor + 1 = \frac{2\log x}{\log\varepsilon} + \theta_1^+\frac{4}{x\log\varepsilon} - 2\theta_2^+ + 1$$

and

$$(2)\qquad u_K(2;x) \ge 2\left\lfloor\frac{\log(x-3)}{\log\varepsilon}\right\rfloor + 1 = \frac{2\log x}{\log\varepsilon} - \theta_1^-\frac{7}{x\log\varepsilon} - 2\theta_2^- + 1,$$

with $0 \le \theta_1^+, \theta_1^-, \theta_2^+, \theta_2^- < 1$. This yields

$$u_K(2;x) = \frac{2\log x}{\log\varepsilon} - \frac{3}{2x\log\varepsilon} + \theta\left(1 + \frac{11}{2x\log\varepsilon}\right)$$

with $|\theta| < 1$. ∎

REMARK.

- The estimates (1) and (2) will yield in most cases an exact value for $u_K(2;x)$, since in most cases $\lfloor\log(x+2)/\log\varepsilon\rfloor = \lfloor\log(x-3)/\log\varepsilon\rfloor$.

- In the case $m=3$ we are not able to compute an exact value for $u_K(3;x)$ from the bounds N_1 and N_2, since there are $2\lfloor N\rfloor^2+2\lfloor N\rfloor+1 = 2N^2+O(N)$ representations with $k_3\le N$ and the order of the error term cannot be improved.

4. Medium sized $\boldsymbol{m}$**.** In this section we investigate the case $m\le\varepsilon/2$. The main theorem of this section is

THEOREM 4. *Let* $K=\mathbb{Q}(\sqrt{d})$ *with* $1<d\in\mathbb{Z}$ *squarefree. Then for* $m\le\varepsilon/2$,

$$u_K(m;x)=\frac{1}{(m-1)!}\left(\frac{2\log x}{\log\varepsilon}\right)^{m-1}+\varLambda_{\varepsilon^6/4}\left(\frac{17\left(\frac{14\log x}{5\log\varepsilon}\right)^{m-2}}{(m-2)!}\right).$$

REMARK. The condition $m\le\varepsilon/2$ yields $m\le 1$ only in the cases $d=2$, $d=3$, $d=5$ and $d=13$.

First, note that every representation of an integer as a sum of m units is unique if $m\le\varepsilon/2$. This is easy to see if we interpret the representation as a digit expansion with basis ε and digit set $\{-m,-m+1,\dots,m\}$.

Now let us compute N_1. We assume $k_{m-r+1}=\dots=k_m$, $0=k_1=\dots=k_s$ and $k_m\ge1$. Then we obtain

$$\begin{aligned}
|\mathrm{N}_{K/\mathbb{Q}}(\alpha)|&=|(1+\dots+(-1)^{l_m}\varepsilon^{k_m})(1+\dots+(-1)^{l'_m}\varepsilon^{-k_m})|\\
&\le(r\varepsilon^{k_m}+(m-r-s)\varepsilon^{k_m-1}+s)(s+(m-r-s)\varepsilon^{-1}+r\varepsilon^{-k_m})\\
&\le(r\varepsilon^{k_m}+(m-r)\varepsilon^{k_m-1})(s+(m-s)\varepsilon^{-1})\\
&\le\varepsilon^{k_m}\left(rs+\frac{m(r+s)-2rs}{\varepsilon}+\frac{(m-r)(m-s)}{\varepsilon^2}\right)\\
&\le\varepsilon^{k_m}\left(\frac{m^2}{4}+\frac{m}{2}+\frac14\right)=\varepsilon^{k_m}\left(\frac{m+1}{2}\right)^2\le\frac{\varepsilon^{k_m+2}}{4}.
\end{aligned}$$

