ACTA ARITHMETICA
144.4 (2010)

On the linear independence of the values of Gauss hypergeometric function

by

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1. Introduction. The aim of this paper is to investigate the linear independence of values of functions defined by the series

$$ {}_2F_1(1,B;C;\gamma_\nu) = \sum_{k=0}^{\infty} \frac{(B)_k}{(C)_k}\gamma_\nu^k, \quad B,C \in \mathbb{Q}^* \setminus \mathbb{Z}^-, \tag{1.1} $$

with $\gamma_\nu \in \mathbb{K}^*$, $1 \le \nu \le D$, where $\mathbb{K}$ is an algebraic number field.

The hypergeometric series ${}_2F_1(1,B;C;z)$ satisfies the first order differential equation

$$ z(1-z)\frac{d}{dz}y(z) + (C - Bz - 1)y(z) + 1 - C = 0 $$

with three regular singularities, $\{0,1,\infty\}$. It defines an analytic function in the unit disc but which, as is well known, may be continued along any path in $\mathbb{C} \setminus [1,\infty)$. The arithmetic properties of this series have been widely studied in particular interesting cases such as the binomial ([2], [3], [6], [8], [18]) and logarithmic function ([13], [14], [28], [29], [35]), as well as a part of a wider class of G-functions ([5], [9], [10], [12], [32], [33]; see also [4]). Naturally, the references above represent only a fraction of the works in the literature. Nevertheless, there are no linear independence results with explicit approximations for ${}_2F_1(1,B;C;z)$ in the generality of this paper. In [34] Vasilenko employed first type Padé approximations to obtain the linear independence over $\mathbb{Q}$ of values (1.1) at distinct points, all sufficiently close to the origin (with respect to normal absolute value). Ivankov proposed in [16] a combination of an effective construction of a linear approximating form and Siegel–Shidlovskiĭ's method to prove linear independence of more general hypergeometric E- and G-functions, comprising also (1.1), over an

2010 *Mathematics Subject Classification*: Primary 11J72; Secondary 11J61.
Key words and phrases: Gauss hypergeometric series, linear independence, algebraic number field, valuation, nonvanishing determinant.

DOI: 10.4064/aa144-4-3

imaginary (archimedean) quadratic field. For irrationality questions, general theorems are given in [15]. In addition, note also the article [30] by Sorokin, based on the works [23]–[27] of Nikishin. In that paper the question of linear independence of values of a hypergeometric function connected to a certain Markov function is studied with lower bounds that depend only on the moduli of the coefficients of related linear forms.

In the first part of this paper we compute explicit second type Padé approximations for the set of functions $\{{}_2F_1(1, B; C; \gamma_\nu z)\}$, $\nu = 1, \ldots, D$, in the spirit of Stihl [31], an approach originating from Maier [19] and thereafter generalized by Chudnovsky [7]. See also [22], where linear independence is considered for values of functions, and their derivatives, deriving from the Heine series, a q-analogue of the Gauss series.

In Section 4 we show that these approximations are linearly independent (Theorem 4.2), i.e. a certain determinant formed by the approximation polynomials does not vanish identically. This is the most important contribution of this work since the method of proof is new, and it applies also to the case of Heine series considered in [22]. The essential ingredient of the Padé construction, the application of the binomial (or q-binomial) theorem, is also the foundation of the determinant consideration. Hence the method may be applied, at least in some cases, to prove the independence of approximations for more general hypergeometric and q-hypergeometric series presented in [20].

The last part of this work consists of analytic and arithmetic estimates of the approximations, together with the proofs of the remaining theorems and corollaries.

2. Notations and results. Let $\mathbb{K}$ be an algebraic number field of degree κ over $\mathbb{Q}$. In particular, we write $\mathbb{K} = \mathbb{I}$ in the case of an imaginary quadratic field. For a finite place v of $\mathbb{K}$ over the prime p, we use the notation $v|p$, whereas we write $v|\infty$ for the infinite place. Let us normalize the absolute value $|*|_v$ of $\mathbb{K}$ so that if $v|p$, then $|p|_v = p^{-1}$, and if $v|\infty$, then $|x|_v = |x|$ for $x \in \mathbb{Q}$. Here, by $|*|$ we mean the usual absolute value in $\mathbb{Q}$. We shall adopt the notation

$$\|*\|_v = |*|_v^{\kappa_v/\kappa} \quad \text{with} \quad \kappa_v = [\mathbb{K}_v : \mathbb{Q}_v],$$

where $\mathbb{K}_v$ is the completion of $\mathbb{K}$ with respect to v. Now, for every $\alpha \in \mathbb{K}^*$ we have the product formula

$$\prod_{w \in \mathcal{M}} \|\alpha\|_w = 1,$$

where $\mathcal{M}_{\mathbb{K}} = \mathcal{M}$ is the set of places on $\mathbb{K}$. The height $H(\alpha)$ of $\alpha \in \mathbb{K}$ is defined by the equation

$$H(\alpha) = \prod_{w \in \mathcal{M}} \|\alpha\|_w^*, \quad \|\alpha\|_w^* = \max\{1, \|\alpha\|_w\},$$

and more generally for a vector $\overline{B} = (b_0, \ldots, b_D) \in \mathbb{K}^{D+1}$ by

$$H(\overline{B}) = \prod_{w \in \mathcal{M}} \|\overline{B}\|_w^*, \quad \|\overline{B}\|_w^* = \max\{1, \|\overline{B}\|_w\},$$

where

$$|\overline{B}|_w = \max_{0 \le i \le D} |b_i|_w, \quad \text{thus} \quad \|\overline{B}\|_w = \max_{0 \le i \le D} \|b_i\|_w.$$

Let $\xi_i \in \mathbb{K}_v$, $0 \le i \le D$, be linearly independent over the field $\mathbb{K}$. Then, by a *linear independence measure* of the numbers $\xi_0, \ldots, \xi_D$, we mean an exponent $\mu > 0$ in the lower bound

$$|b_0 \xi_0 + \cdots + b_D \xi_D|_v > H^{-\mu-\varepsilon} \quad \forall \varepsilon > 0,$$

for every $\overline{B} = (b_0, \ldots, b_D) \in \mathbb{K}^{D+1} \setminus \{\overline{0}\}$ and $H = \max\{H(\overline{B}), H_0\}$, where $H_0 = H_0(\varepsilon) \in \mathbb{Z}^+$ is an effectively computable constant. When $D = 1$, we set $\xi_1 = 1$, $b_0 = 1$ and call μ an *irrationality measure* of ξ_0, i.e.

$$|\xi_0 + b_1|_v > H^{-\mu-\varepsilon} \quad \forall \varepsilon > 0,$$

with $\overline{B} = (1, b_1) \in \mathbb{K}^2$ and $H = \max\{H(b_1), H_0\}$, $H_0 \in \mathbb{Z}^+$.

As is customary, we make use of the Pochhammer notation

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1)$$

for the generalized factorial, and more generally we denote

$$(a;b)_0 = 1, \quad (a;b)_n = a(a+b)\cdots(a+b(n-1))$$

for every $n \in \mathbb{Z}^+$.

Let $B, C \in \mathbb{Q}^*$, say $B = a/b$, $C = c/d$, $b/d = f/g$, $a, b, c, d, f, g \in \mathbb{Z}$, $b, d \ge 1$, $\gcd(a, b) = \gcd(c, d) = \gcd(f, g) = 1$. In addition, for $v|p$, let $\operatorname{ord}_p(d) \ge \operatorname{ord}_p(b)$ and $\delta_v = 0$, while $\delta_v = 1$ for $v|\infty$. We shall also write

$$E_m(a, b) = \operatorname{lcm}\{a, a+b, \ldots, a+b(m-1)\}, \quad \omega_n(d) = \prod_{p|d} p^{\operatorname{ord}_p(n!)} \tag{2.1}$$

for $m, n \in \mathbb{Z}^+$ and define

$$\mu_d = \prod_{p|d} p^{1/(p-1)}, \quad \lambda(b) = \frac{b}{\phi(b)} \sum_{\substack{i=1 \\ \gcd(i,b)=1}}^{b} \frac{1}{i}, \quad b, d \in \mathbb{Z}^+,$$

with ϕ the Euler totient function. We can now state the following theorem.

