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Linear forms in two logarithms and interpolation determinants II

by

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Dédié à Wolfgang Schmidt, à l'occasion de son soixante quinzième anniversaire. Avec beaucoup d'admiration pour l'œuvre accomplie

1. Introduction and results. We improve our previous results [4, 5] on linear forms in two logarithms of complex algebraic numbers by introducing a new ingredient in the theory. Since the underlying idea has a wider scope than its present application, let us start with some comments on the techniques employed in effective diophantine approximation for bounding from below the absolute value of some non-vanishing quantity, say Λ. When using the method of auxiliary functions, one needs to require that $|\Lambda|$, which has to be viewed as an error term, should be much smaller than the absolute value of all non-zero values of the auxiliary function which occur in the proof. More flexibility is permitted when we use the method of interpolation determinants. Larger values of $|\Lambda|$ may then be admissible. We introduce an additional positive parameter μ which takes into account the relative magnitude of $|\Lambda|$ compared with the various interpolation determinants occurring in the proof. Our previous work [4], as well as the subsequent papers [5, 6], correspond to the case $\mu = 1$. However, values $\mu < 1$ are possible. The goal of the paper is to employ this idea in the context of [4], which leads to a significant reduction of the numerical constants obtained. The same plan could as well be applied to closely related topics, such as linear forms in one logarithm [7, 8], or more generally the theory of linear forms in any number of logarithms [9], and could also be adapted to the p-adic theory [2, 1].

We have kept the framework of the papers [4, 5, 6]. We first give a rather general statement involving all parameters of the construction (Theorem 1). Next, we specialize these parameters (Theorem 2) to obtain totally explicit results. The application of Theorem 2 finally produces lower bounds for $|\Lambda|$,

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which are formulated in the usual style of the theory of linear forms in logarithms. We have preserved the notations of the corresponding statements in [5, 6], referring mainly to [5] for the points which remain unchanged.

For any algebraic number α of degree d over $\mathbb{Q}$, we define as usual the *absolute logarithmic height* of α by the formula

$$\mathrm{h}(\alpha) = \frac{1}{d}\Big(\log|a| + \sum_{i=1}^{d} \log\max(1, |\alpha^{(i)}|)\Big),$$

where a is the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$, and the $\alpha^{(i)}$'s are the conjugates of α in the field $\mathbb{C}$ of complex numbers.

Let α_1, α_2 be two non-zero algebraic numbers, viewed as elements of $\mathbb{C}$, and let $\log\alpha_1$ and $\log\alpha_2$ be any determinations of their logarithms. We consider the linear form

$$\Lambda = b_2\log\alpha_2 - b_1\log\alpha_1,$$

where b_1 and b_2 are positive integers. Without loss of generality, we suppose that $|\alpha_1|, |\alpha_2| \geq 1$. Put

$$D = [\mathbb{Q}(\alpha_1,\alpha_2):\mathbb{Q}]/[\mathbb{R}(\alpha_1,\alpha_2):\mathbb{R}].$$

THEOREM 1. *Let K be an integer ≥ 2, and L, R_1, R_2, S_1, S_2 be positive integers. Let ϱ and μ be real numbers with $\varrho > 1$ and $1/3 \leq \mu \leq 1$. Put*

$$R = R_1 + R_2 - 1, \quad S = S_1 + S_2 - 1, \quad N = KL, \quad g = \frac{1}{4} - \frac{N}{12RS},$$

$$\sigma = \frac{1 + 2\mu - \mu^2}{2}, \quad b = \frac{(R-1)b_2 + (S-1)b_1}{2}\Big(\prod_{k=1}^{K-1} k!\Big)^{-2/(K^2-K)}.$$

Let a_1, a_2 be positive real numbers such that

$$a_i \geq \varrho|\log\alpha_i| - \log|\alpha_i| + 2D\mathrm{h}(\alpha_i)$$

for $i = 1, 2$. Suppose that

$$(1)\qquad \begin{aligned} &\mathrm{Card}\{\alpha_1^r\alpha_2^s;\ 0 \leq r < R_1,\ 0 \leq s < S_1\} \geq L,\\ &\mathrm{Card}\{rb_2 + sb_1;\ 0 \leq r < R_2,\ 0 \leq s < S_2\} > (K-1)L \end{aligned}$$

and

$$(2)\qquad K(\sigma L - 1)\log\varrho - (D+1)\log N - D(K-1)\log b - gL(Ra_1 + Sa_2) > \varepsilon(N),$$

where

$$\varepsilon(N) = 2\log(N!N^{-N+1}(e^N + (e-1)^N))/N.$$

Then

$$|\Lambda'| > \varrho^{-\mu KL} \quad \textit{with} \quad \Lambda' = \Lambda\max\Big\{\frac{LSe^{LS|\Lambda|/(2b_2)}}{2b_2}, \frac{LRe^{LR|\Lambda|/(2b_1)}}{2b_1}\Big\}.$$

We now consider specifically the case of multiplicatively independent algebraic numbers α_1, α_2. We specialize the values of the above parameters K, L, R_1, R_2, S_1, S_2 to obtain a more concrete result.

THEOREM 2. *Let a_1, a_2, h, ϱ and μ be real numbers with $\varrho > 1$ and $1/3 \le \mu \le 1$. Set*

$$\sigma = \frac{1+2\mu-\mu^2}{2}, \qquad \lambda = \sigma \log \varrho, \qquad H = \frac{h}{\lambda} + \frac{1}{\sigma},$$
$$\omega = 2\left(1+\sqrt{1+\frac{1}{4H^2}}\right), \qquad \theta = \sqrt{1+\frac{1}{4H^2}} + \frac{1}{2H}.$$

Consider the linear form $\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1$, where b_1 and b_2 are positive integers. Suppose that α_1 and α_2 are multiplicatively independent. Put $D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]/[\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}]$, and assume that

$$(3) \quad h \ge \max\left\{D\left(\log\left(\frac{b_1}{a_2}+\frac{b_2}{a_1}\right)+\log\lambda+1.75\right)+0.06, \lambda, \frac{D\log 2}{2}\right\},$$
$$(4) \quad a_i \ge \max\{1, \varrho|\log\alpha_i| - \log|\alpha_i| + 2D\mathrm{h}(\alpha_i)\} \quad (i=1,2),$$
$$(5) \quad a_1 a_2 \ge \lambda^2.$$

Then

$$\log|\Lambda| \ge -C\left(h+\frac{\lambda}{\sigma}\right)^2 a_1 a_2 - \sqrt{\omega\theta}\left(h+\frac{\lambda}{\sigma}\right) - \log\left(C'\left(h+\frac{\lambda}{\sigma}\right)^2 a_1 a_2\right)$$

with

$$C = \frac{\mu}{\lambda^3\sigma}\left(\frac{\omega}{6}+\frac{1}{2}\sqrt{\frac{\omega^2}{9}+\frac{8\lambda\omega^{5/4}\theta^{1/4}}{3\sqrt{a_1a_2}\,H^{1/2}}+\frac{4}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\frac{\lambda\omega}{H}}\right)^2,$$
$$C' = \sqrt{\frac{C\sigma\omega\theta}{\lambda^3\mu}}.$$

REMARK. The constant 1.75 occurring in (3) may be reduced if we assume that h is large enough. Its asymptotic value is equal to $3/2+\log(3/4) = 1.21\ldots$, as can be easily seen from the computations in Section 3.2.2 below. The interested reader is directed to [6], where this remark is expanded.

For fixed values of the parameters μ and ϱ, the leading coefficient C tends to

$$\frac{16\mu}{9\lambda^3\sigma} = \frac{16}{9(\log\varrho)^3}\cdot\frac{16\mu}{(1+2\mu-\mu^2)^4}$$

when h tends to infinity. The first factor $(16/9)(\log\varrho)^{-3}$ already occurred in Théorème 2 of [5], while the second is equal to 1 for $\mu = 1$. When h is large, the optimal values for μ are thus close to $0.63\ldots$ where the factor $16\mu/(1+2\mu-\mu^2)^4$, viewed as a function of μ, has a local minimum with value 0.83 . . . . Tables 2 and 3 in Section 4 illustrate the convergence of μ to 0.63 . . . as h grows.

In order to make the comparison with the results in [5, 6] more apparent, we give analogues of Corollaires 1 and 2 of [5]. Set

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1},$$

where A_1 and A_2 are real numbers > 1 such that

$$\log A_i \geq \max\{\mathrm{h}(\alpha_i), |\log \alpha_i|/D, 1/D\} \qquad (i = 1, 2).$$

For $m = 10, 12, \ldots, 30$, define coefficients $C_1 = C_1(m)$ and $C_2 = C_2(m)$ by the following table.

Table 1. Main constants

m	10	12	14	16	18	20	22	24	26	28	30
C_1	32.3	29.9	28.2	26.9	26.0	25.2	24.5	24.0	23.5	23.1	22.8
C_2	25.2	23.4	22.1	21.1	20.3	19.7	19.2	18.8	18.4	18.1	17.9

COROLLARY 1. *Suppose that α_1 and α_2 are multiplicatively independent. Then*

$$\log |\Lambda| \geq -C_1 D^4 (\max\{\log b' + 0.21, m/D, 1\})^2 \log A_1 \log A_2$$

for each pair $(m, C_1(m))$ from Table 1.

