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On primitive lattice points in planar domains

by

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1. Introduction and statements of results. Let $\mathcal{D}$ denote a compact convex subset of $\mathbb{R}^2$ which contains the origin as an inner point. Suppose that the boundary $\partial\mathcal{D}$ of $\mathcal{D}$ is smooth with finite nonzero curvature throughout, and define a canonical map $\mathcal{M}$ from $\partial\mathcal{D}$ to the unit circle, which maps every point $\mathbf{u}$ of $\partial\mathcal{D}$ to the outward normal vector of $\partial\mathcal{D}$ at $\mathbf{u}$ of length one. Assume that $\mathcal{M}$ is one-one and of class C^4. Let F denote the distance function of $\mathcal{D}$, i.e.

$$F(\mathbf{u}) = \inf\{\tau > 0 : \mathbf{u}/\tau \in \mathcal{D}\} \quad (\mathbf{u} \in \mathbb{R}^2),$$

and $Q = F^2$, thus Q is homogeneous of degree 2.

For a large real variable x, define $A_{\mathcal{D}}(x)$ as the number of lattice points of $\mathbb{Z}_*^2 := \mathbb{Z}^2 \setminus \{(0,0)\}$ in the *blown up* domain $\sqrt{x}\,\mathcal{D}$, i.e.,

$$A_{\mathcal{D}}(x) = \#(\sqrt{x}\,\mathcal{D} \cap \mathbb{Z}_*^2) = \#\{\mathbf{m} \in \mathbb{Z}_*^2 : Q(\mathbf{m}) \le x\},$$

and $P_{\mathcal{D}}(x)$ as the "lattice rest",

$$P_{\mathcal{D}}(x) = A_{\mathcal{D}}(x) - a(\mathcal{D})x,$$

where $a(\mathcal{D})$ is the area of $\mathcal{D}$. In his deep work [6], Huxley proved that

$$P_{\mathcal{D}}(x) = O(x^{23/73}(\log x)^{315/146}). \tag{1.1}$$

A bit earlier, Nowak [13] proved that

$$\int_0^T P_{\mathcal{D}}^2(x)\,dx \ll T^{3/2}, \qquad \int_0^T |P_{\mathcal{D}}(x)|\,dx \ll T^{5/4}. \tag{1.2}$$

It is also interesting to study the number $B_{\mathcal{D}}(x)$ of primitive lattice points in $\sqrt{x}\,\mathcal{D}$, i.e.,

$$B_{\mathcal{D}}(x) = \#\{(u_1,u_2) : (u_1,u_2) \in \sqrt{x}\,\mathcal{D} \cap \mathbb{Z}_*^2,\ \gcd(u_1,u_2) = 1\}.$$

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By the usual device we have

$$B_{\mathcal{D}}(x) = \sum_{m\in\mathbb{N}} \mu(m) A_{\mathcal{D}}\left(\frac{x}{m^2}\right), \tag{1.3}$$

where $\mu(m)$ is the Möbius function. We can easily derive the result

$$B_{\mathcal{D}}(x) = \frac{6}{\pi^2}\, a(\mathcal{D})x + O(x^{1/2}\omega(x)) \tag{1.4}$$

from the bound

$$\sum_{m\le y} \mu(m) \ll y\omega(y) \tag{1.5}$$

combined with (1.1) and (1.3), where

$$\omega(y) = \exp(-c(\log y)^{3/5}(\log\log y)^{-1/5})$$

for some $c > 0$. The exponent $1/2$ in the error term of (1.4) is closely connected with the zero of the Riemann zeta-function $\zeta(s)$. At present we cannot reduce the exponent $1/2$ since $\zeta(s)$ could have zeros with real part arbitrarily close to the line $\Re s = 1$.

In order to get a sharper bound, it is therefore natural to assume the truth of the Riemann Hypothesis (RH). Moroz [11] first proved that if RH is true, then

$$B_{\mathcal{D}}(x) = \frac{6}{\pi^2}\, a(\mathcal{D})x + O(x^{41/91+\varepsilon}). \tag{1.6}$$

The exponent 41/91 comes from Huxley's result (1.1). Huxley and Nowak [7] proved that the error term in (1.6) can be sharpened to $O(x^{5/12+\varepsilon})$ if RH is true. Müller [12] obtained the estimate $O(x^{9/22})$ under RH.

In this paper, we prove the following

THEOREM 1. *If RH is true, then*

$$B_{\mathcal{D}}(x) = \frac{6}{\pi^2}\, a(\mathcal{D})x + O(x^{\frac{33349}{84040}+\varepsilon}).$$

Actually, we can study the same problem for a much larger class of planar domains $\mathcal{D}$: suppose that $C = \partial\mathcal{D}$ is a closed piecewise smooth curve which can be written in polar coordinates (r, λ) as

$$C: \quad r = \varrho(\lambda), \quad 0 \le \lambda \le 2\pi,$$

where ϱ is continuous on $[0, 2\pi]$ and $\varrho(0) = \varrho(2\pi)$. Assume further that $[0, 2\pi]$ can be subdivided by an increasing sequence

$$0 = \lambda_0 < \lambda_1 < \ldots < \lambda_J = 2\pi$$

such that each restriction ϱ_j to $[\lambda_{j-1}, \lambda_j]$ has four continuous derivatives on $[\lambda_{j-1}, \lambda_j]$. Suppose finally that each of the curves

$$C_j: \quad r = \varrho_j(\lambda), \quad \lambda_{j-1} \le \lambda \le \lambda_j,\ j = 1, \ldots, J,$$

has finite nonvanishing curvature throughout and satisfies the tangent condition (see Nowak [14, pp. 498 and 500]). Let

$$\mathcal{S} = \{\mathcal{D} : \mathcal{D} \text{ satisfies the above conditions}\}.$$

REMARK 1.1. The curve C defined above may have corners and may even possess an asteroid-like shape. For example, $C = \{(\psi,\eta) : |\psi|^{1/2} + |\eta|^{1/2} = 1\}$.

For any $\mathcal{D} \in \mathcal{S}$, let $A_{\mathcal{D}}(x), B_{\mathcal{D}}(x), P_{\mathcal{D}}(x)$ be as defined before. Now (1.1) is still true (see Huxley [6]). The proof of (1.2) can be found in Nowak [14]. Thus following the arguments of Huxley and Nowak [7] without any modifications, we can deduce under RH that the asymptotic formula

$$B_{\mathcal{D}}(x) = \frac{6}{\pi^2}\, a(\mathcal{D})x + O(x^{5/12+\varepsilon})$$

holds for any $\mathcal{D} \in \mathcal{S}$. And by the arguments of Müller [12], the exponent $5/12$ in the above formula can be replaced by $9/22$ for any $\mathcal{D} \in \mathcal{S}$. We shall prove that for all $\mathcal{D} \in \mathcal{S}$, the exponent $9/22$ can be improved.

By Nowak [14, p. 502],

$$P_{\mathcal{D}}(x) = \sum_{j=1}^{J^*} e_j S_j(\sqrt{x}) + O(1), \tag{1.7}$$

for some finite integer $J^* > 0$, where

$$S_j(t) = \sum_{a_j t < n \le b_j t,\, n \in \mathbb{Z}} \psi\Big(t f_j\Big(\frac{n}{t}\Big)\Big), \tag{1.8}$$

ψ is a row-of-teeth function satisfying

$$\begin{aligned} \psi(t) &= t - [t] - 1/2 \quad && \text{for } t \notin \mathbb{Z}, \\ -1/2 \le \psi(t) &\le 1/2 && \text{for } t \in \mathbb{Z}, \end{aligned} \tag{1.9}$$

and for each $1 \le j \le J^*$, f_j is a real-valued function defined on an interval $[a_j, b_j]$ with continuous derivatives up to order 4 and f_j'' has no zero on $[a_j, b_j]$; e_j is $+$ or $-$. Define

$$\mathcal{G}_{\mathcal{D}} = \bigcup_{j=1}^{J^*} \{f_j'(a_j), f_j'(b_j)\}.$$

Now recall a few facts from the theory of Diophantine approximation: By the (approximation) *type* $t(\alpha)$ of an irrational real number α we denote the infimum of all reals r for which there exists a constant $c(r,\alpha)$ such that

$$|\alpha - p/q| \ge c(r,\alpha)/q^{r+1}$$

for all integers p and all positive integers q. Let

$$\mathbb{R}(1) = \{\alpha \in \overline{\mathbb{Q}} : t(\alpha) = 1\}.$$

By Roth's theorem [17], $\mathbb{R}(1)$ contains all algebraic irrationals. Also, the Lebesgue measure of $\mathbb{R}\setminus\mathbb{R}(1)$ is 0 (due to Khinchin). Namely, $\mathbb{R}(1)$ contains almost all irrationals.