THEOREM 2.1. *Let $\gamma_\nu \in \mathbb{K}^*$, $1 \le \nu \le D$, be distinct algebraic numbers and $|\overline{\Gamma}|_v < 1$ where $\overline{\Gamma} = (\gamma_1, \ldots, \gamma_D)$. If*

$$\log |\overline{\Gamma}|_v^{-1} > \frac{(\kappa - \kappa_v \delta_v) \log \mathcal{C} + \kappa(D+\eta) \log H(\overline{\Gamma}) + \kappa_v \delta_v \log \mathcal{C}_r}{\kappa_v (D+\eta+1)} \tag{2.2}$$

with $\eta \in \mathbb{R}^+ \cup \{0\}$ *and*

$$\mathcal{C} = 5^D 4^{\eta+1} d\mu_d (g\mu_g e^{\lambda(b)})^{D+\eta}, \quad \mathcal{C}_r = 3^D 2^{\eta+1} d\mu_d (g\mu_g e^{\lambda(b)})^{D+\eta},$$

then the set of $D+1$ *numbers* $\{1, {}_2F_1(1,B;C;\gamma_1), \dots, {}_2F_1(1,B;C;\gamma_D)\}$ *is linearly independent over* $\mathbb{K}$ *with a linear independence measure*

$$\mu \le \frac{\kappa(D+\eta+1)\log|\overline{\Gamma}|_v - \kappa\delta_v \log((5/3)^D 2^{\eta+1})}{(\kappa-\kappa_v\delta_v)\log\mathcal{C} + \kappa(D+\eta)\log H(\overline{\Gamma}) + \kappa_v(D+\eta+1)\log|\overline{\Gamma}|_v + \kappa_v\delta_v \log\mathcal{C}_r}.$$

Corollary 2.2. *Let* $\gamma_\nu \in \mathbb{I}^*$ *be such that* (2.2) *holds. Then, for* $\overline{B} \in \mathbb{Z}_{\mathbb{I}}^{D+1} \setminus \{\overline{0}\}$,

$$\mu \le -\frac{(D+\eta)\log H(\overline{\Gamma}) + \log\mathcal{C}}{\log\mathcal{C}_r + (D+\eta)\log H(\overline{\Gamma}) + (D+\eta+1)\log|\overline{\Gamma}|}.$$

In particular, let $\gamma_\nu = \xi_\nu/q$ *with* $\xi_\nu \in \mathbb{I}^*$ *and* $q \in \mathbb{Z}_{\mathbb{I}} \setminus \{0\}$. *Choose* $\eta = 0$ *and let*

$$|q| > 2d\mu_d|\overline{\Xi}|(3g\mu_g e^{\lambda(b)}|\overline{\Xi}|_* H(\overline{\Xi}))^D, \tag{2.3}$$

where $\overline{\Xi} = (\xi_1, \dots, \xi_D)$ *and* $|\overline{\Xi}|_* = \min\{1, |\overline{\Xi}|\}$. *Then the numbers in the set*

$$\{1, {}_2F_1(1,B;C;\xi_1/q), \dots, {}_2F_1(1,B;C;\xi_D/q)\}$$

are linearly independent over $\mathbb{Z}_{\mathbb{I}}$ *and*

$$\mu \le D + \frac{\log(\mathcal{C}\mathcal{C}_r^D(H(\overline{\Xi})|\overline{\Xi}|_*)^{D(D+1)})}{\log|q| - \log(\mathcal{C}_r|\overline{\Xi}|_*^D H(\overline{\Xi})^D|\overline{\Xi}|)} = D + \varepsilon(|q|).$$

Further, if $\xi_\nu \in \mathbb{Z}_{\mathbb{I}}$, *then condition* (2.3) *becomes*

$$|q| > \mathcal{C}_r|\overline{\Xi}|^{D+1},$$

and in the case of the logarithmic function ($B=1$, $C=2$) *over* $\mathbb{K} = \mathbb{Q}$,

$$s > 3^D 2^{\eta+1} e^{D+\eta}|r|^{D+\eta+1}, \quad \eta \ge 0, \tag{2.4}$$

with $\gamma_\nu = r_\nu/s$, $s > \max|r_\nu| = |r| \ge 1$. *When* (2.4) *holds, then the numbers*

$$1, \log(1-r_1/s), \dots, \log(1-r_D/s)$$

are linearly independent over $\mathbb{Z}$ *with*

$$\mu \le \frac{\log(5^D 4^{\eta+1} e^{D+\eta}) + (D+\eta)\log s}{\log s - \log(3^D 2^{\eta+1} e^{D+\eta}) - (D+\eta+1)\log|r|}.$$

Corollary 2.3. *Consider* $v|p$ *and let* $\gamma_\nu = r_\nu/s \in \mathbb{Q}^*$, $\gcd(r_\nu, s) = 1$, $\max|r_\nu| = |r|$, $|\overline{\Gamma}|_p < 1$, $\overline{B} = (b_0, \dots, b_D) \in \mathbb{Z}^{D+1} \setminus \{\overline{0}\}$. *If*

$$\log|\overline{\Gamma}|_p^{-1} > \frac{\log(5^D 4^{\eta+1} d\mu_d (g\mu_g e^{\lambda(b)})^{D+\eta}) + (D+\eta)\log\max\{s, |r|\}}{D+\eta+1},$$

then the set $\{1, {}_2F_1(1,B;C;r_1/s),\dots,{}_2F_1(1,B;C;r_D/s)\}$ *of* p*-adic numbers is linearly independent over* $\mathbb{Z}$ *with*

$$\mu \le \frac{(D+\eta+1)\log|\overline{\Gamma}|_p}{\log(5^D 4^{\eta+1} d\mu_d(g\mu_g e^{\lambda(b)}\max\{s,|r|\})^{D+\eta})+(D+\eta+1)\log|\overline{\Gamma}|_p}.$$

An interesting choice of parameters is $B=-1/q$, $q>0$, $C=1$, $\gamma_\nu=p^m\cdot r_\nu$, $r_\nu\in\mathbb{Z}\setminus\{0\}$, $m\ge 1$, $p\nmid r_\nu$, $|\overline{\Gamma}|_p=p^{-m}$. *Now, if*

$$\log p > \frac{\log(5^D 4e^{\lambda(q)D})+D\log|r|}{m} \qquad (\eta=0),$$

then

$$|b_0+b_1(1-p^m r_1)^{1/q}+\dots+b_D(1-p^m r_D)^{1/q}|_p \ge H^{-\mu-\varepsilon}$$

with

$$\begin{aligned}\mu &\le \frac{(D+1)m\log p}{m\log p-\log(5^D 4e^{\lambda(q)D})-D\log|r|}\\ &= D+1+\frac{(D+1)\log(5^D 4e^{\lambda(q)D}|r|^D)}{m\log p-\log(5^D 4e^{\lambda(q)D}|r|^D)} = D+1+\varepsilon_p(m)\end{aligned}$$

for $H=\max\{|b_0|,\dots,|b_D|\}\ge H_0$.

Remark 2.4. In [16] Ivankov employed effective Padé approximations and Siegel's method to obtain good lower bounds for linear forms in the values of hypergeometric functions. For ${}_2F_1(1,B;C;z)$ the general theorem for hypergeometric G-functions implies the linear independence over $\mathbb{Z}_{\mathbb{I}}$ of the values

$$1, {}_2F_1(1,B;C;\xi_1/q),\dots,{}_2F_1(1,B;C;\xi_D/q)$$

when $\xi_\nu\in\mathbb{I}^*$, $|q|>e^{C_1 D(D+1)\Omega}$, $C_1>0$, $q\in\mathbb{Z}_{\mathbb{I}}$ and

$$\Omega\ge\max\{|A+1|,|B+1|,|\xi_n|,1/|\xi_n|,1/|\xi_n-\xi_m|\},\quad n\ne m,\quad \Omega\ge 3.$$

However, note that the dependence of the lower bound of $|q|$ on the number of points in the linear form is of magnitude $e^{\mathcal{O}(D^2)}$, whereas in our Corollary 2.2 it is $e^{\mathcal{O}(D)}$ by the inequality (2.3).

Galochkin showed in [12], as a special case of a more general theorem for a class of G-functions, that the set of numbers

$$\left\{1,\ln\left(1-\frac{1}{q}\right),\ln\left(1+\frac{1}{q}\right)\right\} \tag{2.5}$$

is linearly independent over $\mathbb{Q}$ for any natural number $q>q_0$ where $q_0\ge 25e^{56/5}>e^{14}$ (at least). Corollary 2.2, on the other hand, implies the linear independence of the numbers in (2.5) when $b>18e^2$ $(<e^5)$. Note, however, that in [14] Hata shows, in particular, that $q\ge 54$ suffices for the linear independence of the values in (2.5).

3. Second type Padé approximations. In the following we shall compute second type Padé approximations for the functions

$$g_\nu(z) = \sum_{k=0}^{\infty} \frac{(B)_k}{(C)_k} (\gamma_\nu z)^k, \quad \nu = 1, \dots, D. \tag{3.1}$$

LEMMA 3.1. *Let $\sigma_{i,l}$, $i = 0, \dots, Dl$, be defined by the equation*

$$\prod_{t=1}^{D} (\gamma_t - w)^l = \sum_{i=0}^{Dl} \sigma_{i,l} w^i. \tag{3.2}$$

Then the coefficient $\sigma_{i,l}$ may be expressed as a sum

$$\sigma_{i,l} = (-1)^i \sum_{i_1+\dots+i_D=i} \binom{l}{i_1} \cdots \binom{l}{i_D} \gamma_1^{l-i_1} \cdots \gamma_D^{l-i_D} \tag{3.3}$$

for $i = 0, 1, \dots, Dl$.