COROLLARY 2. *Suppose moreover that the numbers $\alpha_1, \alpha_2, \log \alpha_1, \log \alpha_2$ are real and positive. Then*

$$\log |\Lambda| \geq -C_2 D^4 (\max\{\log b' + 0.38, m/D, 1\})^2 \log A_1 \log A_2$$

for each pair $(m, C_2(m))$ from Table 1.

A look at the analogous Tableaux 1 and 2 on pages 319–320 of [5] reveals that, for each m, the corresponding constants $C_1(m)$ and $C_2(m)$ have actually been reduced by about twenty percent. Notice, however, that a direct application of Theorem 2 will usually provide a better result when dealing with a specific linear form.

To conclude the introduction, let us mention that Theorem 1 can also be applied to the case of multiplicatively dependent numbers α_1 and α_2, leading for instance to a sharpening of Théorème 3 in [5].

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2. Proof of Theorem 1. We follow the proof of the corresponding Théorème 1 in [5, Section 4] ([1]). We need a new analytic estimate, while the other parts of the proof remain unchanged. All the notations employed here are consistent with those of [5].

Let $\mathcal{M}$ be the $KL \times RS$ matrix whose entries are

$$\binom{rb_2 + sb_1}{k} \alpha_1^{lr} \alpha_2^{ls},$$

where (k, l) $(0 \le k < K,\ 0 \le l < L)$ is the row index and (r, s) $(0 \le r < R,\ 0 \le s < S)$ the column index. By [5, Lemme 5], under the assumption (1), the rank of $\mathcal{M}$ is $N = KL$. Let Δ be a non-zero $N \times N$ minor of $\mathcal{M}$. After numbering the rows and columns of Δ, we can write

$$\Delta = \det\biggl(\binom{r_j b_2 + s_j b_1}{k_i} \alpha_1^{l_i r_j} \alpha_2^{l_i s_j} \biggr)_{1 \le i, j \le N}$$

for some integer sequences $(k_i, l_i)_{1 \le i \le N}$ and $(r_j, s_j)_{1 \le j \le N}$.

2.1. *Arithmetical lower bound.* Under the assumptions of Theorem 1, Lemme 6 of [5] provides us with the following lower bound for $|\Delta|$.

Lemma 1. *Put*

$$g = \frac{1}{4} - \frac{N}{12RS}, \qquad G_1 = gLRN/2, \qquad G_2 = gLSN/2,$$
$$M_1 = (L-1)(r_1 + \cdots + r_N)/2, \qquad M_2 = (L-1)(s_1 + \cdots + s_N)/2.$$

Then

$$\log|\Delta| \ge -\frac{D-1}{2} N \log N + (M_1 + G_1)\log|\alpha_1| + (M_2 + G_2)\log|\alpha_2| - 2DG_1 \mathrm{h}(\alpha_1) - 2DG_2 \mathrm{h}(\alpha_2) - \frac{1}{2}(D-1)(K-1)N\log b.$$

2.2. *Analytic upper bound.* Let us now state our new analytic estimate which essentially reduces to Lemme 7 of [5] when $\mu = 1$.

Lemma 2. *Let ϱ and μ be real numbers. Assume that $\varrho > 1$, $1/3 \le \mu \le 1$ and*

$$|\Lambda'| \le \varrho^{-\mu N}. \tag{6}$$

Put $\sigma = (1 + 2\mu - \mu^2)/2$. Then

$$|\Delta| \le \varrho^{-(\sigma N^2 - N)/2} N (e^N + (e-1)^N)(N!)(\varrho b)^{(K-1)N/2} \times |\alpha_1|^{M_1} |\alpha_2|^{M_2} e^{\varrho(G_1|\log\alpha_1| + G_2|\log\alpha_2|)}.$$

([1]) Notice that the stronger assumptions $K \ge 3$ and $L \ge 2$ made in [5, Théorème 1] are unnecessary in our present proof. They ensure the lower bound $N \ge 6$, which is used in Section 4.5 of [5], but not here. Compare with the earlier Theorem 3 in [4].

The proof of Lemma 2 rests on a refinement of the analytic argument introduced in [4, Lemma 6]. The determinant Δ may be written as an interpolation determinant (also called alternant) of N analytic functions in two variables, say x and y, evaluated at N points (x_j, y_j) $(1 \le j \le N)$. Condition (6) means that the supremum of the $|y_j|$'s is small. To estimate such a determinant, we use the device given in the remark on p. 194 of [4]. One has to expand the interpolation determinant into power series of the $2N$ variables x_j, y_j $(1 \le j \le N)$, and next estimate the non-zero summands. Compared with the previous Lemma 6 of [4], we make use here of the whole power series expansion of Δ, instead of the truncated series to order one in the variables $y_1, \ldots, y_N$.

2.2.1. *A combinatorial lemma.* To prove Lemma 2, we begin with the following result.

Lemma 3. *Let ℓ be a positive integer, let $\nu_1, \ldots, \nu_\ell$ be a sequence of positive integers and let μ be a real number with $1/3 \le \mu \le 1$. Put $\sigma = (1 + 2\mu - \mu^2)/2$ and $N = \sum_{k=1}^{\ell} \nu_k$. Then*

$$\sum_{k=1}^{\ell} \binom{\nu_k}{2} + \mu N \sum_{k=1}^{\ell} (k-1)\nu_k \ge \frac{\sigma N^2 - N}{2}.$$

Proof. Consider the polynomial

$$P(x_1, \ldots, x_\ell) = \frac{1}{2}\Big(\sum_{k=1}^{\ell} x_k^2\Big) + \mu\Big(\sum_{k=1}^{\ell} (k-1) x_k\Big),$$

together with the simplex $\mathcal{S} \subset \mathbb{R}^\ell$ consisting of the points $\underline{x} = (x_1, \ldots, x_\ell)$ which satisfy

$$\sum_{k=1}^{\ell} x_k = 1 \quad \text{and} \quad x_k \ge 0 \quad (1 \le k \le \ell).$$

Since $(\nu_1/N, \ldots, \nu_\ell/N)$ belongs to $\mathcal{S}$, it clearly suffices to show that $P(\underline{x}) \ge \sigma/2$ for any $\underline{x} = (x_1, \ldots, x_\ell)$ in $\mathcal{S}$.

Let $\underline{\xi} = (\xi_1, \ldots, \xi_\ell)$ be a point in $\mathcal{S}$ where P reaches its minimal value on $\mathcal{S}$. Observe first that

$$1 \ge \xi_1 \ge \cdots \ge \xi_\ell \ge 0,$$

since otherwise, permuting coordinates $\xi_i < \xi_j$ with $i < j$ would produce a point $\underline{\xi}'$ for which $P(\underline{\xi}')$ is smaller. We remark now that for any index k with $2 \le k \le \ell$ and any real number y in the interval $-\xi_k \le y \le \xi_1$, the point $(\xi_1 - y, \ldots, \xi_k + y, \ldots)$ obtained from $\underline{\xi}$ by modifying only the first and kth coordinates lies in $\mathcal{S}$. Since P attains its minimal value on $\mathcal{S}$ at $\underline{\xi}$, the partial derivative

$$-\xi_1+\xi_k+(k-1)\mu = -\frac{\partial P}{\partial x_1}(\underline{\xi})+\frac{\partial P}{\partial x_k}(\underline{\xi}) = \frac{\partial}{\partial y}P(\xi_1-y,\dots,\xi_k+y,\dots)\Big|_{y=0}$$

is always ≥ 0 since $\xi_1 > 0$, and moreover it vanishes whenever $\xi_k > 0$. Consequently, for any index k with $1 \le k \le \ell$, either $\xi_k = 0$, or $\xi_k = \xi_1 - (k-1)\mu > 0$. Let $m \le \ell$ be the greatest integer k for which $\xi_k > 0$. The relation $\sum_{k=1}^{m} \xi_k = 1$ then implies

$$\xi_k = \frac{1}{m}+\Big(\frac{m+1}{2}-k\Big)\mu, \qquad 1 \le k \le m.$$

Writing now $\xi_m > 0$, we see that $\mu < 2/(m(m-1))$. Since we have assumed that $\mu \ge 1/3$, it follows that $m \le 2$. For $m = 1$, we have $\underline{\xi} = (1, 0, \dots)$ and $P(\underline{\xi}) = 1/2 \ge \sigma/2$. For $m = 2$, we find

$$\underline{\xi} = \Big(\frac{1+\mu}{2}, \frac{1-\mu}{2}, 0, \dots\Big) \quad \text{and} \quad P(\underline{\xi}) = \frac{1+2\mu-\mu^2}{4} = \frac{\sigma}{2}. \ \blacksquare$$

2.2.2. *Expanding the interpolation determinant Δ.* Permuting possibly α_1 with α_2 and b_1 with b_2, we may assume that

$$b_1|\log\alpha_1| \le b_2|\log\alpha_2|.$$

We shall then prove the required upper bound for $|\Delta|$, assuming that

$$|\Lambda''| \le \varrho^{-\mu N}, \tag{7}$$

where $\Lambda'' := (LS\Lambda/(2b_2))e^{LS|\Lambda|/(2b_2)}$. Lemma 2 will obviously follow.

