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On the k-free divisor problem (II)

by

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1. Introduction. Let $d(n)$ denote the divisor function. Dirichlet first proved that the error term

$$\Delta(x) := {\sum_{n\le x}}' d(n) - x\log x - (2\gamma-1)x, \qquad x\ge 2,$$

satisfies $\Delta(x) = O(x^{1/2})$, where $\sum'_{n\le x}$ means that the term for $n=x$ should be halved when x is an integer. The exponent $1/2$ was improved by many authors. The latest result is due to Huxley [4], who proved that

$$\Delta(x) \ll x^{131/416}(\log x)^{26957/8320}.$$

It is conjectured that

$$\Delta(x) = O(x^{1/4+\varepsilon}), \tag{1.1}$$

which is supported by the classical mean-square result

$$\int_1^T \Delta^2(x)\,dx = \frac{(\zeta(3/2))^4}{6\pi^2\zeta(3)}T^{3/2} + O(T\log^5 T) \tag{1.2}$$

proved by Tong [12].

Let $k\ge 2$ denote a fixed integer. An integer n is called *k-free* if p^k does not divide n for any prime p. Let $d^{(k)}(n)$ denote the number of k-free divisors of the positive integer n and define

$$D^{(k)}(x) := {\sum_{n\le x}}' d^{(k)}(n).$$

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Then the expected asymptotic formula for $D^{(k)}(x)$ is

$$D^{(k)}(x) = C_1^{(k)} x \log x + C_2^{(k)} x + \Delta^{(k)}(x), \tag{1.3}$$

where $C_1^{(k)}, C_2^{(k)}$ are two constants, and $\Delta^{(k)}(x)$ is the error term. In 1874 Mertens [7] proved that $\Delta^{(2)}(x) \ll x^{1/2} \log x$. In 1932 Perron [9] proved that

$$\Delta^{(k)}(x) \ll \begin{cases} x^{1/2} & \text{if } k = 2, \\ x^{1/3} & \text{if } k = 3, \\ x^{33/100} & \text{if } k \geq 4. \end{cases}$$

For $k = 2, 3$, it is very difficult to improve the exponent $1/k$ in the bound $\Delta^{(k)}(x) \ll x^{1/k}$, unless we have substantial progress in the study of the zerofree region of $\zeta(s)$. Therefore it is reasonable to get better improvements by assuming the truth of the Riemann Hypothesis (RH). Such results have been given in [1, 2, 6, 8, 10, 11]. In particular, in [2] R. C. Baker proved $\Delta^{(2)}(x) \ll x^{4/11+\varepsilon}$ and in [6] Kumchev proved $\Delta^{(3)}(x) \ll x^{27/85+\varepsilon}$ under RH. For $k \geq 4$, it is easy to show that if $\Delta(x) \ll x^{\alpha}$, then $\Delta^{(k)}(x) \ll x^{\alpha} \log x$.

We believe that the estimate

$$\Delta^{(k)}(x) \ll x^{1/4+\varepsilon} \tag{1.4}$$

is true for any $k \geq 2$, which is an analogue of (1.1). For $k \geq 4$ the conjecture (1.4) is partly supported by the asymptotic formula

$$\int_1^T |\Delta^{(k)}(x)|^2\, dx = \frac{B_k}{6\pi^2} T^{3/2} + \begin{cases} O(T^{3/2} e^{-c\delta(T)}) & \text{for } k = 4, \\ O(T^{\delta_k+\varepsilon}) & \text{for } k \geq 5, \end{cases} \tag{1.5}$$

proved in [3], where $c > 0$ is an absolute constant and

$$B_k := \sum_{m=1}^{\infty} g_k^2(m) m^{-3/2}, \qquad g_k(m) := \sum_{m=nl^k} \mu(l) d(n) l^{k/2},$$
$$\delta(u) := (\log u)^{3/5} (\log\log u)^{-1/5},$$
$$\delta_5 := 29/20, \qquad \delta_k := 3/2 - 1/2k + 1/k^2 \quad (k \geq 6).$$

The approach in [3] fails for $k = 3$ and gives only a weak result for $k = 4$. However, if RH is true, we can do much better. In this short note, we shall prove the following

Theorem. *If RH is true, then*

$$\int_1^T |\Delta^{(k)}(x)|^2\, dx = \frac{B_k}{6\pi^2} T^{3/2} + O(T^{3/2-\eta_k+\varepsilon}) \tag{1.6}$$

with $\eta_k := (k-2)/(12k-8)$ $(k = 3, 4, 5, 6)$, *where the implied constant depends only on* ε.