Proof. Use the binomial identity

$$(\gamma_\nu - w)^l = \sum_{i=0}^{l} (-1)^i \binom{l}{i} \gamma_\nu^{l-i} w^i. \ \blacksquare$$

THEOREM 3.2. *For all $l \in \mathbb{Z}^+$, $\lambda \in \mathbb{N}$, and $1 \le \nu \le D$,*

$$A_{l,\lambda}(z) g_\nu(z) - B_{l,\lambda}^{(\nu)}(z) = R_{l,\lambda}^{(\nu)}(z),$$

where

$$A_{l,\lambda}(z) = \sum_{i=0}^{Dl} \sigma_{i,l} \frac{(C)_{i+\rho-Dl}}{(B)_{i+\lambda}} z^{Dl-i}, \tag{3.4}$$

$$B_{l,\lambda}^{(\nu)}(z) = \sum_{n=0}^{Dl+\lambda-1} g_{l,\lambda,n}^{(\nu)} z^n, \tag{3.5}$$

$$R_{l,\lambda}^{(\nu)}(z) = \sum_{n=0}^{\infty} g_{l,\lambda,n}^{(\nu)} z^n \tag{3.6}$$

with $\operatorname{ord}_{z=0} R_{l,\lambda}^{(\nu)}(z) \ge (D+1)l + \lambda = \rho + 1$ *and*

$$g_{l,\lambda,n}^{(\nu)} = \sum_{i=0}^{\min\{n,Dl\}} \sigma_{Dl-i,l} \frac{(C)_{\rho-i}(B)_{n-i}}{(B)_{Dl-i+\lambda}(C)_{n-i}} \gamma_\nu^{n-i}.$$

Proof. We set the common approximation (denominator) polynomial

$$A_{l,\lambda}(z) = \sum_{i=0}^{Dl} \sigma_{i,l} \frac{(C)_{i+\rho-Dl}}{(B)_{i+\lambda}} z^{Dl-i}$$

with $\rho = l + Dl + \lambda - 1$. By the Cauchy product, the coefficients of

$$A_{l,\lambda}(z)g_\nu(z) = \sum_{n=0}^{\infty} g_{l,\lambda,n}^{(\nu)} z^n$$

are determined as

$$g_{l,\lambda,n}^{(\nu)} = \sum_{i=0}^{\min\{n,Dl\}} \sigma_{Dl-i,l} \frac{(C)_{\rho-i}(B)_{n-i}}{(B)_{Dl-i+\lambda}(C)_{n-i}} \gamma_\nu^{n-i}.$$

For the values $\lambda + Dl \le n \le \rho$, we compute

$$\begin{aligned} g_{l,\lambda,n}^{(\nu)} &= \sum_{i=0}^{Dl} \sigma_{i,l}(C+n-Dl+i)_{\rho-n}(B+\lambda+i)_{n-Dl-\lambda}\gamma_\nu^{n-Dl+i} \\ &= \gamma_\nu^{n-Dl} \sum_{j=0}^{l-1} p_j \sum_{i=0}^{Dl} \sigma_{i,l}\gamma_\nu^i i^j, \end{aligned}$$

where p_j is defined by the equation

$$(C+n-Dl+i)_{\rho-n}(B+\lambda+i)_{n-Dl-\lambda} = \sum_{j=0}^{l-1} p_j i^j.$$

By applying the operator ϑ_w^j, where $\vartheta_w = w(d/dw)$, to both sides of (3.2), we obtain

$$\sum_{i=0}^{Dl} \sigma_{i,l}\gamma_\nu^i i^j = \vartheta_w^j \prod_{t=1}^{D} (\gamma_t - w)^l{}_{|w=\gamma_\nu} = 0 \quad \forall \begin{cases} \nu = 1, \ldots, D, \\ j = 0, \ldots, l-1. \end{cases} \tag{3.7}$$

Therefore,

$$g_{l,\lambda,n}^{(\nu)} = 0 \quad \forall n = \lambda + Dl, \ldots, \rho,$$

and we set

$$\begin{aligned} B_{l,\lambda}^{(\nu)}(z) &:= \sum_{n=0}^{\lambda+Dl-1} g_{l,\lambda,n}^{(\nu)} z^n \\ &= \sum_{n=0}^{\lambda+Dl-1} \sum_{i=0}^{\min\{n,Dl\}} \sigma_{Dl-i,l} \frac{(C+n-i)_{\rho-n}}{(B+n-i)_{Dl+\lambda-n}} \gamma_\nu^{n-i} z^n, \\ R_{l,\lambda}^{(\nu)}(z) &:= \sum_{n=\rho+1}^{\infty} g_{l,\lambda,n}^{(\nu)} z^n = z^{\rho+1} \sum_{k=0}^{\infty} g_{l,\lambda,\rho+k+1}^{(\nu)} z^k, \end{aligned}$$

in order to get

$$A_{l,\lambda}(z)g_\nu(z) - B_{l,\lambda}^{(\nu)}(z) = R_{l,\lambda}^{(\nu)}(z), \quad \nu = 1, \ldots, D. \ \blacksquare \tag{3.8}$$

4. The nonvanishing determinant. In this section we provide the proof for the independence of the approximations computed in Section 3. The core of the proof is Theorem 4.2 concerning the order of the determinant as a polynomial in z connected to the systems of approximations. This promptly yields the validity of Theorem 4.3, i.e. the nonvanishing of the determinant. First, let us introduce a useful lemma the proof of which goes back to [21] (see also [11] and [17] for further discussions on generalized Vandermonde determinants).

LEMMA 4.1. *Let n be a nonnegative integer, and let $B_m(x)$ denote an $n \times m$ matrix formed by applying $x(d/dx)$ subsequently on its columns:*

$$B_m(x) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ x & x & x & \dots & x \\ x^2 & 2x^2 & 4x^2 & \dots & 2^{m-1}x^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x^i & ix^i & i^2x^i & \dots & i^{m-1}x^i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x^{n-1} & (n-1)x^{n-1} & (n-1)^2x^{n-1} & \dots & (n-1)^{m-1}x^{n-1} \end{bmatrix}.$$

If $n = m_1 + \dots + m_d$ is a partition of n, then

$$\det[B_{m_1}(\alpha_1) \dots B_{m_d}(\alpha_d)] = \prod_{j=1}^{d} \alpha_j^{\binom{m_j}{2}} \prod_{i=1}^{m_j-1} i! \prod_{1 \le j < i \le d} (\alpha_i - \alpha_j)^{m_i m_j}.$$

THEOREM 4.2. *If $\gamma_n \neq \gamma_m \neq 0$ for every $n \neq m$, $1 \le n, m \le D$ and $C - B \neq 0, -1, -2, \dots,$ then*

$$\det[g^{(\nu)}_{l,\lambda+j-1,\rho+j}]_{j\nu} \neq 0, \quad 1 \le j \le D, 1 \le \nu \le D,$$

where $g^{(\nu)}_{l,\lambda+j-1,\rho+j}$ is the coefficient of the lowest term in $R^{(\nu)}_{l,\lambda+j-1}(z)$.

Proof. Firstly, let us compute

$$\begin{aligned} g^{(\nu)}_{l,\lambda,\rho+1} &= \gamma_\nu^{l+\lambda} \sum_{i=0}^{Dl} \sigma_{i,l} \frac{(B+\lambda+i)_l}{C+\lambda+i+l-1} \gamma_\nu^i \\ &= \gamma_\nu^{l+\lambda} \sum_{i=0}^{Dl} \sigma_{i,l} \frac{(B+\lambda+i)_{l-1}(C+\lambda+i+l-1+B-C)}{C+\lambda+i+l-1} \gamma_\nu^i \\ &= \gamma_\nu^{l+\lambda} \sum_{i=0}^{Dl} \sigma_{i,l} (B+\lambda+i)_{l-1} \left(1 + \frac{B-C}{C+\lambda+i+l-1}\right) \gamma_\nu^i \end{aligned}$$

$$
\begin{aligned}
&= \gamma_\nu^{l+\lambda} \sum_{i=0}^{Dl} \sigma_{i,l} (B+\lambda+i)_{l-1} \gamma_\nu^i + (B-C) \sum_{i=0}^{Dl} \sigma_{i,l} \frac{(B+\lambda+i)_{l-1}}{C+\lambda+i+l-1} \gamma_\nu^{i+l+\lambda} \\
&= (B-C)\gamma_\nu^{l+\lambda} \sum_{i=0}^{Dl} \sigma_{i,l} \frac{(B+\lambda+i)_{l-1}}{C+\lambda+i+l-1} \gamma_\nu^i = \cdots \\
&= (-1)^l (C-B)_l \gamma_\nu^{l+\lambda} \sum_{i=0}^{Dl} \frac{\sigma_{i,l}}{C+\lambda+i+l-1} \gamma_\nu^i,
\end{aligned}
$$

by applying (3.7) repeatedly. Let us introduce an auxiliary variable x, denote

$$
G^{(\nu)}(\lambda, x) = (x\gamma_\nu)^\lambda \sum_{i=0}^{Dl} \sigma_{i,l} \frac{(\gamma_\nu x)^i}{C+\lambda+i+l-1},
$$

and form the matrix

$$
\Omega(x) = \begin{bmatrix}
G^{(1)}(\lambda, x) & \dots & G^{(D)}(\lambda, x) \\
\vdots & \ddots & \vdots \\
G^{(1)}(\lambda+j, x) & \dots & G^{(D)}(\lambda+j, x) \\
\vdots & \ddots & \vdots \\
G^{(1)}(\lambda+D-1, x) & \dots & G^{(D)}(\lambda+D-1, x)
\end{bmatrix}.
$$

Thus,

$$
\det[g^{(\nu)}_{l,\lambda+j-1,\rho+j}]_{j\nu} = (-1)^{Dl}(C-B)_l^D (\gamma_1 \cdots \gamma_D)^l \det \Omega(1). \tag{4.1}
$$