If $k_m=0$ then $\mathrm{N}_{K/\mathbb{Q}}(\alpha)=m^2\le\varepsilon^{k_m+2}/4$. Therefore

$$N_1=\frac{\log x+\log 4}{\log\varepsilon}-2.$$

Now we compute N_2:

$$\begin{aligned}
|\mathrm{N}_{K/\mathbb{Q}}(\alpha)|&=|(1+\dots+(-1)^{l_m}\varepsilon^{k_m})(1+\dots+(-1)^{l'_m}\varepsilon^{-k_m})|\\
&\ge(r\varepsilon^{k_m}-(m-r-s)\varepsilon^{k_m-1}-s)(s-(m-r-s)\varepsilon^{-1}-r\varepsilon^{-k_m})\\
&\ge(r\varepsilon^{k_m}-(m-r)\varepsilon^{k_m-1})(s-(m-s)\varepsilon^{-1})\\
&\ge\varepsilon^{k_m}\left(rs-\frac{m(r+s)-2rs}{\varepsilon}+\frac{(m-r)(m-s)}{\varepsilon^2}\right)\\
&\ge\varepsilon^{k_m}\left(rs-\frac{r+s}{2}+\frac{2rs}{\varepsilon}\right)\ge\varepsilon^{k_m}\frac{2}{\varepsilon}\ge2\varepsilon^{k_m-1},
\end{aligned}$$

and thus

$$N_2 = \frac{\log x - \log 2}{\log \varepsilon} + 1.$$

Next, we want to find an asymptotic expansion for the number $A_m(N)$ of non-associated integers with $k_m \leq N$. In particular, we prove

Proposition 1.

$$A_m(N) = \frac{(2N)^{m-1}}{(m-1)!} + \Lambda_6(54).$$

Proof. To prove this proposition we establish a formula for $A_m(N)$. Thus we count the possibilities to choose admissible pairs of exponents (l_i, k_i) for $i = 2, \ldots, m-1$:

(1) First, we have the possibility that all exponents are zero, i.e. each unit is 1.
(2) Assume that there are j units with exponents $k_{m-j+1}, \ldots, k_m > 0$, which are distributed on n fixed pairs of exponents (k, l). Then there are $\binom{m-j-1}{n-1}$ possibilities to choose a distribution such that each unit corresponds to at least one pair of exponents.
(3) Now we determine how many possibilities we have to choose n pairs of exponents (k, l). Note that there are $\binom{N}{n}$ possibilities to choose k, and for each choice of k we have 2 choices for l.

Altogether we obtain

$$\begin{aligned} A_m(N) &= 1 + \sum_{j=1}^{m-1} \sum_{n=1}^{m-j} \binom{m-j-1}{n-1} \binom{N}{n} 2^n \\ &= 1 + \sum_{n=1}^{m-1} \binom{N}{n} 2^n \sum_{j=1}^{m-j} \binom{m-j-1}{n-1} \\ &= 1 + \sum_{n=1}^{m-1} \binom{m-1}{n} \binom{N}{n} 2^n = \sum_{n=0}^{m-1} \binom{m-1}{n} \binom{N}{n} 2^n. \end{aligned}$$

Since we are looking for an asymptotic expansion in N we have to compute the coefficients of N^a. The coefficient of N^a in the expression $N(N-1) \cdots (N-n+1)$ can be estimated by $\binom{n-1}{n-1-a} \frac{(n-1)!}{a!}$, and therefore the coefficient c_a of N^a in $A_m(N)$ can be estimated by

$$\sum_{n=a}^{m-1} \binom{m-1}{n} \binom{n}{a} \frac{2^n (n-1)!}{a! n!} \leq \sum_{n=a}^{m-1} \binom{m-1}{n} \binom{n}{a} \frac{2^n}{a!} = \frac{2^a 3^{m-a-1} \binom{m-1}{a}}{a!}.$$

Thus we have

$$\begin{aligned}A_m(N) &= \frac{2N^{m-1}}{(m-1)!} + \theta\biggl(N^{m-2}\sum_{a=0}^{m-1}\frac{2^a 3^{m-a-1}\binom{m-1}{a}}{a!N^{m-2-a}}\biggr)\\ &= \frac{2N^{m-1}}{(m-1)!} + \theta N\biggl(\frac{3^{m-1}L_{m-1}(-2x/3)}{N^{m-1}} - \frac{2^{m-1}}{(m-1)!}\biggr)\end{aligned}$$

for some $|\theta| \le 1$, where $L_n(x)$ is the nth Laguerre polynomial. Note that the second summand decreases while N increases. If we assume $N \ge 6$, we obtain

$$\begin{aligned}A_m(N) &= \frac{2N^{m-1}}{(m-1)!} + \Lambda_6\biggl(6\,\frac{3^{m-1}L_{m-1}(-4)}{6^{m-1}} - 6\,\frac{2^{m-1}}{(m-1)!}\biggr)\\ &= \frac{2N^{m-1}}{(m-1)!} + \Lambda_6\biggl(6\,\frac{L_{m-1}(-4)}{2^{m-1}}\biggr).\end{aligned}$$