As in [4, Lemma 6] and in [5, Lemme 7], we first express Δ as an interpolation determinant. Put $\beta = b_1/b_2$. For any complex number η, linear combinations of rows enable us to write

$$\Delta = \det\Big(\frac{b_2^{k_i}}{k_i!}(r_j+s_j\beta-\eta)^{k_i}\alpha_1^{\ell_i r_j}\alpha_2^{\ell_i s_j}\Big).$$

We choose $\eta = ((R-1)+\beta(S-1))/2$. It is also convenient to center the exponents ℓ_i around their average value $(L-1)/2$. We get

$$\Delta = \alpha_1^{M_1}\alpha_2^{M_2}\det\Big(\frac{b_2^{k_i}}{k_i!}(r_j+s_j\beta-\eta)^{k_i}\alpha_1^{\lambda_i r_j}\alpha_2^{\lambda_i s_j}\Big),$$

where $\lambda_i = \ell_i - (L-1)/2$ $(1 \le i \le N)$. From the relation $\log\alpha_2 = \beta\log\alpha_1 + \Lambda/b_2$, we may write

$$\alpha_1^{\lambda_i r_j}\alpha_2^{\lambda_i s_j} = \alpha_1^{\lambda_i(r_j+s_j\beta-\eta)}e^{\lambda_i s_j\Lambda/b_2}\alpha_1^{\lambda_i\eta}.$$

Noting that $\sum_{i=1}^{N}\lambda_i = 0$, we finally obtain the formula

$$\begin{aligned}\Delta &= \alpha_1^{M_1}\alpha_2^{M_2}\det\biggl(\frac{b_2^{k_i}}{k_i!}\,(r_j+s_j\beta-\eta)^{k_i}\alpha_1^{\lambda_i(r_j+s_j\beta-\eta)}e^{\lambda_i s_j\Lambda/b_2}\biggr)\\ &= \alpha_1^{M_1}\alpha_2^{M_2}\det(\varphi_i(z_j)e^{\lambda_i s_j\Lambda/b_2}),\end{aligned}$$

with

$$\varphi_i(x) = \frac{b_2^{k_i}}{k_i!}\,x^{k_i}\alpha_1^{\lambda_i x} \quad\text{and}\quad z_j = r_j + s_j\beta - \eta \quad (1\le i,j\le N).$$

Thus, $\alpha_1^{-M_1}\alpha_2^{-M_2}\Delta$ has just been written as the interpolation determinant of the N functions $\varphi_i(x)e^{\lambda_i y}$ $(1\le i\le N)$, evaluated at the N points $(z_j, s_j\Lambda/b_2)$ $(1\le j\le N)$.

We now expand each factor

$$e^{\lambda_i s_j\Lambda/b_2} = \sum_{n_i\ge 0}\frac{(\lambda_i s_j\Lambda/b_2)^{n_i}}{n_i!}$$

into a power series in $s_j\Lambda/b_2$. By the multilinearity of the determinant, we get the formula

$$\Delta = \alpha_1^{M_1}\alpha_2^{M_2}\sum_{n_1\ge 0}\cdots\sum_{n_N\ge 0}\Delta_{\underline{n}}\,,$$

where we have set

$$\underline{n} = (n_1,\ldots,n_N) \quad\text{and}\quad \Delta_{\underline{n}} = \det\biggl(\varphi_i(z_j)\,\frac{(\lambda_i s_j\Lambda/b_2)^{n_i}}{n_i!}\biggr).$$

Let $m_1,\ldots,m_\ell$ be the distinct values taken by the n_i's in the N-tuple $\underline{n}$. These values are numbered in order of increasing magnitude, $m_1<\cdots<m_\ell$. For each integer k with $1\le k\le \ell$, we denote by I_k the subset of indices i for which $n_i = m_k$, and by $\nu_k = \operatorname{Card} I_k$ the number of repetitions of the value m_k in the sequence $\underline{n}$.

Lemma 4. *For any N-tuple $\underline{n}=(n_1,\ldots,n_N)$ of non-negative integers and any real number $\varrho>1$, we have the upper bound*

$$|\Delta_{\underline{n}}| \le \Omega\varrho^{-\sum_{k=1}^{\ell}\binom{\nu_k}{2}}\biggl(\frac{LS|\Lambda|}{2b_2}\biggr)^{\sum_{i=1}^{N}n_i}\Bigl(\prod_{i=1}^{N}n_i!\Bigr)^{-1}$$

with

$$\Omega = N!(\varrho b)^{(K-1)N/2}e^{\varrho(G_1|\log\alpha_1|+G_2|\log\alpha_2|)}.$$

Proof. We consider the analytic function

$$\Delta_{\underline{n}}(x) = \det\biggl(\varphi_i(xz_j)\,\frac{(\lambda_i s_j\Lambda/b_2)^{n_i}}{n_i!}\biggr)$$

of the complex variable x. Obviously $\Delta_{\underline{n}} = \Delta_{\underline{n}}(1)$.

Let us first show that $\Delta_{\underline{n}}(x)$ has a zero at the origin $x = 0$ with multiplicity greater than or equal to $\sum_{k=1}^{\ell} \binom{\nu_k}{2}$. For that purpose, expand each $\varphi_i(x) = \sum_{h_i \ge 0} p_{i,h_i} x^{h_i}$ into Taylor's series about the origin and substitute $\varphi_i(xz_j) = \sum_{h_i \ge 0} p_{i,h_i} (xz_j)^{h_i}$ in the determinant $\Delta_{\underline{n}}(x)$. As above, we use the multilinearity of the determinant to find that

$$\Delta_{\underline{n}}(x) = \sum_{(h_1,\dots,h_N)\in\mathbb{N}^N} \Big(\prod_{i=1}^{N} p_{i,h_i}\Big) \det\bigg(\frac{z_j^{h_i}(\lambda_i s_j \Lambda/b_2)^{n_i}}{n_i!}\bigg) x^{\sum_{i=1}^{N} h_i}.$$

Observe that the summand $\det(z_j^{h_i}(\lambda_i s_j \Lambda/b_2)^{n_i}/n_i!)$ vanishes when $h_i = h_{i'}$ for some pair of indices $i \ne i'$ belonging to the same subset I_k, since in that case rows i and i' in the matrix are proportional. It follows that for any non-zero term in the above sum,

$$\sum_{i=1}^{N} h_i \ge \sum_{k=1}^{\ell} \sum_{h=0}^{\nu_k - 1} h = \sum_{k=0}^{\ell} \binom{\nu_k}{2}.$$

We now expand the determinant $\Delta_{\underline{n}}(x)$. On bounding $|\lambda_i s_j| \le LS/2$ for any $1 \le i, j \le N$, we obtain the estimate

$$|\Delta_{\underline{n}}(x)| \le N! \max_{\tau} \Big\{\prod_{i=1}^{N} |\varphi_i(xz_{\tau(i)})|\Big\} \Big(\frac{LS|\Lambda|}{2b_2}\Big)^{\sum_{i=1}^{N} n_i} \Big(\prod_{i=1}^{N} n_i!\Big)^{-1},$$

where τ runs over all substitutions of $\{1, \dots, N\}$. For any such τ, the upper bound

$$\prod_{i=1}^{N} |\varphi_i(xz_{\tau(i)})| \le (|x|b)^{(K-1)N/2} \exp\{|x|(G_1 + \beta G_2)|\log \alpha_1|\}$$

has been established in the proofs of Lemma 8 in [4] and of Lemme 7 in [5] (2). The assumption $\beta|\log \alpha_1| \le |\log \alpha_2|$ then implies that

$$\max_{|x|\le\varrho} |\Delta_{\underline{n}}(x)| \le \Omega \Big(\frac{LS|\Lambda|}{2b_2}\Big)^{\sum_{i=1}^{N} n_i} \Big(\prod_{i=1}^{N} n_i!\Big)^{-1}.$$

The required upper bound finally follows from the usual Schwarz lemma. ∎

2.2.3. *Proof of Lemma 2.* Recall that we have associated to each N-tuple $\underline{n}$ of non-negative integers the two sequences $m_1, \dots, m_\ell$ and $\nu_1, \dots, \nu_\ell$. Notice that there are exactly $\binom{N}{\nu_1,\dots,\nu_\ell}$ N-tuples $\underline{n}$ giving rise to the same couple of sequences $(\nu_1, \dots, \nu_\ell)$ and $(m_1, \dots, m_\ell)$ satisfying $\nu_k \ge 1$ for $1 \le k \le \ell$, $\nu_1 + \dots + \nu_\ell = N$ and $0 \le m_1 < \dots < m_\ell$, since the number of ordered partitions $\{1, \dots, N\} = \coprod_{k=1}^{\ell} I_k$ with $\operatorname{Card} I_k = \nu_k, 1 \le k \le \ell$, is equal to the multinomial coefficient $\binom{N}{\nu_1,\dots,\nu_\ell}$.

(2) Beware that the b in [4] corresponds to $2b$ in [5].