Finally define

$$\alpha(\mathcal{D}) = \begin{cases} 0 & \text{if } \mathcal{G}_{\mathcal{D}} \subset \mathbb{Q}, \\ \dfrac{2c^2}{2c^2+3c+3} & \text{if } \mathcal{G}_{\mathcal{D}} \cap \overline{\mathbb{Q}} \text{ is not empty,} \end{cases}$$

where $c := \max\{t(\alpha) : \alpha \in \mathcal{G}_{\mathcal{D}} \cap \overline{\mathbb{Q}}\}$.

We then have the following

THEOREM 2. *Let $\mathcal{D} \in \mathcal{S}$. If RH is true, we have*

$$B_{\mathcal{D}}(x) = \frac{6}{\pi^2}\, a(\mathcal{D})x + O(x^{\frac{33349}{84040}+\varepsilon} + x^{\frac{749-146\alpha(\mathcal{D})}{2082-584\alpha(\mathcal{D})}+\varepsilon}).$$

Theorem 1 is a special case of Theorem 2. Since the tangent of the curve C is continuous, we can always divide C into finite pieces such that in each piece we have $\{f'(a), f'(b)\} \subset \mathbb{Q}$. Accordingly, $\mathcal{G}_{\mathcal{D}} \subset \mathbb{Q}$ and $\alpha(\mathcal{D}) = 0$.

COROLLARY 1.1. *For any $\mathcal{D} \in \mathcal{S}$, we have*

$$B_{\mathcal{D}}(x) = \frac{6}{\pi^2}\, a(\mathcal{D})x + O(x^{603/1498+\varepsilon}).$$

REMARK 1.2. Theorem 2 shows that there exists a constant $c_0 = 14.46\ldots$ such that for $c \le c_0$, the error term reads $O(x^{33349/84040+\varepsilon})$. This is true for almost all $\mathcal{D} \in \mathcal{S}$. The worst case is $O(x^{603/1498+\varepsilon})$.

For comparison, we have

$$\frac{41}{91} = 0.4505\ldots, \qquad \frac{5}{12} = 0.4166\ldots, \qquad \frac{9}{22} = 0.40909\ldots,$$
$$\frac{33349}{84040} = 0.3968\ldots, \qquad \frac{603}{1498} = 0.4025\ldots$$

Much better results can be obtained if $\mathcal{D}$ has a nice form. For example, we consider the case that $\mathcal{D}$ is a rational ellipse disc. In this case, Nowak [15] proved under RH that

$$B_{\mathcal{D}}(x) = \frac{6}{\pi^2}\, a(\mathcal{D})x + O(x^{15/38+\varepsilon}). \tag{1.10}$$

In particular, if $\mathcal{D}$ is the unit disc, Zhai and Cao [21] proved that the exponent 15/38 can be replaced by 11/30. And recently, Wu [20] obtained the exponent 221/608.

In this paper, we give the following Theorem 3 without proof since the proof is almost the same as that of Wu [20].

THEOREM 3. *Suppose $\mathcal{D}$ is a rational ellipse disc. If RH is true, then*

$$B_{\mathcal{D}}(x) = \frac{6}{\pi^2}\, a(\mathcal{D})x + O(x^{221/608+\varepsilon}).$$

It is also interesting to study the asymptotic behaviour of the quantity $B_{\mathcal{D}}(x+U)-B_{\mathcal{D}}(x)$, where U is another large real parameter but of order smaller than x. When $\mathcal{D}$ is a convex planar domain containing the origin in its interior with $\partial\mathcal{D}$ of class C^4 and has finite nonvanishing curvature throughout, Krätzel and Nowak [8] proved that

$$B_{\mathcal{D}}(x+x^\theta)-B_{\mathcal{D}}(x)=\frac{6}{\pi^2}\,a(\mathcal{D})x^\theta(1+o(1)) \tag{1.11}$$

for $\theta>11/29$. They also remarked that if $\mathcal{D}$ is the unit disc, then (1.11) is true for $\theta>29/80$.

In this paper we shall generalize this statement to any $\mathcal{D}\in\mathcal{S}$. Our argument is slightly different. We have the following theorems.

THEOREM 4. *Suppose $\mathcal{D}\in\mathcal{S}$. Then* (1.11) *is true for*

$$\theta>\max\left(\frac{11}{29},\frac{2}{6-\alpha(\mathcal{D})}\right).$$

COROLLARY 1.2. *For almost all $\mathcal{D}\in\mathcal{S}$,* (1.11) *is true for $\theta>11/29$.*

COROLLARY 1.3. *For any $\mathcal{D}\in\mathcal{S}$,* (1.11) *is true for $\theta>2/5$.*

THEOREM 5. *Suppose $Q(\mathbb{Z}_*^2)\subset\mathbb{N}$ and for any η,*

$$r_Q(n)=\sum_{n=Q(m,l)\in\mathbb{Z}_*^2}1\ll n^\eta.$$

Suppose further that $A_{\mathcal{D}}(x)=a(\mathcal{D})x+O(x^{\theta_0+\varepsilon})$ for some $\theta_0>1/4$. Then (1.11) *is true for $\theta>\theta_0$.*

COROLLARY 1.4. *Suppose $\mathcal{D}$ is a rational ellipse disc. Then* (1.11) *is true for $\theta>23/73$.*

Before going into technical details, we sketch the ideas of our proof. Write

$$\begin{aligned} B_{\mathcal{D}}(x) &= \sum_{m\le y}\mu(m)A_{\mathcal{D}}\left(\frac{x}{m^2}\right)+\sum_{m>y}\mu(m)A_{\mathcal{D}}\left(\frac{x}{m^2}\right) \\ &= a(\mathcal{D})x\sum_{m\le y}\mu(m)/m^2+S_1+S_2, \end{aligned} \tag{1.12}$$

where

$$S_1=\sum_{m\le y}\mu(m)P_{\mathcal{D}}\left(\frac{x}{m^2}\right), \tag{1.13}$$

$$S_2=\sum_{m>y}\mu(m)A_{\mathcal{D}}\left(\frac{x}{m^2}\right). \tag{1.14}$$

In order to deal with S_2, Moroz [11] used an elementary argument. Huxley and Nowak [7] used the Perron formula. Müller [12] used a similar argument to deal with S_2 and obtained a better result.

In order to deal with S_1, Moroz used the upper bound of $P_{\mathcal{D}}(x)$ directly. Huxley and Nowak [7] used the mean square estimate (1.2) to deal with S_1. Müller [12] used the seventh power moment of $P_{\mathcal{D}}(x)$.

The exponents $5/12$ and $9/22$ are sharp. Firstly, the zeta-function of $\mathcal{D}$ has no functional equation. Secondly, as mentioned by Huxley and Nowak in their paper, it is not clear how to use the method of exponential sums to deal with S_1.

In this paper, we shall use the method of exponential sums to deal with S_1. The work of Nowak [14] and Kühleitner and Nowak [9] supplies the foundations of using exponential sums.

In order to estimate S_2, we first obtain a better mean square estimate of the zeta-function of $\mathcal{D}$. Finally we can get a better estimate of S_2.