It remains to show $6L_{m-1}(-4)/2^{m-1} \le 54$ for all $m \ge 2$. This is easy to verify for, say, $m \le 10$. For the other m's we use the upper bound

$$L_n(-4) \le 16.51\biggl(\frac{7+\sqrt{17}}{6}\biggr)^n \qquad (n \ge 7),$$

which follows immediately by induction from the recurrence

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x).$$

The proposition now follows immediately from

$$6\,\frac{L_{m-1}(-4)}{2^{m-1}} \le 99.06\biggl(\frac{7+\sqrt{17}}{12}\biggr)^{m-1} \le 54$$

for $m \ge 9$. ∎

Proposition 1 implies

$$\frac{\bigl(2\frac{\log x+\log 4}{\log\varepsilon} - 4\bigr)^{m-1}}{(m-1)!} - 54 \le u(m;x) \le \frac{\bigl(2\frac{\log x-\log 2}{\log\varepsilon} + 2\bigr)^{m-1}}{(m-1)!} + 54$$

for $x \ge \varepsilon^6/4$. We use the following estimate:

$$\begin{aligned}(x+r)^n &= x^n + (x+r)^n - x^n\\ &= x^n + rx^{n-1}((1+r/x)^{n-1} + \cdots + 1)\\ &= x^n + \Lambda_0(rnx^{n-1}(1+r/x)^{n-1}).\end{aligned}$$

This yields

$$\begin{aligned} u(m;x) &= \frac{\left(2\frac{\log x}{\log\varepsilon}+\theta 4\right)^{m-1}}{(m-1)!} + \Lambda_{\varepsilon^6/4}(54) \\ &= \frac{\left(\frac{2\log x}{\log\varepsilon}\right)^{m-1}}{(m-1)!} + \Lambda_{\varepsilon^6/4}\left(54 + \frac{4(m-1)\left(\frac{2\log x}{\log\varepsilon}\right)^{m-2}(7/5)^{m-2}}{(m-1)!}\right) \\ &= \frac{1}{(m-1)!}\left(\frac{2\log x}{\log\varepsilon}\right)^{m-1} + \Lambda_{\varepsilon^6/4}\left(\frac{17\left(\frac{14\log x}{5\log\varepsilon}\right)^{m-2}}{(m-2)!}\right) \end{aligned}$$

with $|\theta| \le 1$.

5. Large m**.** In this section we treat the general case, in particular we are interested in the case $m > \varepsilon/2$ which has not been treated yet. The aim of this section is to complete the proof of Theorem 1. In the case $m > \varepsilon/2$ new phenomena occur. First, an integer α may have two different representations as a sum of units. Furthermore, we may have

$$|\mathrm{N}_{K/\mathbb{Q}}(\alpha)| < \varepsilon^{k_m(1-\delta)}$$

for some $\delta > 0$, where $\alpha = 1 + \cdots + \varepsilon^{k_m}$. However, we will show that these phenomena occur only finitely many times and so they do not affect the asymptotic behavior of $u_K(m;x)$.

LEMMA 2. *There are only finitely many algebraic integers $\alpha \in K$ that admit two different representations as a sum of units.*

Proof. Without loss of generality we may assume

$$\alpha = a_0 + a_1\varepsilon^{k_1} + \cdots + a_k\varepsilon^{k_k} = b_0\varepsilon^{l_0} + b_1\varepsilon^{l_1} + \cdots + b_l\varepsilon^{l_l},$$

with $a_0, a_1, \ldots, a_k, b_0, \ldots, b_l \in \mathbb{Z}$ and $0 = k_0 \le k_1 \le \cdots \le k_k$ respectively $0 \le l_0 \le \cdots \le l_l$. Furthermore, we may assume that the a's and b's have absolute value less than m. Since we assume that the representations are not identical we obtain an S-unit equation of the form

$$1 = c_1\varepsilon^{j_1} + \cdots + c_j\varepsilon^{j_j}$$

with $j \ge 1$ and the c's are rational numbers with numerator and denominator at most $2m$. Therefore (see [5]) there are only finitely many solutions to this equation. Note that the number of solutions depends on m. ∎