COROLLARY. *If RH is true, then*

$$\Delta^{(3)}(x) = \Omega(x^{1/4}).$$

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2. Proof of Theorem

2.1. *Mean square of $\Delta_{2,y}^{(k)}(x)$.* Suppose RH is true. It is obvious that

$$\sum_{n=1}^{\infty} \frac{d^{(k)}(n)}{n^s} = \frac{\zeta^2(s)}{\zeta(ks)} \qquad (\Re s > 1),$$

which implies that

$$d^{(k)}(n) = \sum_{n=l^k m} \mu(l) d(m).$$

Let $y > 2$ be a parameter. Define

$$d_{1,y}^{(k)}(n) := \sum_{\substack{n=l^k m \\ l \le y}} \mu(l) d(m), \qquad d_{2,y}^{(k)}(n) := \sum_{\substack{n=l^k m \\ l > y}} \mu(l) d(m).$$

Then

$$d^{(k)}(n) = d_{1,y}^{(k)}(n) + d_{2,y}^{(k)}(n).$$

It is easy to see that for $\Re s > 1$ we have

$$\sum_{n=1}^{\infty} \frac{d_{2,y}^{(k)}(n)}{n^s} = \zeta^2(s) f_y(ks), \tag{2.1}$$

where $f_y(s) := \sum_{l>y} \mu(l)/l^s$. It is well-known that under RH, the function $f_y(s)$ can be analytically continued to $\Re s > 1/2$ and that uniformly in the strip $1/2 + \varepsilon < \Re s \le 1$ the estimate

$$f_y(s) \ll y^{1/2-\sigma+\varepsilon}(1 + |t|)^{\varepsilon} \tag{2.2}$$

holds.

Let

$$D_{i,y}^{(k)}(x) := \sideset{}{'}\sum_{n \le x} d_{i,y}^{(k)}(n) \qquad (i = 1, 2).$$

Then

$$D^{(k)}(x) = D_{1,y}^{(k)}(x) + D_{2,y}^{(k)}(x). \tag{2.3}$$

Note that here y is independent of x. We have $D_{2,y}^{(k)}(x) \equiv 0$ when $y > x^{1/k}$.

For $D_{1,y}^{(k)}(x)$, we have

$$\begin{aligned}(2.4)\qquad D_{1,y}^{(k)}(x) &= \sum_{l\le y}\mu(l)\sum_{m\le x/l^k}{}' d(m)\\ &= \sum_{l\le y}\mu(l)\Big\{\frac{x}{l^k}\log\frac{x}{l^k}+(2\gamma-1)\,\frac{x}{l^k}+\Delta\Big(\frac{x}{l^k}\Big)\Big\}\\ &= \mathrm{Res}_{s=1}\Big(\zeta^2(s)\,\frac{x^s}{s}\sum_{l\le y}\frac{\mu(l)}{l^s}\Big)+\sum_{l\le y}\mu(l)\Delta\Big(\frac{x}{l^k}\Big).\end{aligned}$$

By Perron's formula we know that

$$(2.5)\qquad D_{2,y}^{(k)}(x)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\zeta^2(s)\,\frac{x^s}{s}\,f_y(ks)\,ds.$$

Moving the line of integration in (2.5) to some $c<1$ (but close to 1), by the residue theorem we get

$$(2.6)\quad D_{2,y}^{(k)}(x)=\mathrm{Res}_{s=1}\Big(\zeta^2(s)\,\frac{x^s}{s}\,f_y(ks)\Big)+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\zeta^2(s)\,\frac{x^s}{s}\,f_y(ks)\,ds.$$

Let

$$\Delta_{2,y}^{(k)}(x):=\sum_{n\le x}{}' d_{2,y}^{(k)}(n)-\mathrm{Res}_{s=1}\Big(\zeta^2(s)\,\frac{x^s}{s}\,f_y(ks)\Big).$$

Then

$$(2.7)\qquad \Delta_{2,y}^{(k)}(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\zeta^2(s)\,\frac{x^s}{s}\,f_y(ks)\,ds.$$

Since $\zeta^2(s)f_y(ks)s^{-1}\to 0$ uniformly in the strip $1/4<\Re s<1$ when $|t|\to\infty$, (2.7) is true for any $1/4<c<1$. Replacing in (2.7) x by $1/x$, taking $c=1/4+\varepsilon$ and then using Parseval's identity (see for example, (A.5) of Ivić [5]) we get

$$(2.8)\quad \int_0^\infty\frac{|\Delta_{2,y}^{(k)}(x)|^2}{x^{3/2+2\varepsilon}}\,dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\zeta(1/4+\varepsilon+it)|^4|f_y(k(1/4+\varepsilon+it))|^2}{|1/4+\varepsilon+it|^2}\,dt.$$