As a polynomial in x, we have

$$
\begin{aligned}
\det \Omega(x) &= (\gamma_1 \cdots \gamma_D)^\lambda x^{\binom{D}{2}+D\lambda} \begin{vmatrix}
F^{(1)}(0, x) & \dots & F^{(D)}(0, x) \\
\vdots & \ddots & \vdots \\
F^{(1)}(D-1, x) & \dots & F^{(D)}(D-1, x)
\end{vmatrix} \\
&= (\gamma_1 \cdots \gamma_D)^\lambda x^{\binom{D}{2}+D\lambda} \det \widetilde{\Omega}(x)
\end{aligned}
$$

with

$$
F^{(\nu)}(j, x) = \gamma_\nu^j \sum_{i=0}^{Dl} \sigma_{i,l} \frac{(\gamma_\nu x)^i}{C+\lambda+i+j+l-1}, \quad 0 \le j \le D-1.
$$

If $\gamma_n \neq \gamma_m$, $n \neq m$, then $\det \widetilde{\Omega}(0)$ is a nonzero Vandermonde determinant times $(\gamma_1 \cdots \gamma_D)^{Dl}/(C+\lambda+l-1)_D$, and hence

$$
\operatorname{ord}_{x=0} \det \Omega(x) = \binom{D}{2} + D\lambda.
$$

In particular, $\det \Omega(x) \neq 0$ (as a polynomial of x). Consider next the following $D \times D(l+1)$ and $Dl \times D(l+1)$ matrices:

$$A := \begin{bmatrix} \frac{\sigma_{0,l}}{C+\lambda+l-1} & \frac{\sigma_{1,l}}{C+\lambda+l} & \cdots & \frac{\sigma_{Dl,l}}{C+\lambda+Dl+l-1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\sigma_{0,l}}{C+\lambda+l} & \cdots & \frac{\sigma_{Dl-1,l}}{C+\lambda+Dl+l-1} & \frac{\sigma_{Dl,l}}{C+\lambda+Dl+l} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{\sigma_{0,l}}{C+\lambda+D+l-2} & \cdots & \cdots & & \frac{\sigma_{Dl,l}}{C+\lambda+Dl+D+l-2} \end{bmatrix}$$

and $B = [B_1 \dots B_k \dots B_l]^T$ (a column vector with matrix entries), where

$$B_k = [b_{ij}]_{D\times D(l+1)}, \quad b_{ij} = \begin{cases} 0, & 1 \le j \le i-1, \\ \sigma_{j-i,k}, & i \le j \le i+Dk, \\ 0, & i+Dk < j \le D(l+1). \end{cases}$$

Set $\Delta = [A\ B]^T$. Then

$$\Delta V(x) = \begin{bmatrix} A \\ B \end{bmatrix} V(x) = \begin{bmatrix} \widehat{\Omega}(x) & O \\ X(x) & E \end{bmatrix}, \tag{4.2}$$

where by (3.7), O is a $D \times Dl$ zero matrix and $X(x)$ a $Dl \times D$ matrix such that $X(1)$ is zero, while $\det \widehat{\Omega}(x) = x^{\binom{D}{2}} \det \widetilde{\Omega}(x)$. Furthermore,

$$V(x) = [V_0(x) \quad V_1 \quad \dots \quad V_D],$$

where

$$V_0(x) = \begin{bmatrix} 1 & \cdots & 1 \\ \gamma_1 x & \cdots & \gamma_D x \\ \vdots & \ddots & \vdots \\ (\gamma_1 x)^{Dl+D-1} & \cdots & (\gamma_D x)^{Dl+D-1} \end{bmatrix}$$

and

$$V_\nu = [(C+\lambda+i+l-1) i^j \gamma_\nu^i]_{ij}, \quad 0 \le i \le Dl+D-1,\ 0 \le j \le l-1,$$

is a $(Dl+D) \times l$ matrix. The entries of V are polynomials in i and after elementary column operations (repeated cancellations of lower terms of the polynomials, i.e. the terms $(C+\lambda+l-1)i^j\gamma_\nu^i$ in V_ν), Lemma 4.1 yields

$$\det V(1) = \prod_{t=1}^{D} \gamma_t^{\binom{l+1}{2}} \prod_{j=1}^{l} j! \prod_{1\le n<m\le D} (\gamma_m - \gamma_n)^{(l+1)^2} \neq 0$$

when $\gamma_t \neq 0$ and $\gamma_m \neq \gamma_n$. Since $\operatorname{rank} B = Dl$, it follows that $\operatorname{rank} E = Dl$ (E is independent of x), and thus by (4.2),

$$\det \Delta \det V(x) = \det E \det \widehat{\Omega}(x), \quad \det E \neq 0.$$

This implies that $\det \Delta = 0$ if and only if $\det \widehat{\Omega}(1) = 0$. However, $\det \widehat{\Omega}(x) \neq 0$ (as a polynomial in x), which yields $\det \Delta \neq 0$ (does not depend on x).

Furthermore,

$$\det \Omega(x) = \det \Delta \det E^{-1} x^{D\lambda} \det V(x) \prod_{t=1}^{D} \gamma_t^{\lambda},$$

and in particular $\det \Omega(1) \neq 0$, which together with (4.1) proves the theorem. ∎

THEOREM 4.3. *If $\gamma_n \neq \gamma_m \neq 0$ for every $n \neq m$, $1 \leq n, m \leq D$ and $C - B \neq 0, -1, -2, \ldots$, then*

(4.3)

$$\Delta_{l,\lambda}(z) = \begin{vmatrix} -A_{l,\lambda}(z) & -A_{l,\lambda+1}(z) & \ldots & -A_{l,\lambda+D}(z) \\ B_{l,\lambda}^{(1)}(z) & B_{l,\lambda+1}^{(1)}(z) & \ldots & B_{l,\lambda+D}^{(1)}(z) \\ \vdots & \vdots & \ddots & \vdots \\ B_{l,\lambda}^{(\nu)}(z) & B_{l,\lambda+1}^{(\nu)}(z) & \ldots & B_{l,\lambda+D}^{(\nu)}(z) \\ \vdots & \vdots & \ddots & \vdots \\ B_{l,\lambda}^{(D)}(z) & B_{l,\lambda+1}^{(D)}(z) & \ldots & B_{l,\lambda+D}^{(D)}(z) \end{vmatrix}_{(D+1)\times(D+1)} \neq 0.$$

In particular,

$$\mathrm{ord}_{z=0}\, \Delta_{l,\lambda} = \deg_z \Delta_{l,\lambda}.$$

Proof. By (3.8), we may transform the determinant (4.3) into

$$(4.4) \qquad (-1)^{D+1} \Delta_{l,\lambda}(z) = \begin{vmatrix} A_{l,\lambda}(z) & A_{l,\lambda+1}(z) & \ldots & A_{l,\lambda+D}(z) \\ R_{l,\lambda}^{(1)}(z) & R_{l,\lambda+1}^{(1)}(z) & \ldots & R_{l,\lambda+D}^{(1)}(z) \\ \vdots & \vdots & \ddots & \vdots \\ R_{l,\lambda}^{(\nu)}(z) & R_{l,\lambda+1}^{(\nu)}(z) & \ldots & R_{l,\lambda+D}^{(\nu)}(z) \\ \vdots & \vdots & \ddots & \vdots \\ R_{l,\lambda}^{(D)}(z) & R_{l,\lambda+1}^{(D)}(z) & \ldots & R_{l,\lambda+D}^{(D)}(z) \end{vmatrix}.$$

Because

$$\mathrm{ord}_{z=0}\, A_{l,\lambda+j}(z) = 0 \quad \text{and} \quad \mathrm{ord}_{z=0}\, R_{l,\lambda+j}^{(\nu)}(z) \geq \rho + j + 1, \quad j = 0, \ldots, D,$$

where $\rho = (D+1)l + \lambda - 1$, we notice that

$$\mathrm{ord}_{z=0}\, \Delta_{l,\lambda}(z) = \rho \cdot D + \binom{D+1}{2}$$

exactly when

$$(4.5) \qquad \det[g_{l,\lambda+j-1,\rho+j}^{(\nu)}]_{j\nu} = (-1)^{Dl} (C-B)_l^D \prod_{t=1}^{D} \gamma_t^l \det \Omega(1) \neq 0, \qquad 1 \leq \nu, j \leq D.$$

On the other hand, as

$$\deg_z A_{l,\lambda+j}(z) = Dl \quad \text{and} \quad \deg_z B^{(\nu)}_{l,\lambda+j} \le \lambda + DL + j - 1,$$

it follows from (4.3) that

$$\deg_z \Delta_{l,\lambda}(z) \le \rho \cdot D + \binom{D+1}{2}.$$

Thus, if (4.5) is different from zero, then $\operatorname{ord}_z \Delta_{l,\lambda} = \deg_z \Delta_{l,\lambda}$. ∎

5. Estimates for the approximation polynomials and for the remainder

Lemma 5.1. *Let $c, d \in \mathbb{Z} \setminus \{0\}$ with $\gcd(c,d) = 1$ and $p \in \mathbb{P}$ be such that $p \nmid d$. If $p^m \mid i!$, then $p^m \mid (c+dk;d)_i$ for $i, m \in \mathbb{Z}^+$.*

Proof. Since $p \nmid d$, there exists $d^{-1} \in \mathbb{Z}$ such that $dd^{-1} \equiv 1 \pmod{p^m}$. Thus,

$$\begin{aligned}(c+dk)(c+d(k+1))\cdots(c+d(k+i-1))&\\ &\equiv d^i(cd^{-1}+k)(cd^{-1}+k+1)\cdots(cd^{-1}+k+i-1)\\ &= d^i\binom{cd^{-1}+k+i-1}{i} i! \equiv 0 \pmod{p^m}\end{aligned}$$

by the hypothesis $p^m \mid i!$, and so $p^m \mid (c+dk;d)_i$. ∎

As a consequence of Lemma 5.1 we have

$$\prod_{p\mid d} p^{\operatorname{ord}_p(i!)} \frac{(c+dk;d)_i}{i!} \in \mathbb{Z}, \quad \text{as} \quad i! = \prod_{p\nmid d} p^{\operatorname{ord}_p(i!)} \prod_{p\mid d} p^{\operatorname{ord}_p(i!)}.$$

Note also that if $p \mid d$, then $p \nmid (c;d)_i$, since $\gcd(c,d) = 1$.