It should be mentioned that we can only get the upper bound $S_1 \ll x^{1/4+\varepsilon}y^{1/2}$ by using any power moment results for $P_{\mathcal{D}}(x)$ since the best possible upper bound for $P_{\mathcal{D}}(x)$ is $\ll x^{1/4+\varepsilon}$. However, using the method of exponential sums, we can get an upper bound smaller than $x^{1/4+\varepsilon}y^{1/2}$.

Notations. $\mathbb{Z}$ denotes the set of all integers, $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{Q}$ denotes the set of all rational numbers, $\overline{\mathbb{Q}}$ denotes the set of all irrational numbers, $\mathbb{R}(1)$ denotes the set of all irrational numbers with type 1; ε denotes a small positive constant which may be different at each occurrence; $e(t) = e^{2\pi it}$; $\|t\|$ means the distance between t and the integer nearest to t; $m \sim M$ means $M < m \leq 2M$; and $m \asymp M$ means $c_1M < m \leq c_2M$ for two positive constants $c_2 > c_1 > 0$. Finally define

$$E_{\mathcal{D}}(x) = B_{\mathcal{D}}(x) - \frac{6}{\pi^2}\, a(\mathcal{D})x.$$

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2. An estimation of a special sum. Suppose α is an irrational number, and $W > 0$ is a sufficiently large real number. In this section, we shall estimate the sum

$$B(\alpha; W) = \sum_{h=1}^{\infty} h^{-1} \min\left(\frac{1}{\|h\alpha\|}, \frac{W}{h^{1/2}}\right). \tag{2.1}$$

In this section we always suppose that $\varepsilon > 0$ is a fixed sufficiently small real number.

The following lemmas will be needed.

LEMMA 2.1 ([16, Lemma 19.1.5]). *Suppose*

$$\alpha = \frac{a}{q} + \frac{\theta}{q^2}, \qquad \gcd(a,q) = 1,\ q \ge 2,\ |\theta| \le 1. \tag{2.2}$$

Then

$$\sum_{1 \le h \le q/2} 1/\|h\alpha\| \ll q \log q. \ \blacksquare \tag{2.3}$$

LEMMA 2.2. *Suppose* (2.2) *holds. Then for any* $x > 0$ *and integer* $N \ge 2$,

$$\sum_{1 \le h \le N} \min(x, 1/\|h\alpha\|) \ll (N/q + 1)(x + q \log q). \tag{2.4}$$

Proof. This is contained in Lemma 19.1.4 of Pan and Pan [16]. ■

LEMMA 2.3. *Let* $r \ge 1$ *denote the type of* α. *Then there exists a constant* $C = C(\alpha, \varepsilon)$ *such that for any* $Y \ge C(\alpha, \varepsilon), \alpha$ *can be written in the form* (2.2) *with*

$$Y \le q \le Y^{r+\varepsilon/2}.$$

Proof. Let a_n/q_n be the nth convergent of α. Then

$$|\alpha - a_n/q_n| \le \frac{1}{q_n q_{n+1}}. \tag{2.5}$$

This formula can be found in Hua [5, Section 2 of Chapter 10]. By the definition of type we know that the inequality

$$|\alpha - a/q| \le q^{-(r+1+\varepsilon/2)}$$

has only finitely many solutions (a,q). So there exists a constant $C' = C'(\alpha, \varepsilon)$ such that for every $q_n \ge C'(\alpha, \varepsilon)$, we have

$$|\alpha - a_n/q_n| \ge q_n^{-(r+1+\varepsilon/2)}. \tag{2.6}$$

From (2.5) and (2.6), we see that if $q_n \ge C'(\alpha, \varepsilon)$, then

$$q_{n+1} \le q_n^{r+\varepsilon/2}. \tag{2.7}$$

Let q_{n_0} be the smallest q_n with $q_n \ge C'(\alpha, \varepsilon)$, and let $C(\alpha, \varepsilon) = q_{n_0+1}$.

Now suppose Y is a real number with $Y \ge C(\alpha, \varepsilon)$. There must exist an n_Y such that $q_{n_Y} \le Y \le q_{n_Y+1}$. Then by (2.7) we have

$$Y \le q_{n_Y+1} \le q_{n_Y}^{r+\varepsilon/2} \le Y^{r+\varepsilon/2}. \tag{2.8}$$

Taking $a = a_{n_Y+1}$ and $q = q_{n_Y+1}$ completes the proof of Lemma 2.3. ■

We now prove

LEMMA 2.4. *Suppose* $r \ge 1$ *denote the type of* α. *Then*

$$B(\alpha; W) \ll W^{\frac{2r^2}{2r^2+3r+3}+\varepsilon},$$

where the $\ll$ *constant depends on* α, r *and* ε.

Proof. By Lemma 2.3, α has the form (2.2) with

$$W^{\frac{2(r+1)}{2r^2+3r+3}} \le q \le W^{\frac{2(r+1)r}{2r^2+3r+3}+\varepsilon/2}.$$

Then $B(\alpha;W)$ can be written as

$$B(\alpha;W) = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4, \tag{2.9}$$

where

$$\Sigma_1 = \sum_{h\le q^{1/(r+1)}} h^{-1}\min\left(\frac{1}{\|h\alpha\|}, \frac{W}{h^{1/2}}\right),$$

$$\Sigma_2 = \sum_{q^{1/(r+1)}<h\le q/2} h^{-1}\min\left(\frac{1}{\|h\alpha\|}, \frac{W}{h^{1/2}}\right),$$

$$\Sigma_3 = \sum_{q/2<h\le W^{3/2}} h^{-1}\min\left(\frac{1}{\|h\alpha\|}, \frac{W}{h^{1/2}}\right),$$

$$\Sigma_4 = \sum_{h>W^{3/2}} h^{-1}\min\left(\frac{1}{\|h\alpha\|}, \frac{W}{h^{1/2}}\right).$$

By the definition of type again we get

$$\begin{aligned}\Sigma_1 &\ll \sum_{h\le q^{1/(r+1)}} \frac{1}{h\|h\alpha\|} = \sum_{h\le C'(\alpha,\varepsilon)} 1 + \sum_{C'(\alpha,\varepsilon)<h\le q^{1/(r+1)}} \frac{1}{h\|h\alpha\|} \\ &\ll \sum_{h\le q^{1/(r+1)}} h^{r-1+\varepsilon/2} \ll q^{r/(r+1)+\varepsilon/2} \ll W^{\frac{2r^2}{2r^2+3r+3}+\varepsilon}.\end{aligned} \tag{2.10}$$

By Lemma 2.1 we get

$$\begin{aligned}\Sigma_2 &\ll \sum_{q^{1/(r+1)}<h\le q/2} \frac{1}{h\|h\alpha\|} \ll q^{-1/(r+1)} \sum_{h\le q/2} \frac{1}{\|h\alpha\|} \\ &\ll q^{r/(r+1)}\log q \ll W^{\frac{2r^2}{2r^2+3r+3}+\varepsilon}.\end{aligned} \tag{2.11}$$

Let

$$\mathcal{H}_j = \{h : jq/2 < h \le (j+1)q/2\}, \qquad j = 1, \ldots, J = [3W^{3/2}/q].$$

Then by Lemma 2.2 we get

$$\begin{aligned}\Sigma_3 &\ll \sum_{j=1}^{J}\sum_{h\in\mathcal{H}_j} h^{-1}\min\left(\frac{1}{\|h\alpha\|}, \frac{W}{h^{1/2}}\right) \\ &\ll \sum_{j=1}^{J}\frac{1}{jq}\sum_{h\in\mathcal{H}_j}\min\left(\frac{1}{\|h\alpha\|}, \frac{W}{(jq)^{1/2}}\right) \ll \sum_{j=1}^{J}\frac{1}{jq}\left(\frac{W}{(jq)^{1/2}} + q\log q\right) \\ &\ll Wq^{-3/2} + \log W \log q \ll W^{\frac{2r^2}{2r^2+3r+3}+\varepsilon}.\end{aligned} \tag{2.12}$$