From (2.2) we have

$$(2.9)\qquad |f_y(k(1/4+\varepsilon+it))|^2\ll y^{1-k/2-2k\varepsilon+2\varepsilon}(1+|t|)^{2\varepsilon}.$$

Under RH we have, for any $0\le\sigma\le 1/2$,

$$\zeta(\sigma+it)\ll(1+|t|)^{1/2-\sigma+\varepsilon/4}.$$

Thus

$$|\zeta(1/4+\varepsilon+it)|^4\ll(1+|t|)^{1-3\varepsilon},$$

which combined with (2.8) and (2.9) implies

$$\int_0^\infty \frac{|\Delta_{2,y}^{(k)}(x)|^2}{x^{3/2+2\varepsilon}}\,dx \ll y^{1-k/2-2k\varepsilon+2\varepsilon}\int_{-\infty}^{\infty}(1+|t|)^{-1-\varepsilon}\,dt \ll y^{1-k/2}.$$

Hence for any $M > 2$ we have

$$\int_M^{2M} \frac{|\Delta_{2,y}^{(k)}(x)|^2}{x^{3/2+2\varepsilon}}\,dx \ll y^{1-k/2},$$

so

$$\int_M^{2M} |\Delta_2^{(k)}(x)|^2\,dx \ll M^{3/2+\varepsilon}y^{1-k/2}. \tag{2.10}$$

2.2. *Completion of proof.* Suppose $T \geq 10$ is large. It suffices to evaluate the integral $\int_T^{2T} |\Delta^{(k)}(x)|^2\,dx$. From (2.3)–(2.7) we have

$$\Delta^{(k)}(x) = \Delta_{1,y}^{(k)}(x) + \Delta_{2,y}^{(k)}(x), \tag{2.11}$$

where

$$\Delta_{1,y}^{(k)}(x) := \sum_{l\leq y}\mu(l)\Delta(x/l^k).$$

Let $T^\varepsilon \ll y \ll T^{1/4-\varepsilon}$ and $T^\varepsilon \ll z \ll T^{1-\varepsilon}$ be two parameters to be determined later. Let

$$\Delta_1(u) := \frac{u^{1/4}}{\sqrt{2}\,\pi}\sum_{n\leq z}\frac{d(n)}{n^{3/4}}\cos(4\pi\sqrt{nu}-\pi/4), \qquad \Delta_2(u;z) := \Delta(u) - \Delta_1(u).$$

Then we can write

$$\Delta_{1,y}^{(k)}(x) = R_1^{(k)}(x) + R_2^{(k)}(x), \tag{2.12}$$

where

$$R_1^{(k)}(x) := \frac{x^{1/4}}{\sqrt{2}\,\pi}\sum_{l\leq y}\frac{\mu(l)}{l^{k/4}}\sum_{n\leq z}\frac{d(n)}{n^{3/4}}\cos\left(4\pi\sqrt{\frac{nx}{l^k}}-\frac{\pi}{4}\right),$$

$$R_2^{(k)}(x) := \sum_{l\leq y}\mu(l)\Delta_2\left(\frac{x}{l^k};z\right).$$

Taking $z = T^{1-\varepsilon}$ we deduce from (3.3) of [3] that

$$\int_T^{2T} |R_2^{(k)}(x)|^2\,dx \ll \begin{cases} T^{3/2}z^{-1/2}y^{1/2}\log^4 T + Ty^2\log^6 T & \text{if } k = 3,\\ T^{3/2}z^{-1/2}\log^5 T + Ty^2\log^6 T & \text{if } k \geq 4,\end{cases} \tag{2.13}$$

$$\ll Ty^2\log^6 T \qquad (k \geq 3).$$

Now we consider the mean square of $R_1^{(k)}(x)$. By the elementary formula

$$\cos u \cos v = \frac{1}{2}\left(\cos(u-v) + \cos(u+v)\right)$$

we may write

$$|R_1^{(k)}(x)|^2 = \frac{x^{1/2}}{2\pi^2} \sum_{l_1,l_2\le y} \frac{\mu(l_1)\mu(l_2)}{(l_1 l_2)^{k/4}} \sum_{n_1,n_2\le z} \frac{d(n_1)d(n_2)}{(n_1 n_2)^{3/4}} \times \cos\left(4\pi\sqrt{\frac{n_1 x}{l_1^k}} - \frac{\pi}{4}\right)\cos\left(4\pi\sqrt{\frac{n_2 x}{l_2^k}} - \frac{\pi}{4}\right) = S_1(x) + S_2(x) + S_3(x), \tag{2.14}$$