Let us indicate by $\sum'_{\underline{n}}$ a sum over all N-tuples $\underline{n}$ for which at least one of the n_i vanishes (equivalently $m_1 = 0$), and by $\sum''_{\underline{n}}$ the sum over the complementary set of N-tuples $\underline{n}$ for which $m_1 \geq 1$. Our purpose now is to bound $\sum'_{\underline{n}} |\Delta_{\underline{n}}|$ and $\sum''_{\underline{n}} |\Delta_{\underline{n}}|$.

Let $\underline{n}$ be an N-tuple for which $m_1 = 0$. When $\ell = 1$, we have $\underline{n} = (0, \ldots, 0)$. When $\ell \geq 2$, write $m_k = k - 1 + m'_k$ for $2 \leq k \leq \ell$, so that $0 \leq m'_2 \leq \cdots \leq m'_\ell$. Then we have

$$\sum_{i=1}^{N} n_i = \sum_{k=2}^{\ell} (k-1)\nu_k + \sum_{k=2}^{\ell} m'_k \nu_k$$

and

$$\prod_{i=1}^{N} n_i! = \prod_{k=2}^{\ell} (m_k!)^{\nu_k} \geq \prod_{k=2}^{\ell} (k-1)!^{\nu_k} \cdot \prod_{k=2}^{\ell} (m'_k!)^{\nu_k}.$$

Plugging the above estimates into the upper bound furnished by Lemma 4, we find

$$\begin{aligned}\sum_{\underline{n}}{}' |\Delta_{\underline{n}}| &\leq \Omega \varrho^{-\binom{N}{2}} \\ &\quad + \Omega \sum_{\ell=2}^{N} \Bigg\{ \sum_{\substack{\nu_1+\cdots+\nu_\ell=N \\ \nu_1\geq 1,\ldots,\nu_\ell\geq 1}} \Bigg(\frac{\binom{N}{\nu_1,\ldots,\nu_\ell} \varrho^{-\sum_{k=1}^{\ell} \binom{\nu_k}{2}} (LS|\Lambda|/2b_2)^{\sum_{k=2}^{\ell}(k-1)\nu_k}}{\prod_{k=2}^{\ell}(k-1)!^{\nu_k}} \\ &\qquad \times \sum_{0\leq m'_2\leq\cdots\leq m'_\ell} \frac{(LS|\Lambda|/2b_2)^{\sum_{k=2}^{\ell} m'_k\nu_k}}{\prod_{k=2}^{\ell}(m'_k!)^{\nu_k}} \Bigg) \Bigg\}.\end{aligned}$$

We roughly bound the last sum as follows:

$$\begin{aligned}\sum_{0\leq m'_2\leq\cdots\leq m'_\ell} \frac{(LS|\Lambda|/2b_2)^{\sum_{k=2}^{\ell} m'_k\nu_k}}{\prod_{k=2}^{\ell}(m'_k)^{\nu_k}} &\leq e^{(LS|\Lambda|/2b_2)\sum_{k=2}^{\ell}\nu_k} \\ &\leq e^{(LS|\Lambda|/2b_2)\sum_{k=1}^{\ell}(k-1)\nu_k}.\end{aligned}$$

We finally obtain the estimate ([3])

$$\begin{aligned}\sum_{\underline{n}}{}' |\Delta_{\underline{n}}| &\leq \Omega \sum_{\ell=1}^{N} \sum_{\nu_1+\cdots+\nu_\ell=N} \frac{\binom{N}{\nu_1,\ldots,\nu_\ell} \varrho^{-\sum_{k=1}^{\ell}\binom{\nu_k}{2}} |\Lambda''|^{\sum_{k=1}^{\ell}(k-1)\nu_k}}{\prod_{k=2}^{\ell}(k-1)!^{\nu_k}} \\ &\leq \varrho^{-(\sigma N^2 - N)/2} \Omega \sum_{\ell=1}^{N} \sum_{\nu_1+\cdots+\nu_\ell=N} \frac{\binom{N}{\nu_1,\ldots,\nu_\ell}}{\prod_{k=2}^{\ell}(k-1)!^{\nu_k}}\end{aligned}$$

([3]) When $\ell = 1$, the sums and products taken over k in the empty interval $2 \leq k \leq \ell$ have to be replaced by 0 and 1 respectively.

$$\le \varrho^{-(\sigma N^2-N)/2}\Omega\sum_{\ell=1}^{N}\Bigl(1+\sum_{k=2}^{\ell}\frac{1}{(k-1)!}\Bigr)^N \le \varrho^{-(\sigma N^2-N)/2}\Omega N e^N,$$

using here the upper bound (7) for $|\Lambda''|$ combined with Lemma 3.

As for the second sum $\sum''_{\underline{n}}|\Delta_{\underline{n}}|$, which is a residual term, we use similar arguments. We now start with the decomposition $m_k = k+m''_k$ for $1\le k\le \ell$, where $0\le m''_1\le\cdots\le m''_\ell$. Replacing in the above display $k-1$ by k and m'_k by m''_k, where the index k runs from 1 to ℓ, we obtain in that case the upper bound

$$\sum_{\underline{n}}{}''|\Delta_{\underline{n}}| \le \varrho^{-(\sigma N^2-N)/2}\Omega N(e-1)^N,$$

the factor $(e-1)^N$ arising from the estimate $\sum_{k=1}^{\ell}1/k!\le e-1$ used in the last step. Since

$$|\Delta|\le |\alpha_1|^{M_1}|\alpha_2|^{M_2}\Bigl(\sum_{\underline{n}}{}'|\Delta_{\underline{n}}|+\sum_{\underline{n}}{}''|\Delta_{\underline{n}}|\Bigr),$$

the proof of Lemma 2 is now complete. ∎

2.3. *Completion of the proof.* Suppose finally that the assumptions (1) and (2) of Theorem 1 are satisfied and that (6) holds. Then Lemmas 1 and 2 provide us with the following estimate for $\log|\Delta|$:

$$-\frac{(D-1)N\log N}{2}+(M_1+G_1)\log|\alpha_1|+(M_2+G_2)\log|\alpha_2|-2DG_1\mathrm{h}(\alpha_1)$$
$$-2DG_2\mathrm{h}(\alpha_2)-\frac{1}{2}(D-1)(K-1)N\log b\le\log|\Delta|$$
$$\le\log(N(e^N+(e-1)^N)(N!))+\frac{1}{2}(K-1)N\log(\varrho b)+M_1\log|\alpha_1|$$
$$+\varrho G_1|\log\alpha_1|+M_2\log|\alpha_2|+\varrho G_2|\log\alpha_2|-\frac{1}{2}(\sigma N^2-N)\log\varrho.$$

The terms in M_1 and M_2 cancel. Replace now G_1 and G_2 by their values. After division by $N/2$, we get the opposite of (2). Therefore (6) cannot hold under the assumptions (1) and (2), and Theorem 1 is proved.

3. Proof of Theorem 2. For the most part, we follow the proof of the corresponding Theorem 2 in [5] and in [6]. Notice however that we have slightly modified the definition of the parameter L. This new choice leads to smaller constants, even in the setting of [5, 6] where $\mu=1$. Compared with [5, 6], we have also split the proof into successive steps, which hopefully should clarify its structure.

3.1. *The parameters.* We define L to be the unique integer belonging to the interval

$$(8)\qquad \sqrt{\frac{\omega}{\theta}}\,H = H + \sqrt{H^2+\frac{1}{4}} - \frac{1}{2} < L \le H + \sqrt{H^2+\frac{1}{4}} + \frac{1}{2} = \sqrt{\omega\theta}\,H.$$

Set now

$$U = \lambda L - h - \frac{\lambda}{\sigma} = \lambda(L-H), \qquad V = \frac{L}{3}, \qquad W = \frac{1}{3}\biggl(\frac{1}{a_1} + \frac{1}{a_2} + 2\sqrt{\frac{L}{a_1a_2}}\biggr),$$

$$k = \biggl(\frac{V}{2U} + \frac{1}{2}\sqrt{\frac{V^2}{U^2} + 4\,\frac{W}{U}}\biggr)^2 = \frac{V^2}{2U^2} + \frac{W}{U} + \frac{V}{2U}\sqrt{\frac{V^2}{U^2} + \frac{4W}{U}}.$$

Hence

$$\sqrt{k} = \frac{V}{2U} + \frac{1}{2}\sqrt{\frac{V^2}{U^2} + 4\,\frac{W}{U}}$$

is the positive root of the polynomial $UX^2 - VX - W$. Put finally

$$K = 1 + \lfloor kLa_1a_2\rfloor, \qquad R_1 = 1 + \lfloor\sqrt{La_2/a_1}\rfloor, \qquad R_2 = 1 + \lfloor\sqrt{(K-1)La_2/a_1}\rfloor,$$
$$S_1 = 1 + \lfloor\sqrt{La_1/a_2}\rfloor, \qquad S_2 = 1 + \lfloor\sqrt{(K-1)La_1/a_2}\rfloor.$$

With the noteworthy exception of L, these parameters were already employed in [5, 6] ([4]). The present choice of L is motivated by the estimate (11) below. We have selected an interval (8) of length 1 along which the function $x \mapsto x^2/(x-H)$ is as small as possible, in order to minimize the value of the coefficient C depending mainly on the quantity $L^2/(L-H)$.