Trivially we get

$$\Sigma_4 \ll \sum_{h>W^{3/2}} Wh^{-3/2} \ll W^{1/4}. \tag{2.13}$$

Combining (2.9)–(2.13) completes the proof of Lemma 2.4. ■

3. Estimates of exponential sums. In this section we shall study the exponential sums which appear in estimating S_1. We first estimate the sum

$$S(D_0, H) = \sum_{d\sim D_0} a(d) \sum_{(m,h)\in\mathcal{T}} b(m,h)e(Ad^\alpha g(m,h)), \tag{3.1}$$

where $D_0 \geq 100$ and $H \geq 10$ are real numbers, $a(d) \ll 1$, $b(m,h) \ll 1$, $A \neq 0$ is a real number, α ($\neq 0, 1, 2, \ldots$) is a real number, $\mathcal{T}$ is a subset of

$$\{(u,v) : |u| \ll H,\ H < v \leq 2H\},$$

and $g(u,v) \neq 0$ is a real-valued function defined on $\mathcal{T}$ with $g(m,h) \asymp H$. Let $F = |A|D_0^\alpha H$. We suppose that $g(u,v)$ satisfies

CONDITION 3.1. *Let $N(H,\Delta)$ denote the number of lattice points $(m,h) \in \mathcal{T}$ with $H - \Delta \leq g(m,h) \leq H$, where $0 < \Delta \leq H/2$. Then*

$$N(H,\Delta) \ll H^{2/3} + H\Delta.$$

In order to estimate $S(D_0, H)$, we need the following lemmas.

LEMMA 3.1. *Suppose $0 < \Delta \leq H/2$ and let $N(\Delta)$ denote the number of quadruples (m_1, h_1, m_2, h_2) with $(m_i, h_i) \in \mathcal{T}$ and*

$$|g(m_1,h_1) - g(m_2,h_2)| \leq \Delta.$$

Then

$$N(\Delta) \ll H^{8/3} + H^3\Delta.$$

Proof. This follows from Condition 3.1. ■

LEMMA 3.2 ([4, Lemma 1]). *Let $N < N_1 \leq 2N$. Then*

$$\sum_{N<N_1\leq 2N} e(\lambda n^\alpha) \ll \min(N, |\lambda|^{-1}N^{1-\alpha}) + (|\lambda|N^\alpha)^{1/2}. \ ■$$

LEMMA 3.3 ([18, Lemma 3]). *Let*

$$L(Q) = \sum_{1\leq j\leq J} C_j Q^{c_j} + \sum_{1\leq k\leq K} D_k Q^{-d_k},$$

where $C_j, c_j, D_k, d_k > 0$. Then for any $0 < Q' \leq Q$, there is some $Q_1 \in [Q', Q]$ such that

$$L(Q_1) \ll \sum_{j=1}^{J}\sum_{k=1}^{K}(C_j^{d_k} D_k^{c_j})^{1/(c_j+d_k)} + \sum_{1\leq j\leq J} C_j Q'^{c_j} + \sum_{1\leq k\leq K} D_k Q^{-d_k}. \ ■$$

Now we prove

LEMMA 3.4. *We have*

$$\mathcal{L}^{-1}S(D_0,H) \ll F^{1/6}D_0^{2/3}H^{16/9} + D_0^{2/3}H^2 + D_0H^{4/3} + F^{-1/2}D_0H^2 + F^{1/4}D_0^{1/2}H^{5/3}$$

where $\mathcal{L} = \log FHD_0$.

Proof. We use the same argument as in Heath-Brown [4]. Suppose $1 \le Q \le H^{4/3}$ is a parameter to be determined. Also suppose $0 < g(m,h) \le CH$. For each $1 \le q \le Q$, define

$$E_q = \{(m,h) : (m,h) \in \mathcal{T},\ (q-1)CH/Q < g(m,h) \le qCH/Q\}.$$

Now we write

$$S(D_0,H) = \sum_{q=1}^{Q} \sum_{d\sim D_0} a(d) \sum_{(m,h)\in E_q} b(m,h)e(Ad^\alpha g(m,h)).$$

By Cauchy's inequality we get

$$\begin{aligned} (3.2)\quad |S(D_0,H)|^2 &\ll QD_0 \sum_q \sum_{d\sim D_0} \Big| \sum_{(m,h)\in E_q} b(m,h)e(Ad^\alpha g(m,h)) \Big|^2 \\ &\ll QD_0 \sum_q \sum_{\substack{(m_1,h_1)\in E_q\\(m_2,h_2)\in E_q}} |b(m_1,h_1)b(m_2,h_2)| \Big| \sum_{d\sim D_0} e(A\lambda d^\alpha) \Big|, \end{aligned}$$

where $\lambda = g(m_1,h_1) - g(m_2,h_2)$. An application of Lemma 3.2 yields

$$\begin{aligned} (3.3)\quad & |S(D_0,H)|^2 \\ &\ll QD_0 \sum_{\substack{(m_1,h_1)\in\mathcal{T}\\(m_2,h_2)\in\mathcal{T}}} \left(\min\left(D_0, \frac{D_0}{|A\lambda|D_0^\alpha}\right) + (|A\lambda|D_0^\alpha)^{1/2} \right), \end{aligned}$$

where m_1,h_1,m_2,h_2 are restricted by $|\lambda| \le CH/Q$. So the contribution of $(|A\lambda|D_0^\alpha)^{1/2}$ to $|S(D_0,H)|^2$ is

$$\begin{aligned} (3.4)\quad &\ll QD_0(F/Q)^{1/2}N(CH/Q) \\ &\ll QD_0(F/Q)^{1/2}H^2(H^{2/3} + H^2/Q) \ll Q^{-1/2}F^{1/2}D_0H^4, \end{aligned}$$

where we used Lemma 3.1 and the assumption $Q \ll H^{4/3}$.

For $|\lambda| \le (|A|D_0^\alpha)^{-1}$, the term D_0 in the minimum produces a contribution

$$\begin{aligned} (3.5)\quad QD_0^2N\left(\frac{1}{|A|D_0^\alpha}\right) &\ll QD_0^2H^2\left(H^{2/3} + \frac{H}{|A|D_0^\alpha}\right) \\ &\ll QD_0^2H^{8/3} + QD_0^2H^4F^{-1} \end{aligned}$$

by Lemma 3.1 again.

Divide the remaining range

$$(|A|D_0^{\alpha})^{-1} < |\lambda| \le CH/Q$$

into $O(\mathcal{L})$ intervals $\Delta/2 < |\lambda| \le \Delta$. We find that the term $D_0^{1-\alpha}|A\lambda|^{-1}$ in the minimum contributes

$$\begin{aligned}(3.6)\qquad &\ll QD_0^{2-\alpha}|A|^{-1}\mathcal{L}\max_{|\Delta|\ge(|A|D_0^{\alpha})^{-1}}\Delta^{-1}N(\Delta)\\ &\ll QD_0^2H^{8/3}\mathcal{L}+QD_0^2H^4F^{-1}\mathcal{L}.\end{aligned}$$

From (3.3)–(3.6) we get

$$(3.7)\qquad \mathcal{L}^{-1}|S(D_0,H)|^2 \ll QD_0^2H^{8/3}+QD_0^2H^4F^{-1}+Q^{-1/2}F^{1/2}D_0H^4.$$

Hence Lemma 3.4 follows from (3.7) via Lemma 3.3 by choosing a best $Q\in[1,H^{4/3}]$. ∎

Now we estimate the exponential sum

$$S^*(W,D_0)=\sum_{d\sim D_0}\mu(d)e\left(\frac{W}{d}\right),$$

where $W, D_0 \ge 10$ are two positive numbers with $D_0 \ll W^{1-\varepsilon}$.