Let us denote the coefficients of the polynomial $A_{l,\lambda}(z)$ by $A_i = \sigma_{Dl-i,l} a_i$, and the coefficients of $g_\nu(z)$ by $f_k^{(\nu)} = f_k \gamma_\nu^k$, i.e.

$$a_i = \frac{(C)_{\rho-i}}{(B)_{Dl+\lambda-i}}, \quad f_k = \frac{(B)_k}{(C)_k}.$$

Then, for $0 \le i \le Dl$, $0 \le i+k \le Dl+\lambda-1$,

$$a_i f_k = \frac{(C)_{Dl+\lambda+l-1-i}(B)_k}{(B)_{Dl+\lambda-i}(C)_k} = \frac{(C+k)_{Dl+\lambda+l-i-1-k}}{(B+k)_{Dl+\lambda-i-k}} = \frac{(C+k)_{\rho-i-k}}{(B+k)_{Dl+\lambda-i-k}}.$$

The partial fraction decomposition

$$\frac{1}{(B+k)_{Dl+\lambda-i-k}} = \sum_{j=0}^{\rho-l-i-k} \frac{b\alpha_j}{a+b(k+j)}, \tag{5.1}$$

$$\alpha_j = \frac{(-1)^j}{(\rho-l-i-k)!}\binom{\rho-l-i-k}{j},$$

gives

$$\frac{(C+k)_{\rho-i-k}}{(B+k)_{Dl+\lambda-i-k}} = \frac{(C+k)_{\rho-i-k}}{(\rho-l-i-k)!}\sum_{j=0}^{\rho-l-i-k}\binom{\rho-l-i-k}{j}\frac{(-1)^j}{B+k+j}, \tag{5.2}$$

where we write

$$\frac{(C+k)_{\rho-i-k}}{(\rho-l-i-k)!} = l!\binom{C+k+l-1}{l}\binom{C+\rho-i-1}{\rho-l-i-k}. \tag{5.3}$$

Altogether, we obtain the expression

$$\frac{(C+k)_{\rho-i-k}}{(B+k)_{Dl+\lambda-i-k}} = l!\binom{C+k+l-1}{l}\binom{C+\rho-i-1}{\rho-l-i-k}\frac{(\rho-l-i-k)!}{(B+k)_{Dl+\lambda-i-k}},$$

where, by (5.1) and the first definition in (2.1),

$$E_{Dl+\lambda-i}(a,b)\frac{(\rho-l-i-k)!}{(B+k)_{Dl+\lambda-i-k}} \in \mathbb{Z}. \tag{5.4}$$

More precisely, since $\gcd(b, E_{Dl+\lambda-i}(a,b)) = \gcd(a,b) = 1$, we have

$$\frac{\prod_{p\nmid b} p^{\mathrm{ord}_p((\rho-l-i-k)!)}}{(a+bk;b)_{Dl+\lambda-i-k}} E_{Dl+\lambda-i}(a,b) \in \mathbb{Z}.$$

This, together with $\gcd(c,d) = 1$, yields

$$E_{Dl+\lambda-i}(a,b)d^{l-2}g^{Dl+\lambda-i-k}\omega_l(d)\omega_{\rho-l-i-k}(g)\frac{a_i f_k}{l!} \in \mathbb{Z},$$

where g is the denominator of b/d. Finally,

$$E_{Dl+\lambda}(a,b)d^{l-2}g^{Dl+\lambda}\omega_l(d)\omega_{\rho-l}(g)\frac{a_i f_k}{l!} \in \mathbb{Z} \quad \forall 0 \le i+k \le Dl+\lambda-1.$$

LEMMA 5.2. *For the coefficients of the approximation polynomials of Theorem* 3.2 *we have*

$$\frac{\Omega_{l,\lambda}^{(D)}(a,b,d)}{l!}a_i f_k \in \mathbb{Z} \quad \forall 0 \le i+k \le Dl+\lambda-1,\ 0 \le i \le Dl,$$

and furthermore

$$\frac{\Omega_{l,\lambda}^{(D)}(a,b,d)}{l!}A_{l,\lambda}(z),\ \frac{\Omega_{l,\lambda}^{(D)}(a,b,d)}{l!}B_{l,\lambda}^{(\nu)}(z) \in \mathbb{Z}[\gamma_1,\ldots,\gamma_D,z],$$

where

$$\Omega_{l,\lambda}^{(D)}(a,b,d) = d^{l-2}g^{Dl+\lambda}\omega_{\rho-l}(g)\omega_l(d)E_{Dl+\lambda}(a,b).$$

Proof. The coefficients of the polynomial $B_{l,\lambda}^{(\nu)}(z)$ are

$$g_{l,\lambda,n}^{(\nu)} = \sum_{i+k=n} A_i f_k^{(\nu)} = \sum_{i+k=n} \sigma_{Dl-i,l}\gamma_\nu^k a_i f_k$$

for $0 \le n \le Dl + \lambda - 1$, and
$$\frac{d^{l-2} g^{Dl+\lambda} \omega_l(d)\omega_{\rho-l}(g) E_{Dl+\lambda}(a,b)}{l!} \cdot a_i f_k \in \mathbb{Z},$$
as can be seen by the previous discussion. ■

LEMMA 5.3. *If $\lambda = \lambda(l) = \lfloor \eta l \rfloor$, $\eta \ge 0$, then for every $w \in \mathcal{M}$ and l sufficiently large,*
$$\max\{|(\Omega_{l,\lambda}^{(D)}(a,b,d)/l!)A_{l,\lambda}(z)|_w, |(\Omega_{l,\lambda}^{(D)}(a,b,d)/l!)B_{l,\lambda}^{(\nu)}(z)|_w\} \\ \le \mathcal{C}^{\delta_w l}(|\overline{\Gamma(z)}|_w^{D+\eta})^{l+c_1 \log l},$$
where $c_1 > 0$ and
$$\mathcal{C} = 5^D 4^{\eta+1} d\mu_d (g\mu_g e^{\lambda(b)})^{\eta+D},$$
and we denote $\overline{\Gamma(z)} = (\gamma_1 z, \dots, \gamma_D z)$ for brevity.

Proof. Firstly, we let $w|\infty$ and we estimate the sum
$$\Big|\sum_{j=0}^{\rho-l-i-k}\binom{\rho-l-i-k}{j}\frac{(-1)^j}{B+k+j}\Big|_w \\ \le \max_{0\le j\le \rho-l-i-k}\Big|\frac{1}{B+k+j}\Big| \sum_{j=0}^{\rho-l-i-k}\binom{\rho-l-i-k}{j} \le M 2^{\rho-l-i-k},$$
where
$$M := \max_{0\le j\le \rho-l-i-k}\Big|\frac{1}{B+k+j}\Big| \quad \text{for } l \text{ large enough.}$$
On the other hand, another simple application of the binomial theorem yields
$$\Big|\binom{C+k+l-1}{l}\binom{C+\rho-i-1}{\rho-l-i-k}\Big| \le 2^{2\lceil |C|\rceil+\rho+l+k-i-2},$$
and therefore by (5.2) and (5.3) we obtain
$$\begin{aligned} (5.5)\quad & |(\Omega_{l,\lambda}^{(D)}(a,b,d)/l!)A_{l,\lambda}(z)|_w \\ &= \Big|\Omega_{l,\lambda}^{(D)}(a,b,d)\sum_{i=0}^{Dl}\sigma_{Dl-i,l}\frac{(C)_{\rho-i}}{l!(B)_{Dl+\lambda-i}}z^i\Big|_w \\ &\le \Omega_{l,\lambda}^{(D)}(a,b,d)\sum_{i=0}^{Dl}|\sigma_{Dl-i,l}|_w\Big|\frac{(C)_{\rho-i}}{l!(B)_{Dl+\lambda-i}}\Big||z|_w^i \\ &\le \Omega_{l,\lambda}^{(D)}(a,b,d)M4^{\lceil |C|\rceil+\rho-1}\sum_{i=0}^{Dl}|\sigma_{Dl-i,l}|_w\Big|\frac{z}{4}\Big|_w^i \end{aligned}$$