LEMMA 3. *Let $\alpha = a_0 + a_1\varepsilon^{k_1} + \cdots + a_n\varepsilon^{k_n}$ with $\sum |a_n| = m$, $a_m \ne 0$, $0 < k_1 < \cdots < k_n$, $a_0, \ldots, a_n, k_1, \ldots, k_n \in \mathbb{Z}$ and m a fixed integer. Let $\delta > 0$ be a fixed real number. Then the inequality*

$$|\mathrm{N}_{K/\mathbb{Q}}(\alpha)| < \varepsilon^{k_n(1-\delta)} \tag{3}$$

has finitely many solutions. The number of solutions depends on m and δ.

Proof. Let σ be the non-identical automorphism of K. Then we write $\sigma\alpha = a_0 + a_1'\varepsilon^{-k_1} + \cdots + a_n'\varepsilon^{-k_n}$ with $a_j' = \pm a_j$ for $j = 1, \ldots, n$. If (3) is fulfilled, then either

$$|a_0 + a_1\varepsilon^{k_1} + \cdots + a_n\varepsilon^{k_n}| < \varepsilon^{k_n(1-\delta/2)} \tag{4}$$

or

$$|a_0 + a_1'\varepsilon^{-k_1} + \cdots + a_n'\varepsilon^{-k_n}| < \varepsilon^{-k_n\delta/2}. \tag{5}$$

Let us consider the first case. We want to apply Theorem 2 for $S = \{\infty_1, \infty_2\}$, where $\infty_1 = |\cdot|$ and $\infty_2 = |\sigma(\cdot)|$ denote the two places of K at infinity and $L_{\infty_1,n} = a_0x_0 + a_1x_1 + \cdots + a_nx_n$, $L_{\infty_1,j} = x_j$ for $j = 0, \ldots, n-1$ and $L_{\infty_2,j} = x_j$ for $j = 0, \ldots, n$. This yields the inequality

$$|a_0 + \cdots + a_n\varepsilon^{k_n}|\,|1|\,|\varepsilon^{k_1}| \cdots |\varepsilon^{k_{n-1}}|\,|1|\,|\varepsilon^{-k_1}| \cdots |\varepsilon^{-k_n}| < \varepsilon^{-k_n\delta/2}. \tag{6}$$

By the subspace theorem the solutions $(1, \varepsilon^{k_1}, \ldots, \varepsilon^{k_n})$ lie in finitely many subspaces of K^n. Let T be such a subspace. It is defined by an equation of the form

$$t_0x_0 + t_1x_1 + \cdots + t_nx_n = 0.$$

Since $\varepsilon^{k_i} \neq 0$, there exist $i \neq j$ such that $t_i, t_j \neq 0$. Therefore we find an expression for ε^{k_i} with $k_i \neq k_n$. Inserting this expression into inequality (6) and omitting the linear forms $L_{\infty_1,i}$ and $L_{\infty_2,i}$ yields an inequality of the form (6) with new coefficients $b_0, \ldots, b_{n-1}$. In order to apply the subspace theorem to this new inequality we have to prove that the new $n-1$ linear forms in $n-1$ variables are still linearly independent. In particular, we have to prove that $b_{n-1} \neq 0$. But $0 = b_{n-1} = a_n - a_it_n/t_i$ yields $a_nt_i - a_it_n = 0$. If there is no index i such that $a_nt_i - a_it_n \neq 0$, then the vectors $(a_0, \ldots, a_n)$ and $(t_0, \ldots, t_n)$ are dependent, i.e. $(a_0, \ldots, a_n) = \lambda(t_0, \ldots, t_n)$ for some $\lambda \neq 0$. Furthermore, we obtain $a_0 + \cdots + a_n\varepsilon^{k_n} = 0$, which contradicts the condition that no subsum vanishes.