LEMMA 3.5 ([3, Proposition 1]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be two finite sets of real numbers, $\mathcal{X}\subset[-X,X], \mathcal{Y}\subset[-Y,Y]$. Then for any complex functions $u(x)$ and $v(y)$ we have*

$$\begin{aligned}&\Big|\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}}u(x)v(y)e(xy)\Big|^2\\ &\qquad\le 20(1+XY)\sum_{\substack{x,x'\in\mathcal{X}\\ |x-x'|\le Y^{-1}}}|u(x)u(x')|\sum_{\substack{y,y'\in\mathcal{Y}\\ |y-y'|\le X^{-1}}}|v(y)v(y')|.\ ∎\end{aligned}$$

LEMMA 3.6. *Let $\alpha,\alpha_1,\alpha_2,z$ be real numbers such that $z\alpha\alpha_1\alpha_2\ne 0$, $\alpha\notin\mathbb{N}$. Let $M\ge 2, M_1\ge 1, M_2\ge 1$, and let a_m and $b_{m_1m_2}$ be complex numbers with $|a_m|\le 1$, $|b_{m_1m_2}|\le 1$. Let $F_1=|z|M^{\alpha}M_1^{\alpha_1}M_2^{\alpha_2}$. If $F_1\ge M_1M_2$, then*

$$\begin{aligned}&\sum_{m\sim M}\sum_{m_1\sim M_1}\sum_{m_2\sim M_2}a_mb_{m_1m_2}e(zm^{\alpha}m_1^{\alpha_1}m_2^{\alpha_2})\\ &\qquad\ll MM_1M_2\log 2MM_1M_2\{(M_1M_2)^{-1/2}+F_1^{1/6}M^{-1/3}(M_1M_2)^{-1/6}\}.\end{aligned}$$

Proof. This is Theorem 2 of Baker [1] with $(\kappa,\lambda)=(1/2,1/2)$. ∎

LEMMA 3.7. *Suppose $a_m \ll 1$ is any complex number. If $M\ll D_0^{1/3}$ and $D_0\ll MN\ll D_0$, then*

$$S_I=\sum_{m\sim M}a_m\sum_{n\sim N}e\left(\frac{W}{mn}\right)\ll D_0^2W^{-1}+W^{1/6}D_0^{7/12}.$$

Proof. This lemma follows by using the exponent pair $(1/6, 4/6)$ for the sum over n. ∎

LEMMA 3.8. *Suppose $a_m \ll 1$ and b_n are any complex numbers. If $D_0^{1/3} \ll M \ll D_0^{1/2}$ and $D_0 \ll MN \ll D_0$, then*

$$S_{II} = \sum_{m\sim M} a_m \sum_{n\sim N} b_n e\left(\frac{W}{mn}\right) \ll (W^{1/6}D_0^{7/12} + D_0^{5/6} + D_0^{3/2}W^{-1/2})\log D_0.$$

Proof. Let $F_1 = W/D_0$. If $F_1 < N$, then by Lemma 3.5 we get

$$S_{II} \ll MNF_1^{-1/2} \ll D_0^{3/2}W^{-1/2}.$$

If $F_1 \ge N$, by Lemma 3.6 we get (take $m_1 = 1, m_2 = n$)

$$S_{II} \log^{-1} D_0 \ll W^{1/6}D_0^{7/12} + D_0^{5/6}$$

if we notice $D_0^{1/3} \ll M \ll D_0^{1/2}$. ∎

Now we prove

LEMMA 3.9. *We have*

$$D_0^{-\varepsilon} S^*(W, D_0) \ll W^{1/6}D_0^{7/12} + D_0^{5/6} + D_0^{3/2}W^{-1/2}.$$

Proof. We use the skillful decomposition due to Montgomery and Vaughan [10] and write

$$S^*(W, D_0) = \Sigma_1 + \Sigma_2 + \Sigma_3,$$

say, where

$$\Sigma_1 = -\sum_{m\le U} \xi_m \sum_{n\sim D_0/m} e\left(\frac{W}{mn}\right),$$

$$\Sigma_2 = -\sum_{U<m\le U^2} \xi_m \sum_{n\sim D_0/m} e\left(\frac{W}{mn}\right), \quad \xi_m = \sum_{\substack{m=d_1d_2\\ d_1,d_2\le U}} \mu(d_1)\mu(d_2) \ll m^{\varepsilon},$$

$$\Sigma_3 = -\sum_{\substack{m>U,\, n>U\\ mn\sim D_0}} \mu(m)\eta_n e\left(\frac{W}{mn}\right), \quad \eta_n = \sum_{d|n,\, d\le U} \mu(d) \ll n^{\varepsilon}.$$

Take $U = D_0^{1/3}$. Now Lemma 3.9 follows by using Lemma 3.7 to estimate Σ_1 and using Lemma 3.8 to estimate Σ_2 and Σ_3. ∎

4. Estimation of S_1. In this section we estimate

$$S_1 = \sum_{d\le y} \mu(d) P_{\mathcal{D}}\left(\frac{x}{d^2}\right),$$

where $10 \le y \ll \sqrt{x}$ is a parameter to be determined.

By a splitting argument we get, for some $1 \ll D_0 \ll y$,

$$S_1 \log^{-1} x \ll |S_1(D_0, x)|, \tag{4.1}$$

where

$$S_1(D_0, x) = \sum_{d \sim D_0} \mu(d) P_{\mathcal{D}}\Big(\frac{x}{d^2}\Big).$$

By (1.1) we have

$$S_1(D_0, x) \ll x^{23/73} D_0^{27/73} \log^3 x. \tag{4.2}$$

This is our first estimate of $S_1(D_0, x)$.

By (1.7) we have

$$S_1(D_0, x) = \sum_{j=1}^{J^*} e_j \sum_{d \sim D_0} \mu(d) S_j\Big(\frac{\sqrt{x}}{d}\Big) + O(D_0). \tag{4.3}$$

So we only need to estimate

$$S_1(D_0; x, f, a, b) = \sum_{d \sim D_0} \mu(d) S\Big(\frac{\sqrt{x}}{d}\Big), \tag{4.4}$$

where $\{f, a, b\}$ is any one of $\{f_j, a_j, b_j\}_{j=1}^{J^*}$, and $S(u)$ is defined by (1.8) with the function f.

By Vaaler's result [19], we can write

$$\psi(u) = \sum_{1 \le |h| \le H_0} a(h) e(hu) + O\Big(\sum_{1 \le h \le H_0} b(h) e(hu)\Big) + O(1/H_0) \tag{4.5}$$

for any $H_0 \ge 2$, where $a(h) \ll 1/|h|, b(h) \ll 1/H_0$.

Suppose $x^\varepsilon \ll H_0 \ll x^{1/4}$ is a parameter to be determined. By (1.8) and (4.5) we have

$$\sum_{d \sim D_0} \mu(d) S\Big(\frac{\sqrt{x}}{d}\Big) = \Sigma_1 + O(\Sigma_2) + O\Big(\frac{\sqrt{x}}{H_0}\Big), \tag{4.6}$$

where

$$\Sigma_1 = \sum_{d \sim D_0} \mu(d) \sum_{1 \le |h| \le H_0} a(h) \sum_{a\sqrt{x}/d < n \le b\sqrt{x}/d} e\Big(h \frac{\sqrt{x}}{d} f\Big(\frac{nd}{\sqrt{x}}\Big)\Big),$$

$$\Sigma_2 = \sum_{d \sim D_0} \sum_{1 \le h \le H_0} b(h) \sum_{a\sqrt{x}/d < n \le b\sqrt{x}/d} e\Big(h \frac{\sqrt{x}}{d} f\Big(\frac{nd}{\sqrt{x}}\Big)\Big)$$

with $a(h) \ll 1/|h|, b(h) \ll 1/H_0$. We only estimate Σ_1. The proof of Σ_2 is similar and easier.