$$
\begin{aligned}
&= M\Omega_{l,\lambda}^{(D)}(a,b,d)4^{\lceil |C|\rceil+\rho-1-Dl}|z|_w^{Dl}\sum_{i=0}^{Dl}|\sigma_{i,l}|_w\left|\frac{4}{z}\right|_w^i \\
&\le M\Omega_{l,\lambda}^{(D)}(a,b,d)4^{\lceil |C|\rceil+\lambda+l-2}|z|_w^{Dl}\prod_{t=1}^{D}(|\gamma_t|_w+4/|z|_w)^l \\
&= M\Omega_{l,\lambda}^{(D)}(a,b,d)4^{\lceil |C|\rceil+\lambda+l-2}\prod_{t=1}^{D}(|\gamma_t z|_w+4)^l \\
&\le M\Omega_{l,\lambda}^{(D)}(a,b,d)5^{Dl}4^{\lceil |C|\rceil+\lambda+l-2}\prod_{t=1}^{D}|\gamma_t z|_w^*.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|(\Omega_{l,\lambda}^{(D)}(a,b,d)/l!)B_{l,\lambda}^{(\nu)}(z)|_w \\
&\quad= \Omega_{l,\lambda}^{(D)}(a,b,d)\Big|\sum_{n=0}^{\lambda+Dl-1} z^n\sum_{i+k=n}\sigma_{Dl-i,l}(a_if_k/l!)\gamma_\nu^{n-i}\Big|_w \\
&\quad\le \Omega_{l,\lambda}^{(D)}(a,b,d)\sum_{n=0}^{\lambda+Dl-1}|\gamma_\nu z|_w^n\sum_{i+k=n}|\sigma_{Dl-i,l}|_w|(a_if_k/l!)|\,|\gamma_\nu|_w^{-i} \\
&\quad\le M\Omega_{l,\lambda}^{(D)}(a,b,d)\sum_{n=0}^{\lambda+Dl-1}|\gamma_\nu z|_w^n|\gamma_\nu|_w^{-Dl}4^{\lceil |C|\rceil+\lambda+l-2}\sum_{i=0}^{Dl}|\sigma_{i,l}|_w|4\gamma_\nu|_w^i \\
&\quad\le M\Omega_{l,\lambda}^{(D)}(a,b,d)\sum_{n=0}^{\lambda+Dl-1}|\gamma_\nu z|_w^n|\gamma_\nu|_w^{-Dl}4^{\lceil |C|\rceil+\lambda+l-2}\prod_{t=1}^{D}(|\gamma_t|_w+4|\gamma_\nu|_w)^l \\
&\quad\le M\Omega_{l,\lambda}^{(D)}(a,b,d)4^{\lceil |C|\rceil+\lambda+l-2}\sum_{n=0}^{\lambda+Dl-1}|\gamma_\nu z|_w^{n-Dl}\prod_{t=1}^{D}(|z\gamma_t|_w+4|z\gamma_\nu|_w)^l \\
&\quad\le M\Omega_{l,\lambda}^{(D)}(a,b,d)5^{Dl}4^{\lceil |C|\rceil+\lambda+l-2}(Dl+\lambda)(|\overline{\Gamma(z)}|_w^*)^{Dl+\lambda-1}.
\end{aligned}
$$

If $w|p$, then by Lemma 5.2,

$$
\begin{aligned}
&|(\Omega_{l,\lambda}^{(D)}(a,b,d)/l!)B_{l,\lambda}^{(\nu)}(z)|_w \\
&\quad= \max_{0\le n\le Dl+\lambda-1}\Big|z^n\sum_{\substack{i+k=n\\0\le i\le Dl}}\sigma_{Dl-i,l}\gamma_\nu^k(\Omega_{l,\lambda}^{(D)}(a,b,d)/l!)a_if_k\Big|_w \\
&\quad\le \max_{0\le n\le Dl+\lambda-1}|z|_w^n\max_{\substack{i+k=n\\0\le i\le Dl}}|\overline{\Gamma}|_w^k|\sigma_{Dl-i,l}|_w \\
&\quad\le \max_{0\le n\le Dl+\lambda-1}(|z|_w|\overline{\Gamma}|_w)^n\max_{\substack{i+k=n\\0\le i\le Dl}}\max_{i_1+\cdots+i_D=Dl-i}|\gamma_1|_w^{l-i_1}\cdots|\gamma_D|_w^{l-i_D}|\overline{\Gamma}|_w^{-i}
\end{aligned}
$$

$$= \max_{0\le n\le Dl+\lambda-1} |\overline{\Gamma(z)}|_w^n \times \max_{\substack{i+k=n\\ 0\le i\le Dl}} \max_{i_1+\cdots+i_D=Dl-i} (|\gamma_1|_w/|\overline{\Gamma}|_w)^{l-i_1}\cdots(|\gamma_D|_w/|\overline{\Gamma}|_w)^{l-i_D}$$
$$\le (|\overline{\Gamma(z)}|_w^*)^{Dl+\lambda-1},$$

and also

$$|(\Omega_{l,\lambda}^{(D)}(a,b,d)/l!)A_{l,\lambda}(z)|_w \le \max_{i_1+\cdots+i_D=Dl-i} |\gamma_1|_w^{l-i_1}\cdots|\gamma_D|_w^{l-i_D}|z|_w^i \le (|\overline{\Gamma(z)}|_w^*)^{Dl}.$$

Thus, for every $w\in\mathcal{M}$ we are left with the upper bound

$$\max\left\{\left|\frac{\Omega_{l,\lambda}^{(D)}(a,b,d)}{l!}A_{l,\lambda}(z)\right|_w, \left|\frac{\Omega_{l,\lambda}^{(D)}(a,b,d)}{l!}B_{l,\lambda}^{(\nu)}(z)\right|_w\right\} \le \mathcal{C}_l^{\delta_w}(|\overline{\Gamma(z)}|_w^*)^{Dl+\lambda}, \tag{5.6}$$

where

$$\mathcal{C}_l = M\Omega_{l,\lambda}^{(D)}(a,b,d)(Dl+\lambda)5^{Dl}4^{\lceil|C|\rceil+\lambda+l-2},$$

and $\delta_w=1$ if $w|\infty$, while $\delta_w=0$ if $w|p$.

By setting $\lambda=\lfloor\eta l\rfloor$, $\eta\ge 0$, and by making use of the well-known estimates (the first inequality follows directly from (2.1) and we refer to Lemma 1 of [1] for the asymptotic bound)

$$\omega_N(d)\le\mu_d^N,\qquad \lim_{M\to\infty}\frac{\ln(E_M(a,b))}{M} = \frac{b}{\phi(b)}\sum_{\substack{j=1\\ \gcd(j,b)=1}}^{b}\frac{1}{j} = \lambda(b),$$

we are led to the estimate

$$\Omega_{l,\lambda}^{(D)}(a,b,d) = d^{l-2}g^{Dl+\lambda}\omega_{\rho-l}(g)\omega_l(d)E_{Dl+\lambda}(a,b) \le c_0(g\mu_g e^{\lambda(b)})^{(\eta+D)l}(d\mu_d)^l$$

with $c_0>0$. From the upper bound (5.6) we obtain in turn

$$\max\{|(\Omega_{l,\lambda}^{(D)}(a,b,d)/l!)A_{l,\lambda}(z)|_w, |(\Omega_{l,\lambda}^{(D)}(a,b,d)/l!)B_{l,\lambda}^{(\nu)}(z)|_w\} \le c_0(Dl+\lambda)\mathcal{C}^{\delta_w l}(|\overline{\Gamma(z)}|_w^*)^{(D+\eta)l} \le \mathcal{C}^{\delta_w l}(|\overline{\Gamma(z)}|_w^*)^{(D+\eta)l+c_1\log l}, \tag{5.7}$$

where $c_1>0$ and $\mathcal{C}=5^D4^{\eta+1}d\mu_d(g\mu_g e^{\lambda(b)})^{\eta+D}$, proving the lemma. ■

Lemma 5.4. *Let $|\overline{\Gamma(z)}|_v<1$ and $\operatorname{ord}_p(d)\ge\operatorname{ord}_p(b)$ if $v|p$. Then*

$$|(\Omega_{l,\lambda}^{(D)}(a,b,d)/l!)R_{l,\lambda}^{(\nu)}(z)|_v \le \mathcal{C}_r^{\delta_v l}|\overline{\Gamma(z)}|_v^{(D+\eta+1)l-c_3\log l},\qquad c_3>0,$$

for every $v\in\mathcal{M}$, where

$$\mathcal{C}_r = d\mu_d(g\mu_g e^{\lambda(b)})^{\eta+D}2^{\eta+1}3^D.$$

Proof. Let $v|\infty$. Then

$$|R_{l,\lambda}^{(\nu)}(z)|_v \le |z|_v^{\rho+1} \sum_{k=0}^{\infty} |g_{l,\lambda,\rho+k+1}^{(\nu)}|_v |z|_v^k$$