Continuing this process of variable elimination we arrive at the inequality

$$|c\varepsilon^{k_n}| < \varepsilon^{-k_n\delta/2},$$

which yields only finitely many solutions.

The second case runs analogously. Here our linear forms are $L_{\infty_1,0} = a_0x_0 + a_1'x_1 + \cdots + a_n'x_n$, $L_{\infty_1,j} = x_j$ for $j = 1, \ldots, n$ and $L_{\infty_2,j} = x_j$ for $j = 0, \ldots, n$. In this case we get the inequality

$$|a_0 + \cdots + a_n'\varepsilon^{-k_n}|\,|\varepsilon^{-k_1}| \cdots |\varepsilon^{-k_{n-1}}|\,|1|\,|\varepsilon^{k_1}| \cdots |\varepsilon^{k_n}| < \varepsilon^{-k_n\delta/2}.$$

From the finitely many subspaces we find expressions for ε^{-k_i} with $k_i \neq 0$. Inserting these expressions into the inequality above we find new linear forms, which are by the same arguments linearly independent. This process of variable elimination terminates in the inequality

$$|c\varepsilon^{-k_n}|\,|\varepsilon^{k_n}| = c \le \varepsilon^{-k_n\delta}.$$

Again we find only finitely many solutions. ∎

Next, we compute the bounds N_1 and N_2. The inequality

$$|\mathrm{N}_{K/\mathbb{Q}}(\alpha)| = |(1+\cdots+(-1)^{l_m}\varepsilon^{k_m})(1+\cdots+(-1)^{l'_m}\varepsilon^{-k_m})| \le m^2\varepsilon^{k_m}$$

yields $N_1 = (\log x - 2\log m)/\log\varepsilon$. On the other hand, by Lemma 3 we have

$$|\mathrm{N}_{K/\mathbb{Q}}(\alpha)| \ge \varepsilon^{k_n(1-\delta)}$$

except for finitely many α. Thus $N_2^{(\infty)} = (\log x)/(1-\delta)\log\varepsilon$ with $k_m > N_2^{(\infty)}$ implies $|\mathrm{N}_{K/\mathbb{Q}}(\alpha)| > x$ except for finitely many integers α. Note that the number of exceptions depends on m and δ, but not on x.

By the discussion in Section 2, Proposition 1 and Lemma 2, we find

$$\frac{(2N_1)^{m-1}}{(m-1)!} - c_m^{(1)} \le u_K(m;x) \le \frac{(2N_2^{(\infty)})^{m-1}}{(m-1)!} + c_{m,\delta}^{(1)},$$

where the constants $c_m^{(1)}, c_m^{(2)}, \dots$ respectively $c_{m,\delta}^{(1)}, c_{m,\delta}^{(2)}, \dots$ depend only on m, respectively m and δ. This yields

$$1 - \frac{c_m^{(2)}}{\log x} \le \frac{u_K(m;x)(\log\varepsilon)^{m-1}(m-1)!}{(2\log x)^{m-1}} \le 1 + c_m^{(3)}\delta + \frac{c_{m,\delta}^{(2)}}{\log x}$$

for every $\delta > 0$. Thus Theorem 1 is proved.

Remark. The same method works also for biquadratic fields of the form $K = \mathbb{Q}(\sqrt[4]{-d^2})$. Note that in this case $\mathrm{N}_{K/\mathbb{Q}}(\alpha) = |\alpha\sigma(\alpha)|^2$, where σ is the automorphism induced by $\sqrt{2d} \mapsto -\sqrt{2d}$ and $\sqrt{-1} \mapsto \sqrt{-1}$. Hence

$$u_K(m;x) = \frac{1}{(m-1)!}\left(\frac{2Q\log x}{\log\eta}\right)^{m-1}(1+o(1)),$$

with η the fundamental unit of $\mathbb{Q}(\sqrt{2d})$ and $Q = [U : WU^+]$, where U is the unit group of K, U^+ the unit group of the maximal real subfield of K and W the group of roots of unity of K. For more details concerning the unit sum number problem in biquadratic fields we refer to our recent paper [6].

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