By a splitting argument we get

$$\Sigma_1 \log^{-1} x \ll |\Sigma_1(D_0, H, x)| \tag{4.7}$$

for some $1 \ll H \ll H_0$, where

$$\varSigma_1(D_0,H,x)=\sum_{d\sim D_0}\mu(d)\sum_{h\sim H}a^*(h)\sum_{a\sqrt{x}/d<n\le b\sqrt{x}/d}e\biggl(h\,\frac{\sqrt{x}}{d}\,f\biggl(\frac{nd}{\sqrt{x}}\biggr)\biggr)$$

with $a^*(h) \ll 1/h$. Now we use the B-process to the sum over n. Since Kühleitner and Nowak [9] has used this procedure, we use their result directly. By formula (3.2) of Kühleitner and Nowak [9], we get (take $t=\sqrt{x}/d$)

$$\varSigma_1(D_0,H,x)=\varSigma_{11}(D_0,H,x)+O(\varSigma_{12}(D_0,H,x))+O(D_0\log x), \tag{4.8}$$

where

$$\begin{aligned}\varSigma_{11}(D_0,H,x)&=x^{1/4}\sum_{d\sim D_0}\frac{\mu(d)}{d^{1/2}}\sum_{h\sim H}a^*(h)\,\frac{1}{h^{1/2}}\sideset{}{''}\sum_{-hf'(a)\le m\le -hf'(b)}\kappa(m,h)\\&\qquad\times e\biggl(\frac{\sqrt{x}\,G(m,h)}{d}-\frac{1}{8}\biggr),\\ \varSigma_{12}(D_0,H,x)&=\sum_{d\sim D_0}\sum_{h\sim H}h^{-1}(r_h(a)+r_h(b)),\end{aligned}$$

and where for $c=a$ or b,

$$r_h(c)=\begin{cases}0, & hf'(c)\in\mathbb{Z},\\ \min\biggl(\dfrac{1}{\|hf'(c)\|},\dfrac{x^{1/4}}{(dh)^{1/2}}\biggr), & \text{else},\end{cases}$$

and where $''$ means that if $-hf'(a)$ or $-hf'(b)$ is an integer, then the corresponding term should be weighted by $1/2$.

If $f'(a)$ and $f'(b)$ are both rational numbers, then

$$\varSigma_{12}(D_0,H,x)\ll D_0\log x. \tag{4.9}$$

If $\{f'(a),f'(b)\}\not\subset\mathbb{Q}$ then Lemma 2.4 yields

$$\varSigma_{12}(D_0,H,x)\ll x^{\alpha(\mathcal{D})/4+\varepsilon}D_0^{1-\alpha(\mathcal{D})/2}. \tag{4.10}$$

Now we estimate $\varSigma_{11}(D_0,H,x)$. Obviously

$$\varSigma_{11}(D_0,H,x)\ll\frac{x^{1/4}}{D_0^{1/2}H^{3/2}}\,|\varSigma_{11}^*(D_0,H,x)|, \tag{4.11}$$

where

$$\varSigma_{11}^*(D_0,H,x)=\sum_{d\sim D_0}\frac{\mu(d)D_0^{1/2}}{d^{1/2}}\sum_{(m,h)\in\mathcal{T}}\kappa^*(m,h)e\biggl(\frac{\sqrt{x}\,G(m,h)}{d}\biggr),$$

$$\mathcal{T}=\{(u,v)\in\mathbb{R}\times\mathbb{R}^+:-vf'(a)\le u\le -vf'(b),v\sim H\},$$

and $\kappa^*(m,h)\asymp 1, |G(m,h)|\le CH$ for some $C>0$.

By Lemma 4 of Nowak [14], we know that $G(m,h)$ satisfies Condition 3.1. Thus we can use Lemma 3.4 to bound $\Sigma_{11}^*(D_0,H,x)$. Taking $A=x^{1/2}$ and $\alpha=-1$ in Lemma 3.4 we get

$$\begin{aligned}(4.12)\quad &\Sigma_{11}(D_0,H,x)\log^{-2}x\\ &\ll D_0+x^{1/4}D_0^{1/6}H^{1/2}+x^{1/3}H^{4/9}+x^{3/8}D_0^{-1/4}H^{5/12}+x^{1/4}D_0^{1/2}H^{-1/6}.\end{aligned}$$

Now we use Lemma 3.9 to estimate $\Sigma_{11}^*(D_0,H,x)$ over d and estimate the sum over (m,h) trivially. We get

$$\Sigma_{11}(D_0,H,x)x^{-\varepsilon}\ll x^{1/3}D_0^{1/12}H^{2/3}+x^{1/4}D_0^{1/3}H^{1/2}+D_0. \tag{4.13}$$

From (4.12) and (4.13) we have

$$\begin{aligned}\Sigma_{11}(D_0,H,x)x^{-\varepsilon}\ll{}& D_0+x^{1/4}D_0^{1/6}H^{1/2}+x^{1/3}H^{4/9}\\ &+x^{3/8}D_0^{-1/4}H^{5/12}+E_1+E_2,\end{aligned}\tag{4.14}$$

where

$$\begin{aligned}E_1&=\min(x^{1/4}D_0^{1/2}H^{-1/6},x^{1/3}D_0^{1/12}H^{2/3})\\ &\le(x^{1/4}D_0^{1/2}H^{-1/6})^{4/5}(x^{1/3}D_0^{1/12}H^{2/3})^{1/5}=x^{4/15}D_0^{5/12},\\ E_2&=\min(x^{1/4}D_0^{1/2}H^{-1/6},x^{1/4}D_0^{1/3}H^{1/2})\\ &\le(x^{1/4}D_0^{1/2}H^{-1/6})^{3/4}(x^{1/4}D_0^{1/3}H^{1/2})^{1/4}=x^{1/4}D_0^{11/24}.\end{aligned}$$

Collecting (4.6)–(4.11) and (4.14) we get (note $H\ll H_0$)

$$\begin{aligned}x^{-\varepsilon}\sum_{d\sim D_0}\mu(d)S\left(\frac{\sqrt{x}}{d}\right)\ll{}& D_0+x^{1/4}D_0^{1/6}H_0^{1/2}+x^{1/3}H_0^{4/9}+x^{3/8}D_0^{-1/4}H_0^{5/12}\\ &+x^{1/2}H_0^{-1}+x^{4/15}D_0^{5/12}+x^{1/4}D_0^{11/24}+\delta(x,D_0)\end{aligned}$$

where

$$\delta(x,D_0)=\begin{cases}0 & \text{if } \{f'(a),f'(b)\}\subset\mathbb{Q},\\ x^{\alpha(\mathcal{D})/4}D_0^{1-\alpha(\mathcal{D})/2} & \text{otherwise.}\end{cases}$$

Now choose a best $H_0\in[x^{\varepsilon},x^{1/4}]$ via Lemma 3.3. We get

$$\begin{aligned}x^{-\varepsilon}\sum_{d\sim D_0}\mu(d)S\left(\frac{\sqrt{x}}{d}\right)\ll{}& D_0+x^{5/13}+x^{1/3}D_0^{1/9}+x^{7/17}D_0^{-3/17}\\ &+x^{4/15}D_0^{5/12}+x^{1/4}D_0^{11/24}+\delta(x,D_0),\end{aligned}$$

which combined with (4.3) yields

$$\begin{aligned}x^{-\varepsilon}S_1(D_0,x)\ll{}& D_0+x^{5/13}+x^{1/3}D_0^{1/9}+x^{7/17}D_0^{-3/17}\\ &+x^{4/15}D_0^{5/12}+x^{1/4}D_0^{11/24}+\delta^*(x,D_0),\end{aligned}\tag{4.15}$$

where

$$\delta^*(x, D_0) = \begin{cases} 0 & \text{if } \mathcal{G}_{\mathcal{D}} \subset \mathbb{Q}, \\ x^{\alpha(\mathcal{D})/4} D_0^{1-\alpha(\mathcal{D})/2} & \text{otherwise.} \end{cases}$$

This is our second estimate of $S_1(D_0, x)$.