where

$$S_1(x) = \frac{x^{1/2}}{4\pi^2} \sum_{\substack{l_1,l_2\le y;\, n_1,n_2\le z\\ n_1 l_2^k = n_2 l_1^k}} \frac{\mu(l_1)\mu(l_2)}{(l_1 l_2)^{k/4}} \frac{d(n_1)d(n_2)}{(n_1 n_2)^{3/4}},$$

$$S_2(x) = \frac{x^{1/2}}{4\pi^2} \sum_{\substack{l_1,l_2\le y;\, n_1,n_2\le z\\ n_1 l_2^k \ne n_2 l_1^k}} \frac{\mu(l_1)\mu(l_2)}{(l_1 l_2)^{k/4}} \frac{d(n_1)d(n_2)}{(n_1 n_2)^{3/4}} \cos\left(4\pi\sqrt{x}\left(\sqrt{\frac{n_1}{l_1^k}} - \sqrt{\frac{n_2}{l_2^k}}\right)\right),$$

$$S_3(x) = \frac{x^{1/2}}{4\pi^2} \sum_{l_1,l_2\le y;\, n_1,n_2\le z} \frac{\mu(l_1)\mu(l_2)}{(l_1 l_2)^{k/4}} \frac{d(n_1)d(n_2)}{(n_1 n_2)^{3/4}} \sin\left(4\pi\sqrt{x}\left(\sqrt{\frac{n_1}{l_1^k}} + \sqrt{\frac{n_2}{l_2^k}}\right)\right).$$

From (3.7) and (4.4) of [3] we have

$$\int_T^{2T} S_1(x)\,dx = \frac{B_k}{4\pi^2}\int_T^{2T} x^{1/2}\,dx + O(T^{3/2+\varepsilon}y^{-1/2+1/k}). \tag{2.15}$$

From (3.8) and (5.10) of [3] we get

$$\int_T^{2T} S_2(x)\,dx \ll T^{1+\varepsilon}y^2 + T^{1+(k+1)/3k+\varepsilon}. \tag{2.16}$$

From (3.9) of [3] we have

$$\int_T^{2T} S_3(x)\,dx \ll Ty^2\log^4 T. \tag{2.17}$$

From (2.14)–(2.17) we obtain

$$(2.18)\qquad \int_T^{2T} |R_1^{(k)}(x)|^2\,dx = \frac{B_k}{4\pi^2}\int_T^{2T} x^{1/2}\,dx + O(T^{3/2+\varepsilon}y^{-1/2+1/k}) + O(T^{1+\varepsilon}y^2 + T^{1+(k+1)/3k+\varepsilon}).$$

From (2.13), (2.18) and the Cauchy inequality we get

$$(2.19)\qquad \int_T^{2T} R_1^{(k)}(x)R_2^{(k)}(x)\,dx \ll T^{5/4}y\log^3 T.$$

From (2.13), (2.18) and (2.19) we get

$$(2.20)\qquad \int_T^{2T} |\Delta_{1,y}^{(k)}(x)|^2\,dx = \frac{B_k}{4\pi^2}\int_T^{2T} x^{1/2}\,dx + O(T^{3/2+\varepsilon}y^{-1/2+1/k}) + O(T^{5/4}y\log^3 T + T^{1+(k+1)/3k+\varepsilon}),$$

which combining (2.10) with $M = T$ gives

$$(2.21)\qquad \int_T^{2T} \Delta_{1,y}^{(k)}(x)\Delta_{2,y}^{(k)}(x)\,dx \ll T^{3/2+\varepsilon}y^{-(k-2)/4}.$$

From (2.20), (2.21) and (2.10) with $M=T$ and then taking $y=T^{k/2(3k-2)}$ we get

$$(2.22)\qquad \int_T^{2T} |\Delta^{(k)}(x)|^2\,dx = \frac{B_k}{4\pi^2}\int_T^{2T} x^{1/2}\,dx + O(T^{3/2+\varepsilon}y^{-1/2+1/k}) + O(T^{5/4}y\log^3 T + T^{1+(k+1)/3k+\varepsilon}) = \frac{B_k}{4\pi^2}\int_T^{2T} x^{1/2}\,dx + O(T^{3/2-\eta_k+\varepsilon})$$

where η_k was defined in the Theorem upon noting that $(k-2)/4 \ge 1/2-1/k$. Hence our Theorem follows from a splitting argument.

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