where, for sufficiently large l,

$$\begin{aligned}
|g_{l,\lambda,\rho+k+1}^{(\nu)}|_v &= |\gamma_\nu|_v^{\rho+k+1} \left| \sum_{i=0}^{Dl} \sigma_{Dl-i,l} \frac{(C)_{\rho-i}(B)_{\rho+k+1-i}}{(B)_{Dl-i+\lambda}(C)_{\rho+k+1-i}} \gamma_\nu^{-i} \right|_v \\
&= |\gamma_\nu|_v^{l+\lambda+k} \left| \sum_{i=0}^{Dl} \sigma_{i,l}(B+\lambda+i)_{l-1} \frac{(B+\rho+i-Dl)_{k+1}}{(C+\rho+i-Dl)_{k+1}} \gamma_\nu^i \right|_v \\
&= (l-1)! \cdot |\gamma_\nu|_v^{l+\lambda+k} \\
&\quad \times \left| \sum_{i=0}^{Dl} \sigma_{i,l} \binom{B+\lambda+i+l-2}{l-1} \prod_{j=0}^{k} \left(1 + \frac{B-C}{C+\rho-Dl+i+j}\right) \gamma_\nu^i \right|_v \\
&\le (l-1)! \cdot |\gamma_\nu|_v^{l+\lambda+k} 2^{\lceil |B| \rceil + \lambda + l - 2} \\
&\quad \times \sum_{i=0}^{Dl} |\sigma_{i,l}(2\gamma_\nu)^i|_v \prod_{j=0}^{k} \left| 1 + \frac{B-C}{C+\rho-Dl+i+j} \right| \\
&\le (l-1)! \cdot |\gamma_\nu|_v^{l+\lambda+k} 2^{\lceil |B| \rceil + \lambda + l - 2} \left(1 + \frac{|B-C|}{|C+l-1|}\right)^{k+1} \sum_{i=0}^{Dl} |\sigma_{i,l}(2\gamma_\nu)^i|_v \\
&= (l-1)! \cdot |\gamma_\nu|_v^{l+\lambda+k} 2^{\lceil |B| \rceil + \lambda + l - 2} \left(1 + \frac{|B-C|}{|C+l-1|}\right)^{k+1} \prod_{t=1}^{D} (|\gamma_t|_v + 2|\gamma_\nu|_v)^l \\
&\le (l-1)! \cdot |\overline{\Gamma}|_v^{\rho+1} 2^{\lceil |B| \rceil + \lambda + l - 2} 3^{Dl} \\
&\quad \times \left(1 + \frac{|B-C|}{|C+l-1|}\right) \left(|\overline{\Gamma}|_v \left(1 + \frac{|B-C|}{|C+l-1|}\right)\right)^k .
\end{aligned}$$

Thus, we arrive at the estimate

$$\begin{aligned}
|R_{l,\lambda}^{(\nu)}(z)|_v &\le |\overline{\Gamma(z)}|_v^{\rho+1} (l-1)! 2^{\lceil |B| \rceil + \lambda + l - 2} 3^{Dl} \left(1 + \frac{|B-C|}{|C+l-1|}\right) \\
&\quad \times \sum_{k=0}^{\infty} \left(|\overline{\Gamma}|_v \left(1 + \frac{|B-C|}{|C+l-1|}\right)\right)^k |z|_v^k \\
&\le c_2 (l-1)! 2^{\lambda+l} 3^{Dl} |\overline{\Gamma(z)}|_v^{\rho+1}, \qquad c_2 = c_2(z) > 0,
\end{aligned}$$

since

$$\frac{|B-C|}{|C+l-1|} \to 0 \quad \text{and} \quad |\overline{\Gamma(z)}|_v < 1.$$

By this upper bound we obtain

$$|(\Omega_{l,\lambda}^{(D)}(a,b,d)/l!)R_{l,\lambda}^{(\nu)}(z)|_v \le \frac{c_2}{l}(d\mu_d(g\mu_g e^{\lambda(b)})^{D+\eta}2^{\eta+1}3^D|\overline{\Gamma(z)}|_v^{D+\eta+1})^l$$

for $v|\infty$.

Let now $v|p$. Then

$$(\Omega_{l,\lambda}^{(D)}(a,b,d)/l!)R_{l,\lambda}^{(\nu)}(z) = (\Omega_{l,\lambda}^{(D)}(a,b,d)/l!)z^{\rho+1}\sum_{k=0}^{\infty} g_{l,\lambda,\rho+k+1}^{(\nu)} z^k,$$

where

$$g_{l,\lambda,\rho+k+1}^{(\nu)} = \sum_{i=0}^{Dl}\sigma_{Dl-i}a_i f_{\rho+k+1-i}\gamma_\nu^{\rho+k+1-i}, \qquad k=0,1,\ldots.$$

By Lemma 5.2,

$$\begin{aligned}
&|(\Omega_{l,\lambda}^{(D)}(a,b,d)/l!)g_{l,\lambda,\rho+k+1}^{(\nu)}|_v \\
&\qquad \le \max_{0\le i\le Dl}|(\Omega_{l,\lambda}^{(D)}(a,b,d)/l!)\sigma_{Dl-i,l}a_i f_{\rho+k+1-i}\gamma_\nu^{\rho+k+1-i}|_v \\
&\qquad \le \max_{0\le i\le Dl}|\sigma_{Dl-i,l}|_v|\overline{\Gamma}|_v^{-i}|f_{\rho+k+1-i}|_v|\overline{\Gamma}|_v^{\rho+k+1} \\
&\qquad \le |\overline{\Gamma}|_v^{\rho+k+1}\max_{0\le i\le Dl}|f_{\rho+k+1-i}|_v,
\end{aligned}$$

with

$$f_{\rho+k+1-i} = \left(\frac{d}{b}\right)^{\rho+k+1-i}\frac{a(a+b)\cdots(a+b(\rho+k-i))}{c(c+d)\cdots(c+d(\rho+k-i))}.$$

We assume that $\mathrm{ord}_p(d) \ge \mathrm{ord}_p(b)$. Firstly, if $p\,|\,bd$, then $|f_{\rho+k+1-i}|_p \le 1$, and hence we may suppose $p\nmid bd$. If $p^{m_p(x,y)}\,|\,(x;y)_{\rho+k+1-i}$, $x,y\in\mathbb{Z}$, $y>0$, $m_p(x,y)$ being maximal, then

$$\mathrm{ord}_p((\rho+k+1-i)!) \le m_p(x,y) \le \mathrm{ord}_p((\rho+k+1-i)!) + M_p(x,y)$$

by Lemma 5.1, where $M_p(x,y) = \lfloor\log(|x|+y(\rho+k-i))/\log p\rfloor$. This yields

$$|f_{\rho+k+1-i}|_p = p^{m_p(c,d)-m_p(a,b)} \le p^{M_p(c,d)} \le d(|C|+\rho+k)$$

for every $0\le i\le Dl$, due to the inequality

$$M_p(c,d) \le \frac{\log(d(|C|+\rho+k))}{\log p}, \quad k=0,1,2,\ldots.$$

By the above estimates, for l large enough ($|\overline{\Gamma(z)}|_v<1$) we get

$$\begin{aligned}
(5.8)\quad |(\Omega_{l,\lambda}^{(D)}(a,b,d)/l!)R_{l,\lambda}^{(\nu)}(z)|_v &\le |\overline{\Gamma(z)}|_v^{\rho+1}\max_{k\ge 0}|f_{\rho+k+1-i}|_p|\overline{\Gamma(z)}|_v^k \\
&\le d(|C|+2\rho)|\overline{\Gamma(z)}|_v^{\rho+1} \\
&\le 2(|C|+d)(D+\eta+1)l|\overline{\Gamma(z)}|_v^{(D+\eta+1)l} \\
&\le |\overline{\Gamma(z)}|_v^{(D+\eta+1)l-c_3\log l}, \quad c_3>0. \ \blacksquare
\end{aligned}$$

REMARK 5.5. The estimates of the polynomials may, in many special cases of B and C, be sharpened. For example, in the case of the logarithmic function, i.e. $B = 1$, $C = 2$, we have

$$A_{l,\lambda}(z) = \sum_{i=0}^{Dl} \sigma_{i,l} \binom{l+\lambda+i}{l} z^{Dl-i}, \tag{5.9}$$

$$B_{l,\lambda}^{(\nu)}(z) = \sum_{n=0}^{Dl+\lambda-1} z^n \sum_{i+k=n} \sigma_{Dl-i,l} \frac{\binom{l+\lambda+Dl-i}{l}}{k+1} \gamma_\nu^k,$$

$$R_{l,\lambda}^{(\nu)}(z) = (z\gamma_\nu)^{\rho+1} \sum_{n=0}^{\infty} \sum_{i=0}^{Dl} \sigma_{Dl-i,l} \frac{\binom{l+\lambda+Dl-i}{l}}{\rho+n-i+2} \gamma_\nu^{n-i}.$$

This yields $\mathcal{C} = \mathcal{C}_r = 3^D 2^{\eta+1} e^{\eta+D}$ in Lemma 5.3. The polynomial (5.9) is the same as the denominator polynomial $P_{0,n,m}(z)$ appearing in [28] with $r = D$, $l = n$ and $m = \lambda$.

6. Proof of Theorem 2.1 and the corollaries. Let

$$a_{l,\lambda} := (\Omega_{l,\lambda}^{(D)}(a,b,d)/l!) A_{l,\lambda}(1) \in \mathbb{Z}[\gamma_1, \ldots, \gamma_D],$$
$$b_{l,\lambda}^{(\nu)} := (\Omega_{l,\lambda}^{(D)}(a,b,d)/l!) B_{l,\lambda}^{(\nu)}(1) \in \mathbb{Z}[\gamma_1, \ldots, \gamma_D],$$
$$r_{l,\lambda}^{(\nu)} := (\Omega_{l,\lambda}^{(D)}(a,b,d)/l!) R_{l,\lambda}^{(\nu)}(1),$$

where by Lemmas 5.3 and 5.4

$$\max\{\|a_{l,\lambda}\|_w, \|b_{l,\lambda}^{(\nu)}\|_w\} \le \mathcal{C}^{(\kappa_w/\kappa)\delta_w l} (\|\overline{\Gamma}\|_w^*)^{(D+\eta)l+\mathcal{O}(\log l)} =: \mathcal{P}_w(l) \quad \forall w \in \mathcal{M}$$

and

$$\|r_{l,\lambda}^{(\nu)}\|_v \le \mathcal{C}_r^{(\kappa_v/\kappa)\delta_v l} \|\overline{\Gamma}\|_v^{(D+\eta+1)l-\mathcal{O}(\log l)} =: \mathcal{R}_v(l). \tag{6.1}$$