From (4.1), (4.2) and (4.15) we finally get (note $D_0 \ll Y$) the estimate

$$\begin{aligned} x^{-\varepsilon} S_1 &\ll y + x^{1/3} y^{1/9} + x^{4/15} y^{5/12} + x^{1/4} y^{11/24} + \delta^*(x, D_0) \\ &\quad + x^{5/13} + \min(x^{7/17} D_0^{-3/17}, x^{23/73} D_0^{27/73}) \\ &\ll y + x^{1/3} y^{1/9} + x^{4/15} y^{5/12} + x^{1/4} y^{11/24} + \delta^*(x, y) + x^{5/13} \end{aligned} \tag{4.16}$$

for any $x^{\varepsilon} \ll y \ll x^{1/2-\varepsilon}$.

5. Estimation of S_2 and proof of Theorem 2. In this section we shall estimate S_2 and give the proof of Theorem 2. We need a new estimate of the mean square of the zeta-function $Z_{\mathcal{D}}(s)$ of $\mathcal{D}$.

LEMMA 5.1. *Suppose $\mathcal{D} \in \mathcal{S}, 1 \le t \le 10$. Then*

$$\sum_{Q(\mathbf{m}) \le X} P_{\mathcal{D}}(tQ(\mathbf{m})) \ll X^{87/68} \log^2 X.$$

Proof. Obviously, we can suppose $t = 1$. It suffices to show

$$\sum_{Q(\mathbf{m}) \sim N} P_{\mathcal{D}}(Q(\mathbf{m})) \ll N^{87/68} \log^2 N \tag{5.1}$$

for any $X^{1/2} \ll N \ll X$. By (1.7) we have

$$\sum_{Q(\mathbf{m}) \sim N} P_{\mathcal{D}}(Q(\mathbf{m})) \ll \sum_{j=1}^{J^*} \Big| \sum_{Q(\mathbf{m}) \sim N} S_j(Q^{1/2}(\mathbf{m})) \Big|, \tag{5.2}$$

where $S_j(u)$ is defined by (1.8). Let $\{f, a, b\}$ denote any one of $\{f_j, a_j, b_j\}_{j=1}^{J^*}$, and let $S(u)$ denote this $S_j(u)$.

Suppose H_0 is a parameter to be determined. Similarly to (4.6), we have

$$\sum_{Q(\mathbf{m}) \sim N} S_j(Q^{1/2}(\mathbf{m})) = \Sigma_1^* + O(\Sigma_2^*) + O\left(\frac{N^{3/2}}{H_0}\right), \tag{5.3}$$

where

$$\Sigma_1^* = \sum_{Q(\mathbf{m}) \sim N} \sum_{1 \le |h| \le H_0} a(h) \sum_{aQ^{1/2}(\mathbf{m}) < n \le bQ^{1/2}(\mathbf{m})} e\left(hQ^{1/2}(\mathbf{m}) f\left(\frac{n}{Q^{1/2}(\mathbf{m})}\right)\right),$$

$$\Sigma_2^* = \sum_{Q(\mathbf{m}) \sim N} \sum_{1 \le h \le H_0} b(h) \sum_{aQ^{1/2}(\mathbf{m}) < n \le bQ^{1/2}(\mathbf{m})} e\left(hQ^{1/2}(\mathbf{m}) f\left(\frac{n}{Q^{1/2}(\mathbf{m})}\right)\right),$$

with $a(h) \ll 1/|h|, b(h) \ll 1/H_0$. We only consider the contribution of Σ_1^*. The contribution of Σ_2^* is the same. By a splitting argument we have

$$\Sigma_1^* \ll |\Sigma_1^*(N,H)| \log N \tag{5.4}$$

for some $1 \ll H \ll H_0$, where

$$\begin{aligned}\Sigma_1^*(N,H) = &\sum_{Q(\mathbf{m})\sim N} \sum_{h\sim H} a^*(h) \\ &\times \sum_{aQ^{1/2}(\mathbf{m})<n\le bQ^{1/2}(\mathbf{m})} e\left(hQ^{1/2}(\mathbf{m}) f\left(\frac{n}{Q^{1/2}(\mathbf{m})}\right)\right)\end{aligned}$$

with $a^*(h) \ll 1/h$. Using the B-process to the sum over n, we get

$$\begin{aligned}\Sigma_1^*(N,H) = &\sum_{Q(\mathbf{m})\sim N} Q^{1/4}(\mathbf{m}) \sum_{h\sim H} \frac{a^*(h)}{h^{1/2}} \\ &\times \sideset{}{''}\sum_{-hf'(a)\le l\le -hf'(b)} \kappa(l,h) e(Q^{1/2}(\mathbf{m})G(l,h) - 1/8) + O(N^{5/4}) \\ \ll &\frac{N^{1/4}}{H^{3/2}} \max_{N<N'\le 2N} |S^*(N,H)| + N^{5/4},\end{aligned}$$

where

$$S^*(N,H) = \sum_{N<Q(\mathbf{m})\le N'} \sum_{(l,h)\in\mathcal{T}} b(l,h) e(Q^{1/2}(\mathbf{m})G(l,h))$$

with $b(l,h) \ll 1$ and $\mathcal{T}$ is defined by

$$\mathcal{T} = \{(u,v) \in \mathbb{R}\times\mathbb{R}^+ : -vf'(a) \le u \le -vf'(b), v \sim H\}.$$

Now we estimate $S^*(N,H)$ by the same argument of Lemma 3.4. Suppose $1 \le R \le H^{4/3}$ is a parameter to be determined. Also suppose $0 < G(l,h) \le CH$. For each $1 \le r \le Q$, define

$$E_r = \{(l,h) : (l,h) \in \mathcal{T}, (r-1)CH/Q < G(l,h) \le rCH/Q\}.$$

Now we write

$$S^*(N,H) = \sum_{r=1}^{R} \sum_{N<Q(\mathbf{m})\le N'} \sum_{(l,h)\in E_r} b(l,h) e(Q^{1/2}(\mathbf{m})G(l,h)).$$

By Cauchy's inequality we get

$$|S^*(N,H)|^2 \ll RN \sum_r \sum_{N<Q(\mathbf{m})\le N'} \Big| \sum_{(l,h)\in E_r} b(l,h) e(Q^{1/2}(\mathbf{m})G(l,h)) \Big|^2 \tag{5.5}$$

$$\ll RN \sum_q \sum_{\substack{(l_1,h_1)\in E_r\\(l_2,h_2)\in E_r}} |b(l_1,h_1)b(l_2,h_2)| \Big| \sum_{N<Q(\mathbf{m})\le N'} e(Q^{1/2}(\mathbf{m})\lambda)\Big|,$$

where $\lambda = G(l_1,h_1) - G(l_2,h_2)$. Notice $|\lambda| \ll H/R$.