For later use, we now estimate various expressions involving these parameters in terms of the data $a_1, a_2, h, \varrho, \mu$. Here, the quantity $H = h/\lambda + 1/\sigma$ plays an important role. Note that our assumptions imply that $H \ge 2$, since $h \ge \lambda$ (by (3)) and $0 < \sigma \le 1$.

Using the formulas

$$\sqrt{\frac{\omega}{\theta}} = 1 + \sqrt{1 + \frac{1}{4H^2}} - \frac{1}{2H} \quad\text{and}\quad \sqrt{\omega\theta} = 1 + \sqrt{1 + \frac{1}{4H^2}} + \frac{1}{2H},$$

we first deduce from the lower bound $H \ge 2$ the inequalities

$$(9)\qquad \frac{3+\sqrt{17}}{4} \le \sqrt{\frac{\omega}{\theta}} < \sqrt{\omega\theta} \le \frac{5+\sqrt{17}}{4}.$$

We now estimate quantities of the form $L^\alpha/(L-H)$ for exponents α which are half integers.

LEMMA 5. *For any half integer $\alpha \ne 2$, we have*

$$(10)\qquad \omega^{\alpha/2}\theta^{-|\alpha/2-1|}H^{\alpha-1} \le L^\alpha/(L-H) \le \omega^{\alpha/2}\theta^{|\alpha/2-1|}H^{\alpha-1}.$$

([4]) More precisely, the parameters K, R_1, R_2, S_1, S_2 are defined in [5, 6] by the same formulas, but with slightly larger values of the parameter k.

When $\alpha = 2$, *we have*

$$4H \le \frac{L^2}{L-H} \le 2\left(H + \sqrt{H^2 + \frac{1}{4}}\right) = \omega H. \tag{11}$$

Proof. Let us show that the function $x \mapsto x^\alpha/(x-H)$ decreases in the interval (8) when $\alpha \le 3/2$ and increases when $\alpha \ge 5/2$. Differentiating gives

$$\frac{\partial}{\partial x}\frac{x^\alpha}{x-H} = \frac{x^{\alpha-1}(\alpha(x-H)-x)}{(x-H)^2}.$$

For any $\alpha \ge 5/2$ and any x in (8), we bound from below

$$\begin{aligned}\alpha(x-H)-x \ge \frac{5}{2}(x-H)-x = \frac{3}{2}x - \frac{5}{2}H \ge \frac{3}{2}\sqrt{\frac{\omega}{\theta}}\,H - \frac{5}{2}H \\ \ge \frac{H(-11+3\sqrt{17})}{8} > 0,\end{aligned}$$

since $\sqrt{\omega/\theta} \ge (3+\sqrt{17})/4$ by (9). The function $x \mapsto x^\alpha/(x-H)$ is therefore increasing in (8) when $\alpha \ge 5/2$. When $\alpha \le 3/2$, we bound from above

$$\begin{aligned}\alpha(x-H)-x \le \frac{3}{2}(x-H)-x = \frac{1}{2}x - \frac{3}{2}H \le \frac{1}{2}\sqrt{\omega\theta}\,H - \frac{3}{2}H \\ \le \frac{H(-7+\sqrt{17})}{8} < 0,\end{aligned}$$

since $\sqrt{\omega\theta} \le (5+\sqrt{17})/4$ by (9), to conclude that the function strictly decreases in that case.

Therefore we find the estimate

$$\begin{aligned}\omega^{\alpha/2}\theta^{-(\alpha/2-1)}H^{\alpha-1} &= \frac{(H+\sqrt{H^2+1/4}-1/2)^\alpha}{\sqrt{H^2+1/4}-1/2} \le \frac{L^\alpha}{L-H} \\ &\le \frac{(H+\sqrt{H^2+1/4}+1/2)^\alpha}{\sqrt{H^2+1/4}+1/2} = \omega^{\alpha/2}\theta^{\alpha/2-1}H^{\alpha-1}\end{aligned}$$

for any $\alpha \ge 5/2$, while the reverse inequalities, obtained by exchanging the upper and lower bounds, hold true when $\alpha \le 3/2$. The estimate (10) is thus verified for any half integer $\alpha \ne 2$.

When $\alpha = 2$, the function $x \mapsto x^2/(x-H)$ attains its minimal value in (8) at $x = 2H$, and reaches its maximal value at the extremities $H + \sqrt{H^2+1/4} \pm 1/2$ of the interval. The estimate (11) is thus verified. ∎

We now proceed to show that $L \ge 4$ and $K \ge 8$, hence $N \ge 32$. The lower bound $L \ge 4$ immediately follows from (8), since $H \ge 2$. As for K, using the definitions of k, V and W, we can write

$$\sqrt{kLa_1a_2} = \frac{L^{3/2}\sqrt{a_1a_2}}{6U} + \frac{1}{2}\sqrt{\left(\frac{L^{3/2}}{3U}\right)^2 a_1a_2 + \frac{4}{3}\frac{L}{U}(a_1+a_2) + \frac{8}{3}\frac{L^{3/2}}{U}\sqrt{a_1a_2}}.$$

On combining (10) (for $\alpha = 3/2$ and $\alpha = 1$) with the lower bounds $a_1+a_2 \geq 2\sqrt{a_1a_2} \geq 2\lambda$ deduced from (5), we find that

$$\sqrt{kLa_1a_2} \geq \frac{\omega^{3/4}H^{1/2}}{6\theta^{1/4}} + \frac{1}{2}\sqrt{\frac{\omega^{3/2}H}{9\theta^{1/2}} + \frac{8\omega^{3/4}H^{1/2}}{3\theta^{1/4}} + \frac{8\omega^{1/2}}{3\theta^{1/2}}}.$$

Observe that the right hand side of the above inequality, when viewed as a function of H, may be written as a composed function

$$\frac{1}{6}\sqrt{H\omega\sqrt{\frac{\omega}{\theta}}} + \frac{1}{2}\sqrt{\frac{1}{9}H\omega\sqrt{\frac{\omega}{\theta}} + \frac{8}{3}\sqrt{H\omega\sqrt{\frac{\omega}{\theta}}} + \frac{8}{3}\sqrt{\frac{\omega}{\theta}}},$$

where the two functions

$$H\omega = 2H + \sqrt{4H^2+1} \quad \text{and} \quad \sqrt{\frac{\omega}{\theta}} = 1 + \sqrt{1+\frac{1}{4H^2}} - \frac{1}{2H}$$

increase in the range $H \geq 2$. It follows that $\sqrt{kLa_1a_2}$ is greater than or equal to the value of the above expression at $H = 2$. We find that $\sqrt{kLa_1a_2} \geq 2.66$. Hence $K = 1 + \lfloor kLa_1a_2 \rfloor \geq 8$.

3.2. *An intermediate lower bound.* Our goal is to establish the estimate

$$\log|\Lambda'| \geq -C\left(h+\frac{\lambda}{\sigma}\right)^2 a_1a_2 - \sqrt{\omega\theta}\left(h+\frac{\lambda}{\sigma}\right), \tag{12}$$

assuming that

$$b'' := \left(\frac{b_1}{a_2}+\frac{b_2}{a_1}\right) > 2\mu\lambda\sigma^{-1}kL^2\cdot\gcd(b_1,b_2). \tag{13}$$

The lower bound (12) will be furnished by Theorem 1, and thus we have to verify conditions (1) and (2).

3.2.1. *Condition (1).* Let us first record the lower bound

$$\sqrt{k}\,L \geq \frac{VL}{U} = \frac{L^2}{3\lambda(L-H)} \geq \frac{4H}{3\lambda}, \tag{14}$$

deduced from the obvious estimate $\sqrt{k} \geq V/U$ and (11). Put

$$b_1^* = \frac{b_1}{\gcd(b_1,b_2)} \quad \text{and} \quad b_2^* = \frac{b_2}{\gcd(b_1,b_2)}.$$

Our assumption (13) implies that

$$b_1^* > \mu\lambda\sigma^{-1}\sqrt{k}\,L\cdot\sqrt{(K-1)La_2/a_1}$$

or

$$b_2^* > \mu\lambda\sigma^{-1}\sqrt{k}\,L\cdot\sqrt{(K-1)La_1/a_2},$$

since $K-1\le kLa_1a_2$. Note that $\mu/\sigma\ge 3/7$. Then (14) gives $\mu\lambda\sigma^{-1}\sqrt{k}\,L\ge 4H/7>1$, since $H\ge 2$. We then deduce from the estimates $R_2-1\le\sqrt{(K-1)La_2/a_1}$ and $S_2-1\le\sqrt{(K-1)La_1/a_2}$ that

$$b_1^* > R_2-1 \quad\text{or}\quad b_2^* > S_2-1.$$

We infer that there is no linear relation $rb_2+sb_1=0$ with integer coefficients (r,s) satisfying $0<|r|\le R_2-1$ and $0<|s|\le S_2-1$. Otherwise, b_1^* would divide r and b_2^* would divide s, in contradiction with the above lower bounds. It follows that

$$\mathrm{Card}\{rb_2+sb_1;\ 0\le r<R_2,\ 0\le s<S_2\}=R_2S_2$$

and, by the choice of R_2 and S_2, we have $R_2S_2>(K-1)L$. Moreover, since α_1 and α_2 are multiplicatively independent,

$$\mathrm{Card}\{\alpha_1^r\alpha_2^s;\ 0\le r<R_1,\ 0\le s<S_1\}=R_1S_1\ge L.$$

This ends the verification of condition (1).