Furthermore,

$$\prod_{w\in\mathcal{M}} \mathcal{P}_w(l) = \mathcal{C}^l H(\overline{\Gamma})^{(D+\eta)l+\mathcal{O}(\log l)} =: \mathcal{P}(l) \tag{6.2}$$

and suppose

$$\|\overline{\Gamma}\|_v^{D+\eta+1} \mathcal{C} H(\overline{\Gamma})^{D+\eta} < \left(\frac{\mathcal{C}}{\mathcal{C}_r}\right)^{\kappa_v\delta_v/\kappa} =: \mathcal{E}_v, \tag{6.3}$$

i.e.

$$\log \|\overline{\Gamma}\|_v^{-1} > \frac{\log \mathcal{C} + (D+\eta)\log H(\overline{\Gamma}) - \log \mathcal{E}_v}{D+\eta+1}.$$

Since for all $\varepsilon > 0$ there exists $l_0 \in \mathbb{N}$ such that

$$\|\overline{\Gamma}\|_v^{-\mathcal{O}(\log l)} \le \|\overline{\Gamma}\|_v^{-\varepsilon l} \quad \forall l \ge l_0$$

and

$$H(\overline{\Gamma})^{\mathcal{O}(\log l)} \le H(\overline{\Gamma})^{\varepsilon l} \quad \forall l \ge l_0,$$

for every $l \ge l_0$ we have

$$\mathcal{P}(l) \le \mathcal{C}^l H(\overline{\Gamma})^{(D+\eta+\varepsilon)l}, \quad \mathcal{R}_v(l) \le \mathcal{C}_r^{(\kappa_v/\kappa)\delta_v l} \|\overline{\Gamma}\|_v^{(D+\eta+1-\varepsilon)l}.$$

Next, consider the linear form

$$L = b_0 + \sum_{\nu=1}^{D} b_\nu g_\nu(1), \quad \overline{B} = (b_0, \dots, b_D) \in \mathbb{K}^{D+1} \setminus \{\overline{0}\}.$$

Multiplication of this form by a_{l,λ_0} yields

$$a_{l,\lambda_0} L = \tilde{L} + \sum_{\nu=1}^{D} b_\nu r_{l,\lambda_0}^{(\nu)},$$

where

$$\tilde{L} = b_0 a_{l,\lambda_0} + \sum_{\nu=1}^{D} b_\nu b_{l,\lambda_0}^{(\nu)} \in \mathbb{K}^*$$

for some $\lambda_0 = \lambda + \tau$, where $\tau \in [0, D-1]$, by the nonvanishing of the determinant in Theorem 4.3. By the product formula we have

$$\begin{aligned}
1 &= \prod_{w \in \mathcal{M}} \|\tilde{L}\|_w = \prod_{\substack{w \in \mathcal{M} \\ w \ne v}} \Big\| b_0 a_{l,\lambda_0} + \sum_{\nu=1}^{D} b_\nu b_{l,\lambda_0}^{(\nu)} \Big\|_w \Big\| a_{l,\lambda_0} L - \sum_{\nu=1}^{D} b_\nu r_{l,\lambda_0}^{(\nu)} \Big\|_v \\
&\le \prod_{\substack{w \in \mathcal{M} \\ w \ne v}} \|\overline{B}\|_w^* \mathcal{P}_w(l) \{\mathcal{P}_v(l) \|L\|_v + \|\overline{B}\|_v^* \mathcal{R}_v(l)\} \\
&= \mathcal{P}(l) \|\overline{B}\|_v^{*\,-1} H(\overline{B}) \|L\|_v + H(\overline{B}) \mathcal{C}^{-(\kappa_v/\kappa)\delta_v l} \mathcal{P}(l) \mathcal{R}_v(l) \\
&\le \mathcal{C}^l H(\overline{\Gamma})^{(D+\eta+\varepsilon)l} \|\overline{B}\|_v^{*\,-1} H(\overline{B}) \|L\|_v \\
&\quad + \frac{1}{2} H(\overline{B}) (\mathcal{C} H(\overline{\Gamma})^{D+\eta+\varepsilon} \mathcal{E}_v^{-1} \|\overline{\Gamma}\|_v^{D+\eta+1-\varepsilon})^l.
\end{aligned}$$

Since $\varepsilon > 0$ may be chosen arbitrarily small, we know from (6.3) that

$$(\mathcal{C} H(\overline{\Gamma})^{D+\eta+\varepsilon} \mathcal{E}_v^{-1} \|\overline{\Gamma}\|_v^{D+\eta+1-\varepsilon})^l \to 0 \quad \text{as } l \to \infty,$$

and therefore we may find, by choosing $H(\overline{B})$ large enough, say $H(\overline{B}) \ge H_0$, the maximum l_1 of $l \ge l_0$ such that

$$\frac{1}{2} H(\overline{B}) (\mathcal{C} H(\overline{\Gamma})^{D+\eta} \mathcal{E}_v^{-1} \|\overline{\Gamma}\|_v^{D+\eta+1})^{l_1} \ge \frac{1}{2}. \tag{6.4}$$

This in turn implies, by taking the logarithm on both sides of (6.4), that

$$\log H(\overline{B}) + l_1 \log(\mathcal{E}_v^{-1} \|\overline{\Gamma}\|_v^{D+\eta+1} \mathcal{C} H(\overline{\Gamma})^{D+\eta}) \ge 0$$

and

$$l_1 \leq -\frac{\log H(\overline{B})}{\log(\mathcal{E}_v^{-1}\|\overline{\Gamma}\|_v^{D+\eta+1}\mathcal{C}H(\overline{\Gamma})^{D+\eta})}.$$

Thus, for $l = l_1 + 1$,

$$\mathcal{C}^{l_1+1}H(\overline{\Gamma})^{(D+\eta)(l_1+1)}\|\overline{B}\|_v^{*\,-1}H(\overline{B})\|L\|_v \geq \frac{1}{2}$$

by the product formula, and

$$\begin{aligned}\|L\|_v &\geq \frac{\|\overline{B}\|_v^*}{2H(\overline{B})(\mathcal{C}H(\overline{\Gamma})^{D+\eta})^{l_1+1}}\\ &\geq \frac{C_1\|\overline{B}\|_v^*}{H(\overline{B})^{1-\log(\mathcal{C}H(\overline{\Gamma})^{D+\eta})/\log(\mathcal{E}_v^{-1}\|\overline{\Gamma}\|_v^{D+\eta+1}\mathcal{C}H(\overline{\Gamma})^{D+\eta})}}, \quad C_1 > 0.\end{aligned}$$

Finally,

$$|L|_v \geq C|\overline{B}|^* H(\overline{B})^{-\mu}, \quad C > 0, \tag{6.5}$$

where

$$\begin{aligned}\mu &= \frac{\kappa}{\kappa_v}\cdot\frac{\log(\mathcal{E}_v^{-1}\|\overline{\Gamma}\|_v^{D+\eta+1})}{\log(\mathcal{E}_v^{-1}\|\overline{\Gamma}\|_v^{D+\eta+1}\mathcal{C}H(\overline{\Gamma})^{D+\eta})}\\ &= \frac{\kappa(D+\eta+1)\log|\overline{\Gamma}|_v - \kappa\delta_v\log((5/3)^D 2^{\eta+1})}{(\kappa-\kappa_v\delta_v)\log\mathcal{C} + \kappa(D+\eta)\log H(\overline{\Gamma}) + \kappa_v(D+\eta+1)\log|\overline{\Gamma}|_v + \kappa_v\delta_v\log\mathcal{C}_r}\end{aligned}$$

and

$$\mathcal{C} = 5^D 4^{\eta+1} d\mu_d(g\mu_g e^{\lambda(b)})^{D+\eta}, \quad \mathcal{C}_r = 3^D 2^{\eta+1} d\mu_d(g\mu_g e^{\lambda(b)})^{D+\eta}.$$

This proves Theorem 2.1.

Proof of the corollaries. If $\mathbb{K} = \mathbb{Q}$ or $\mathbb{I}$ and $\overline{B} \in \mathbb{Z}_{\mathbb{K}}^{D+1}\setminus\{\overline{0}\}$, then $H(\overline{B}) = \max\{|b_0|, \ldots, |b_D|\}$ and

$$|L|_v \geq CH(\overline{B})^{-\mu+1}, \quad C > 0,$$

by the lower bound (6.5). Set $\gamma_\nu = \xi_\nu/q$, where $\xi_\nu \in \mathbb{I}^*$, $q \in \mathbb{Z}_{\mathbb{I}} \setminus \{0\}$ and write $\overline{\Xi} = (\xi_1, \ldots, \xi_D)$. Since $|\overline{\Gamma}| = |q|^{-1}|\overline{\Xi}| < 1$, we have

$$H(\overline{\Gamma}) \leq \frac{H(\overline{\Xi})H(q^{-1})}{\max\{1, |\overline{\Xi}|\}} = \frac{H(\overline{\Xi})H(q)}{|\overline{\Xi}|^*} = H(\overline{\Xi})|\overline{\Xi}|^{*\,-1}|q|,$$

and the claim follows from (6.3). The remaining results in Corollary 2.2 are straightforward computations, and Corollary 2.3 is a direct consequence of Theorem 2.1.

Acknowledgements. The author would like to thank the anonymous referee for the useful comments which improved the presentation of this article.

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Received on 20.8.2009
and in revised form on 18.1.2010 (6125)