3.2.2. *Condition (2).* We follow the arguments of [5, Section 5.3] which remain mostly valid, since we deal here with the same parameters K, R_1, R_2, S_1, S_2. However, due to our new choice of L, some slight modifications are needed.

Let us quote the estimate

$$b\le\frac{(1+\sqrt{K-1})\sqrt{K}}{2(K-1)\sqrt{k}}\left(\frac{b_1}{a_2}+\frac{b_2}{a_1}\right)\times\exp\left\{\frac{3}{2}-\frac{\log(2\pi(K-1)/\sqrt{e})}{K-1}+\frac{\log K}{6K(K-1)}\right\}$$

from [5, p. 307, line 16]. The inequality $\sqrt{k}\ge V/U$, together with (9), (10), implies the lower bound

$$\sqrt{k}\ge\frac{V}{U}=\frac{L}{3\lambda(L-H)}\ge\frac{1}{3\lambda}\sqrt{\frac{\omega}{\theta}}\ge\frac{3+\sqrt{17}}{12\lambda}.$$

Combining the preceding two estimates gives

$$\log b\le\log(\lambda b'')-\frac{\log(2\pi K/\sqrt{e})}{K-1}+f(K)$$

with

$$f(x) = \log\left(\frac{(1+\sqrt{x-1})\sqrt{x}}{x-1}\right) + \frac{\log x}{6x(x-1)} + \frac{3}{2} + \log\left(\frac{6}{3+\sqrt{17}}\right) + \frac{\log(x/(x-1))}{x-1}.$$

Observe that $f(x)$ is a decreasing function for $x > 1$ (see [5, p. 308] for details). Since $K \geq 8$, it follows that $f(K) \leq f(8) \leq 1.75$. Then we deduce from (3) the upper bound

$$\log b \leq \frac{h - 0.06}{D} - \frac{\log(2\pi K/\sqrt{e})}{K-1}. \tag{15}$$

Next, we quote the upper bound

$$gL(Ra_1 + Sa_2) \leq \frac{1}{3}\, L^{3/2}\sqrt{(K-1)a_1a_2} + \frac{2}{3}\, L^{3/2}\sqrt{a_1a_2} + \frac{1}{3}\, L(a_1+a_2) - \frac{L^{3/2}\sqrt{a_1a_2}}{6(1+\sqrt{K-1})} \tag{16}$$

provided by Lemme 9 of [5], noting that its proof is valid for any integer $L \geq 1$. The estimates (15) and (16) imply that the left hand side of (2) is bounded from below by $\varPhi + \varTheta$, where

$$\varPhi = \lambda KL - K\left(h + \frac{\lambda}{\sigma}\right) - \frac{L^{3/2}\sqrt{(K-1)a_1a_2}}{3} - \frac{2L^{3/2}\sqrt{a_1a_2}}{3} - \frac{L(a_1+a_2)}{3},$$

$$\varTheta = 0.06(K-1) + h + \frac{L^{3/2}\sqrt{a_1a_2}}{6(1+\sqrt{K-1})} + D\log\left(\frac{2\pi K}{\sqrt{e}}\right) - (D+1)\log(KL).$$

We proceed to show that $\varPhi \geq 0$ and $\varTheta > \varepsilon(N)$. Then condition (2) will obviously follow. The inequality $\varPhi \geq 0$ is the main constraint, which justifies our definition of k. On combining (8) and (9), we first notice that

$$\lambda L \geq \lambda\left(\frac{3+\sqrt{17}}{4}\right)H \geq h + \frac{\lambda}{\sigma}.$$

Then the estimate $kLa_1a_2 \leq K \leq 1 + kLa_1a_2$ shows that

$$\varPhi \geq kLa_1a_2\left(\lambda L - h - \frac{\lambda}{\sigma}\right) - \frac{\sqrt{k}\,L^2a_1a_2}{3} - \frac{2L^{3/2}\sqrt{a_1a_2}}{3} - \frac{L(a_1+a_2)}{3} = La_1a_2(kU - \sqrt{k}\,V - W) = 0,$$

as required.

As for $\Theta > \varepsilon(N)$, we use again the estimate $h \geq D(\log(\lambda b'')+1.75)+0.06$ to bound from below $\Theta \geq \Theta_0(D-1)+\Theta_1$, where

$$\begin{aligned}\Theta_0 &= \log(\lambda b'') + 1.75 - \log L + \log\left(\frac{2\pi}{\sqrt{e}}\right),\\ \Theta_1 &= 0.06K - \log K - 2\log L + \frac{L^{3/2}\sqrt{a_1a_2}}{6(1+\sqrt{K-1})}\\ &\quad + \log(\lambda b'') + 1.75 + \log\left(\frac{2\pi}{\sqrt{e}}\right).\end{aligned}$$

It is therefore sufficient to prove that $\Theta_0 \geq 0$ and $\Theta_1 > \varepsilon(N)$, since $D \geq 1$.

Combining (13) and (14) gives

$$\lambda b'' \geq 2\,\frac{\mu}{\sigma}\,\lambda^2 kL^2 \geq \frac{6}{7}\,\lambda^2 kL^2 \geq \frac{32}{21}\,H^2. \tag{17}$$

Bounding $L \leq (5+\sqrt{17})H/4$, by (8) and (9), and plugging the lower bound (17) into Θ_0, we find

$$\Theta_0 \geq \log H + \log\left(\frac{32}{21}\right) + 1.75 + \log\left(\frac{2\pi}{\sqrt{e}}\right) - \log\left(\frac{5+\sqrt{17}}{4}\right) > 3,$$

since $H \geq 2$.

We now prove the inequality $\Theta_1 > \varepsilon(N)$. First, combining (17) and (3) gives

$$h \geq D\left(\log\left(\frac{32H^2}{21}\right) + 1.75\right) + 0.06 \geq 3.6,$$

since $H \geq 2$ and $D \geq 1$. Recalling that $L \geq 4$ and using (5), (8) and (9), we obtain the lower bound

$$L^{3/2}\sqrt{a_1a_2} \geq 2L\lambda \geq 2\sqrt{\frac{\omega}{\theta}}\,H\lambda \geq 2\sqrt{\frac{\omega}{\theta}}\,h \geq 2\left(\frac{3+\sqrt{17}}{4}\right)\cdot 3.6 \geq 12.$$

Then we insert the lower bound (17) and the preceding one into Θ_1. On bounding $L \leq (5+\sqrt{17})H/4$, we find

$$\begin{aligned}\Theta_1 &\geq 0.06K - \log K - 2\log\left(\frac{5+\sqrt{17}}{4}\right) + \log\left(\frac{2\pi}{\sqrt{e}}\right)\\ &\quad + \log\left(\frac{32}{21}\right) + 1.75 + \frac{2}{1+\sqrt{K-1}}.\end{aligned}$$

An elementary numerical verification shows that the right hand side is ≥ 0.4 for any $K \geq 8$. Thus, it suffices to prove $\varepsilon(N) < 0.4$. For that purpose, we use Feller's version [3, Chapter 2] of Stirling's formula

$$N! \leq \sqrt{2\pi}\,N^{N+1/2}e^{-N+1/(12N)},$$

which is valid for any integer $N \geq 1$. It implies the upper bound

$$\varepsilon(N) \leq \frac{2}{N}\left(\frac{3}{2}\log N + \frac{1}{2}\log(2\pi) + \frac{1}{12N} + \log\left(1 + \left(\frac{e-1}{e}\right)^N\right)\right).$$

Observe that the right hand side is a decreasing function of N for $N > e$, whose value at $N = 32$ is < 0.4. Since $N \geq 32$, it follows that $\varepsilon(N) < 0.4$ and condition (2) is verified.

3.2.3. *The coefficient C.* Conditions (1) and (2) having been verified, Theorem 1 provides us with the lower bound

$$\log|\Lambda'| \geq -\mu(\log\varrho)KL = -\mu\lambda\sigma^{-1}KL.$$

From the definition of K, we obviously obtain $KL \leq L + kL^2 a_1 a_2$, and we now proceed to estimate the two terms of the sum.

Using the definitions of $\sqrt{k}$, V and W, we can write

$$\sqrt{k}\,L = \frac{L^2}{6U} + \frac{1}{2}\sqrt{\left(\frac{L^2}{3U}\right)^2 + \frac{8}{3}\frac{1}{\sqrt{a_1a_2}}\frac{L^{5/2}}{U} + \frac{4}{3}\left(\frac{1}{a_1} + \frac{1}{a_2}\right)\frac{L^2}{U}}. \tag{18}$$

Then, putting $U = \lambda(L - H)$ and using the upper bounds provided by (10) and (11), we find

$$\begin{aligned} kL^2 &= (\sqrt{k}\,L)^2 \\ &\leq \left(\frac{\omega H}{6\lambda} + \frac{1}{2}\sqrt{\left(\frac{\omega H}{3\lambda}\right)^2 + \frac{8\omega^{5/4}\theta^{1/4}H^{3/2}}{3\sqrt{a_1a_2}\,\lambda} + \frac{4}{3}\left(\frac{1}{a_1} + \frac{1}{a_2}\right)\frac{\omega H}{\lambda}}\right)^2 \\ &= \mu^{-1}\lambda\sigma C H^2 = \mu^{-1}\lambda^{-1}\sigma C(h + \lambda/\sigma)^2. \end{aligned}$$

We thus obtain the main estimate

$$\mu\lambda\sigma^{-1}kL^2a_1a_2 \leq C(h + \lambda/\sigma)^2 a_1a_2. \tag{19}$$

It follows that

$$\begin{aligned} \log|\Lambda'| &\geq -\mu\lambda\sigma^{-1}L - \mu\lambda\sigma^{-1}kL^2a_1a_2 \\ &\geq -\sqrt{\omega\theta}\,(h + \lambda/\sigma) - C(h + \lambda/\sigma)^2 a_1a_2, \end{aligned}$$

since, by (8),

$$\mu\lambda\sigma^{-1}L \leq \lambda L \leq \lambda\sqrt{\omega\theta}\,H = \sqrt{\omega\theta}\,(h + \lambda/\sigma).$$

The proof of the intermediate lower bound (12) is now complete.

3.3. *The coefficient C'.* In this section we record various estimates involving the coefficient C'. Their proofs being all related, we have collected them here regardless of their forthcoming applications.

First, notice that C' may be expressed in the form

$$C' = \frac{1}{\lambda^3}\left(\frac{\omega^{3/2}\theta^{1/2}}{6} + \frac{1}{2}\sqrt{\frac{\omega^3\theta}{9} + \frac{8\lambda\omega^{9/4}\theta^{5/4}}{3\sqrt{a_1a_2}\,H^{1/2}} + \frac{4}{3}\left(\frac{1}{a_1} + \frac{1}{a_2}\right)\frac{\lambda\omega^2\theta}{H}}\right). \tag{20}$$

Multiplying (18) by L, we can write

$$\sqrt{k}\,L^2 = \frac{1}{6}\,\frac{L^3}{U} + \frac{1}{2}\sqrt{\left(\frac{L^3}{3U}\right)^2 + \frac{8}{3\sqrt{a_1a_2}}\,\frac{L^{9/2}}{U} + \frac{4}{3}\left(\frac{1}{a_1} + \frac{1}{a_2}\right)\frac{L^4}{U}}.$$

Now, putting $U = \lambda(L-H)$ and applying (10) with $\alpha = 3, 4, 9/2$, we deduce from (20) the estimates

$$(21)\qquad \begin{aligned} \frac{\omega^{3/2}\theta^{-1/2}H^2}{3\lambda} &\le \frac{L^3}{3\lambda(L-H)} < \sqrt{k}\,L^2 \\ &\le \frac{\omega^{3/2}\theta^{1/2}H^2}{6\lambda} \\ &\quad + \frac{1}{2}\sqrt{\frac{\omega^3\theta H^4}{9\lambda^2} + \frac{8}{3\sqrt{a_1a_2}}\,\frac{\omega^{9/4}\theta^{5/4}H^{7/2}}{\lambda} + \frac{4}{3}\left(\frac{1}{a_1} + \frac{1}{a_2}\right)\frac{\omega^2\theta H^3}{\lambda}} \\ &= \lambda^2 C' H^2 = C'(h+\lambda/\sigma)^2. \end{aligned}$$

Using (4), (5) and the upper bound for L in (8), it follows that

$$\sqrt{k}\,L^2 a_1a_2 \ge \frac{\omega^{3/2}\theta^{-1/2}H^2\max\{1,\lambda^2\}}{3\lambda} \ge \frac{\omega^{3/2}\theta^{-1/2}H^2}{3} \ge \frac{\omega HL}{3\theta} \ge \frac{2\omega L}{3\theta},$$

since $H \ge 2$. We shall use the above lower bound in the form

$$(22)\qquad L \le \frac{3\theta}{2\omega}\sqrt{k}\,L^2a_1a_2 \le \frac{3}{2}\left(\frac{4}{3+\sqrt{17}}\right)^2\sqrt{k}\,L^2a_1a_2,$$

the last inequality following from (9). Using again (21), (4) and (5), we bound from below

$$(23)\quad C'(h+\lambda/\sigma)^2a_1a_2 \ge \frac{\omega^{3/2}\theta^{-1/2}H^2}{3\lambda}\max\{1,\lambda^2\} \ge \frac{\omega^{3/2}\theta^{-1/2}H^2}{3} > e^2,$$

since $\omega \ge 4$, $H \ge 2$ and $\sqrt{\omega/\theta} \ge (3+\sqrt{17})/4$ by (9).

We shall need an upper bound for the ratio C'/C. For that purpose, write

$$\begin{aligned} \frac{C'}{C} &= \sqrt{\frac{\sigma\omega\theta}{\lambda^3\mu C}} \\ &= \frac{\sigma}{\mu}\sqrt{\omega\theta}\left(\frac{\omega}{6} + \frac{1}{2}\sqrt{\frac{\omega^2}{9} + \frac{8\lambda\omega^{5/4}\theta^{1/4}}{3\sqrt{a_1a_2}\,H^{1/2}} + \frac{4}{3}\left(\frac{1}{a_1} + \frac{1}{a_2}\right)\frac{\lambda\omega}{H}}\right)^{-1}. \end{aligned}$$

Ignoring the second and third terms under the radical, we obtain the bound

$$(24)\qquad \frac{C'}{C} \le 3\,\frac{\sigma}{\mu}\sqrt{\frac{\theta}{\omega}} < 4\,,$$

since $\sigma/\mu \le 7/3$ and $\sqrt{\theta/\omega} \le 4/(3+\sqrt{17})$ by (9).

3.4. *From Λ' to Λ.* Observe that $\sqrt{\omega\theta}\,(h+\lambda/\sigma) \ge D\log 2$, since $\sqrt{\omega\theta} \ge 2$ and $h \ge D(\log 2)/2$ by (3). Recalling (23), we may therefore assume without loss of generality that

$$\log|\Lambda| \le -C(h+\lambda/\sigma)^2 a_1 a_2 - D\log 2 - 2 \le -C(h+\lambda/\sigma)^2 a_1 a_2 - 2.6. \tag{25}$$

Then we show that

$$|\Lambda'| \le |\Lambda| C'(h+\lambda/\sigma)^2 a_1 a_2. \tag{26}$$

To do that, we bound

$$\begin{aligned} R = R_1 + R_2 - 1 &\le 1 + \sqrt{La_2/a_1} + \sqrt{(K-1)La_2/a_1} \\ &\le 1 + (1/\sqrt{7}+1)\sqrt{k}\,La_2, \end{aligned}$$

since $K \ge 8$. Recall that $a_1 \ge 1$ by (4). It follows from (22) and (21) that

$$\begin{aligned} LR &\le L + \Big(\frac{1}{\sqrt{7}}+1\Big)\sqrt{k}\,L^2 a_2 \le \bigg(\frac{3}{2}\Big(\frac{4}{3+\sqrt{17}}\Big)^2 + \frac{1}{\sqrt{7}}+1\bigg)\sqrt{k}\,L^2 a_1 a_2 \\ &\le 1.86\sqrt{k}\,L^2 a_1 a_2 \le 1.86 C'(h+\lambda/\sigma)^2 a_1 a_2. \end{aligned}$$

The same upper bound holds for LS. We thus obtain the estimate

$$\max\{LS, LR\} \le 1.86 C'(h+\lambda/\sigma)^2 a_1 a_2. \tag{27}$$

Notice the lower bound

$$\begin{aligned} C(h+\lambda/\sigma)^2 a_1 a_2 &\ge \frac{\mu}{\sigma}\,\lambda k L^2 a_1 a_2 \ge \frac{3}{7}\,\lambda\Big(\frac{4H}{3\lambda}\Big)^2 \max\{1,\lambda^2\} \\ &\ge \frac{16}{21}\,\frac{\max\{1,\lambda^2\}}{\lambda}\,H^2 \ge 3, \end{aligned}$$

deduced from the inequalities (19), (14), (4), (5) and $H \ge 2$. Now, using (24), (25) and the above lower bound, we first deduce from (27) that

$$\begin{aligned} \max\bigg\{\frac{LS|\Lambda|}{2b_2}, \frac{LR|\Lambda|}{2b_1}\bigg\} &\le \frac{1.86\cdot 4}{2}\,C(h+\lambda/\sigma)^2 a_1 a_2 e^{-C(h+\lambda/\sigma)^2 a_1 a_2 - 2.6} \\ &\le 12 e^{-5.6}, \end{aligned}$$

since the function $x \mapsto xe^{-x}$ is decreasing for $x > 1$. Applying again (27), it follows that

$$\begin{aligned} \max\bigg\{\frac{LSe^{LS|\Lambda|/(2b_2)}}{2b_2}, \frac{LRe^{LR|\Lambda|/(2b_1)}}{2b_1}\bigg\} &\le 0.53\max\{LS, LR\} \\ &\le C'(h+\lambda/\sigma)^2 a_1 a_2, \end{aligned}$$

so that (26) is established.

Combination of (12) and (26) then gives the required lower bound

$$\log|\Lambda| \ge -C(h+\lambda/\sigma)^2 a_1a_2 - \sqrt{\omega\theta}\,(h+\lambda/\sigma) - \log(C'(h+\lambda/\sigma)^2 a_1a_2),$$

if we assume that (13) is satisfied.

3.5. *Liouville inequality.* It remains to deal with the case $b'' \le 2\mu\lambda\sigma^{-1}kL^2 \cdot \gcd(b_1,b_2)$. Alternatively, we can write this inequality in the form

$$\frac{b_1^*}{a_2} + \frac{b_2^*}{a_1} \le 2\mu\lambda\sigma^{-1}kL^2.$$

Recall the lower bound $\sqrt{\omega\theta}\,(h+\lambda/\sigma) \ge D\log 2$ and the estimate (19). Applying the Liouville inequality in the form of [9, Exercise 3.7.b, p. 109] gives

$$\begin{aligned}\log|\Lambda| &\ge \log|b_2^*\log\alpha_2 - b_1^*\log\alpha_1| \ge -b_1^*D\mathrm{h}(\alpha_1) - b_2^*D\mathrm{h}(\alpha_2) - D\log 2\\ &\ge -\frac{1}{2}\left(\frac{b_1^*}{a_2}+\frac{b_2^*}{a_1}\right)a_1a_2 - D\log 2 \ge -\mu\lambda\sigma^{-1}kL^2a_1a_2 - D\log 2\\ &\ge -C(h+\lambda/\sigma)^2a_1a_2 - \sqrt{\omega\theta}\,(h+\lambda/\sigma).\end{aligned}$$

Then the required lower bound

$$\log|\Lambda| \ge -C(h+\lambda/\sigma)^2a_1a_2 - \sqrt{\omega\theta}\,(h+\lambda/\sigma) - \log(C'(h+\lambda/\sigma)^2a_1a_2)$$

obviously follows from (23). This ends the proof of Theorem 2.

4. The corollaries. The recipe for applying Theorem 2 is simple. Observe that for fixed ϱ and μ, the coefficients C and C' are decreasing functions of the parameters h, a_1, a_2, since ω and θ are decreasing functions of H, hence of h. Consequently, if h, a_1 and a_2 are bounded from below, then C and C' will be bounded from above.

We may extend the preceding observation in the following way. Rewrite the lower bound provided by Theorem 2 in the form

$$\log|\Lambda| \ge -C''h^2a_1a_2,$$

where

$$C'' = \left(1+\frac{\lambda}{h\sigma}\right)^2\left(C + \frac{\sqrt{\omega\theta}}{(h+\lambda/\sigma)a_1a_2} + \frac{\log(C'(h+\lambda/\sigma)^2a_1a_2)}{(h+\lambda/\sigma)^2a_1a_2}\right). \tag{28}$$

We now show that C'' is a decreasing function of h, a_1, a_2, for any values of μ and ϱ. It suffices to verify that the term

$$T := \frac{\log(C'(h+\lambda/\sigma)^2a_1a_2)}{(h+\lambda/\sigma)^2a_1a_2}$$

is itself decreasing, since the other two terms C and $\sqrt{\omega\theta}\,(h+\lambda/\sigma)^{-1}(a_1a_2)^{-1}$ are clearly decreasing, as is the factor $(1+\lambda/(h\sigma))^2$. For that purpose, we use (21) to write

$$C'(h+\lambda/\sigma)^2 a_1a_2 = \frac{\omega^{3/2}\theta^{1/2}H^2a_1a_2}{6\lambda} + \frac{1}{2}\sqrt{\frac{\omega^3\theta H^4a_1^2a_2^2}{9\lambda^2} + \frac{8\omega^{9/4}\theta^{5/4}H^{7/2}a_1^{3/2}a_2^{3/2}}{3\lambda} + \frac{4}{3}\,a_1a_2(a_1+a_2)\,\frac{\omega^2\theta H^3}{\lambda}}.$$

This formula shows that $C'(h+\lambda/\sigma)^2a_1a_2$ is an increasing function of h, a_1, a_2, since ωH and θH are increasing functions of H. Note that the function $x \mapsto x/\log x$ decreases for $x > e$ and that $C'(h+\lambda/\sigma)^2a_1a_2 > e^2$, by (23). It follows that the composed function

$$\frac{\log(C'(h+\lambda/\sigma)^2a_1a_2)}{C'(h+\lambda/\sigma)^2a_1a_2}$$

is a decreasing function of h, a_1, a_2 and that it takes positive values. Multiplying the above ratio by the decreasing function C', we obtain T, which is therefore a decreasing function as announced.

We are now ready to prove Corollaries 1 and 2. Recall the notations used in those corollaries. For each $m \in \{10,\dots,30\}$, choose μ and ϱ according to the following table:

Table 2. Parameters for Corollary 1

m	10	12	14	16	18	20	22	24	26	28	30
μ	0.54	0.54	0.55	0.56	0.56	0.56	0.57	0.57	0.57	0.57	0.58
ϱ	5.9	6.0	6.1	6.2	6.3	6.3	6.4	6.4	6.4	6.5	6.5

Fix $m \in \{10,\dots,30\}$. To deduce Corollary 1 from Theorem 2, we make use of the parameters μ and ϱ given by Table 2, together with

$$h = \max\{D(\log b' + 0.21), m, D\},$$
$$a_1 = (\varrho+2)D\log A_1, \qquad a_2 = (\varrho+2)D\log A_2.$$

It follows that

$$h \ge m, \qquad a_1 \ge \varrho + 2, \qquad a_2 \ge \varrho + 2. \tag{29}$$

A numerical computation shows that

$$\begin{aligned} D(\log(\lambda b'') + 1.75) + 0.06 &\le D(\log b' - \log(\varrho+2) + \log\lambda + 1.81) \\ &\le D(\log b' + 0.21) \end{aligned}$$

for any pair (μ, ϱ) provided by Table 2. Condition (3) is therefore satisfied. Recall that $|\alpha_1|, |\alpha_2| \ge 1$. Then the trivial upper bounds

$$\begin{aligned} \varrho|\log\alpha_i| - \log|\alpha_i| + 2D\mathrm{h}(\alpha_i) &\le \varrho|\log\alpha_i| + 2D\mathrm{h}(\alpha_i) \\ &\le (\varrho+2)D\log A_i \quad (i=1,2) \end{aligned} \tag{30}$$

show that the parameters a_1 and a_2 satisfy condition (4). Finally, condition (5) follows from the obvious inequalities

$$a_1 a_2 \geq (\varrho+2)^2 \geq (\log \varrho)^2 \geq (\log \varrho)^2 \sigma^2 = \lambda^2,$$

since $0 < \sigma \leq 1$. Thus, Theorem 2 gives the lower bound

$$\begin{aligned}\log |\Lambda| &\geq -C'' h^2 a_1 a_2 \\ &= -C''(\varrho+2)^2 D^4 (\max\{\log b' + 0.21, m/D, 1\})^2 \log A_1 \log A_2.\end{aligned}$$

Now recall the lower bounds (29). Since C'' is a decreasing function of h, a_1, a_2, it follows that $C''(\varrho+2)^2 \leq C_1$, where $C_1/(\varrho+2)^2$ is the constant obtained on substituting the values $h = m, a_1 = \varrho+2, a_2 = \varrho+2$ into the expression (28) giving C''. A numerical computation then gives rise to the constants $C_1(m)$ listed in Table 1. We thus obtain the desired estimate

$$\log |\Lambda| \geq -C_1 D^4 (\max\{\log b' + 0.21, m/D, 1\})^2 \log A_1 \log A_2.$$

Of course, the values (μ, ϱ) given by Table 2 have been determined in order that the constants $C_1(m)$ should be minimal. The computations were performed using Mathematica.

As for the real case, the proof is similar. We apply Theorem 2 with

$$\begin{gathered} h = \max\{D(\log b' + 0.38), m, D\}, \\ a_1 = (\varrho+1) D \log A_1, \qquad a_2 = (\varrho+1) D \log A_2, \end{gathered}$$

and with μ and ϱ given by the following table:

Table 3. Parameters for Corollary 2

m	10	12	14	16	18	20	22	24	26	28	30
μ	0.52	0.53	0.54	0.55	0.55	0.56	0.56	0.56	0.57	0.57	0.57
ϱ	5.0	5.1	5.2	5.2	5.3	5.3	5.4	5.4	5.4	5.4	5.5

Since $\log \alpha_1$ and $\log \alpha_2$ are positive real numbers, we can replace (30) by the sharper estimate

$$\varrho |\log \alpha_i| - \log |\alpha_i| + 2D\mathrm{h}(\alpha_i) = (\varrho-1)\log \alpha_i + 2D\mathrm{h}(\alpha_i) \leq (\varrho+1) D \log A_i = a_i$$

for $i = 1, 2$. We now use the lower bounds

$$h \geq m, \qquad a_1 \geq \varrho+1, \qquad a_2 \geq \varrho+1.$$

Then the preceding arguments give rise to the constants $C_2(m)$ listed in Table 1.

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