By Stieltjes integration we have

$$\begin{aligned}\sum_{N<Q(\mathbf{m})\le N'} e(Q^{1/2}(\mathbf{m})\lambda) &= \int_N^{N'} e(\sqrt{u}\,\lambda)\, dA_{\mathcal{D}}(u)\\ &= a(\mathcal{D})\int_N^{N'} e(\sqrt{u}\,\lambda)\, du - \pi i\lambda \int_N^{N'} P_{\mathcal{D}}(u)e(\sqrt{u}\,\lambda)u^{-1/2}\, du\\ &\quad + P_{\mathcal{D}}(u)e(\sqrt{u}\,\lambda)|_N^{N'}\\ &= \int_1 + \int_2 + O(N^{1/3}),\end{aligned}$$

say, where we used the estimate $P_{\mathcal{D}}(u) \ll u^{1/3}$. We have

$$\int_1 \ll \min(N, N^{1/2}/|\lambda|).$$

By (1.2) we have (for the proof of (1.2) for general $\mathcal{D}\in\mathcal{S}$, see Nowak [14])

$$\int_2 \ll |\lambda| N^{3/4}.$$

This yields

$$\begin{aligned}\sum_{N<Q(\mathbf{m})\le N'} e(Q^{1/2}(\mathbf{m})\lambda) &\ll \min(N, N^{1/2}/|\lambda|) + |\lambda|N^{3/4} + N^{1/3}\\ &\ll \min(N, N^{1/2}/|\lambda|) + |\lambda|N^{3/4},\end{aligned} \tag{5.6}$$

where $N^{1/3}$ can be absorbed because

$$\min(N, N^{1/2}/|\lambda|) + |\lambda|N^{3/4} \gg N^{5/8}.$$

Similarly to the proof of Lemma 3.4, the contribution of $\min(N, N^{1/2}/|\lambda|)$ to $|S^*(N,H)|^2$ is

$$\ll (RN^2H^{8/3} + RN^{3/2}H^3)\log N.$$

The contribution of $|\lambda|N^{3/4}$ to $|S^*(N,H)|^2$ is

$$\ll R^{-1}N^{7/4}H^5.$$

Combining the above we get

$$|S^*(N,H)|^2 \log^{-1} N \ll RN^2H^{8/3} + RN^{3/2}H^3 + R^{-1}N^{7/4}H^5.$$

Choosing a best $R\in[10, H^{4/3}]$ via Lemma 3.3 we get

$$\begin{aligned}(5.7)\qquad S^*(N,H)\log^{-1}N &\ll N^{15/16}H^{23/12}+N^{13/16}H^2+NH^{4/3}+N^{3/4}H^{3/2}+N^{7/8}H^{11/6}\\ &\ll N^{15/16}H^{23/12}+NH^{4/3},\end{aligned}$$

if we notice

$$N^{3/4}H^{3/2}\ll N^{13/16}H^2\ll N^{15/16}H^{23/12},\qquad N^{7/8}H^{11/6}\ll N^{15/16}H^{23/12}.$$

Inserting (5.7) into (5.5), we get

$$\Sigma_1^*(N,H)\log^{-1}N\ll N^{19/16}H^{5/12}+N^{5/4}. \tag{5.8}$$

Now Lemma 5.1 follows from (5.1)–(5.4) and (5.8) by choosing $H_0=N^{15/68}$.

LEMMA 5.2. *Suppose $\mathcal{D}\in\mathcal{S}$. For any $T\geq 10$, we have*

$$\int_T^{2T}|Z_{\mathcal{D}}(749/1168+it)|^2\,dt\ll T\log^3 T.$$

Proof. The zeta-function of $\mathcal{D}$ is defined by

$$Z_{\mathcal{D}}(s)=\sum_{\mathbf{m}\in\mathbb{Z}_*^2}Q^{-s}(\mathbf{m}),$$

which is absolutely convergent for $\Re s>1$. Suppose X is a large real number not attainable by $Q(\mathbf{m})$ as $\mathbf{m}$ runs through $\mathbb{Z}_*^2$. For $\Re s>1$, by Stieltjes integration we have

$$\begin{aligned}(5.9)\qquad Z_{\mathcal{D}}(s)&=\sum_{Q(\mathbf{m})\leq X}Q^{-s}(\mathbf{m})+\int_X^\infty\frac{dA_{\mathcal{D}}(\omega)}{\omega^s}\\ &=\sum_{Q(\mathbf{m})\leq X}Q^{-s}(\mathbf{m})+a(\mathcal{D})\,\frac{X^{1-s}}{s-1}-\frac{P_{\mathcal{D}}(X)}{X^s}+s\int_X^\infty\frac{P_{\mathcal{D}}(\omega)}{\omega^{s+1}}\,d\omega.\end{aligned}$$

By (1.2) we know that $Z_{\mathcal{D}}(s)$ has a continuation to the half-plane $\Re s>1/4$ with a simple pole at $s=1$ with residue $a(\mathcal{D})$.

Suppose $87/136<\sigma<1$ is fixed, $100T\leq X\leq T^2$ is a parameter to be determined. Then by (5.9) we have

$$\int_T^{2T}|Z_{\mathcal{D}}(\sigma+it)|^2\,dt\ll W_1+T^2W_2+X^{2-2\sigma}T^{-1}+T, \tag{5.10}$$

where

$$W_1=\int_T^{2T}\Big|\sum_{Q(\mathbf{m})\leq X}\frac{1}{Q^{\sigma+it}(\mathbf{m})}\Big|^2\,dt,\qquad W_2=\int_T^{2T}\Big|\int_X^\infty\frac{P_{\mathcal{D}}(\omega)}{\omega^{\sigma+1+it}}\Big|^2\,dt.$$

We first estimate W_1. Squaring and integrating, we have

$$\begin{aligned} W_1 &\ll T \sum_{Q(\mathbf{m})\le X} Q^{-2\sigma}(\mathbf{m}) \\ &\quad + \sum_{Q(\mathbf{m})<Q(\mathbf{n})\le X} Q^{-\sigma}(\mathbf{m})Q^{-\sigma}(\mathbf{n}) \min\left(T, \frac{1}{\log \frac{Q(\mathbf{n})}{Q(\mathbf{m})}}\right) \\ &= \Sigma_1 + \Sigma_2. \end{aligned} \tag{5.11}$$

For Σ_1, we have

$$\Sigma_1 \ll T \int_1^X u^{-2\sigma}\, dA_{\mathcal{D}}(u) \ll T. \tag{5.12}$$

We write Σ_2 as

$$\Sigma_2 = \Sigma_{21} + \Sigma_{22} + \Sigma_{23}, \tag{5.13}$$

where

$$\begin{aligned} \Sigma_{21} &= T \sum_{Q(\mathbf{m})\le X} Q^{-\sigma}(\mathbf{m}) \sum_{Q(\mathbf{m})<Q(\mathbf{n})\le e^{1/T}Q(\mathbf{m})} Q^{-\sigma}(\mathbf{n}), \\ \Sigma_{22} &= \sum_{Q(\mathbf{m})\le X} Q^{-\sigma}(\mathbf{m}) \sum_{e^{1/T}Q(\mathbf{m})<Q(\mathbf{n})\le 2Q(\mathbf{m})} Q^{-\sigma}(\mathbf{n}) \frac{1}{\log \frac{Q(\mathbf{n})}{Q(\mathbf{m})}}, \\ \Sigma_{23} &= \sum_{Q(\mathbf{m})\le X} Q^{-\sigma}(\mathbf{m}) \sum_{Q(\mathbf{n})>2Q(\mathbf{m})} Q^{-\sigma}(\mathbf{n}) \frac{1}{\log \frac{Q(\mathbf{n})}{Q(\mathbf{m})}}. \end{aligned}$$

For Σ_{23}, we trivially have

$$\Sigma_{23} \ll \Big(\sum_{Q(\mathbf{m})\le X} Q^{-\sigma}(\mathbf{m})\Big)^2 \ll X^{2-2\sigma}. \tag{5.14}$$

For Σ_{21}, we have

$$\begin{aligned} \Sigma_{21} &\ll T \sum_{Q(\mathbf{m})\le X} Q^{-2\sigma}(\mathbf{m}) \sum_{Q(\mathbf{m})<Q(\mathbf{n})\le e^{1/T}Q(\mathbf{m})} 1 \\ &= T \sum_{Q(\mathbf{m})\le X} Q^{-2\sigma}(\mathbf{m})(A_{\mathcal{D}}(e^{1/T}Q(\mathbf{m})) - A_{\mathcal{D}}(Q(\mathbf{m}))) \\ &\ll T \sum_{Q(\mathbf{m})\le X} Q^{-2\sigma}(\mathbf{m})((e^{1/T}-1)Q(\mathbf{m})) \\ &\quad + T\Big|\sum_{Q(\mathbf{m})\le X} P_{\mathcal{D}}(e^{1/T}Q(\mathbf{m}))Q^{-2\sigma}(\mathbf{m})\Big| \\ &\quad + T\Big|\sum_{Q(\mathbf{m})\le X} P_{\mathcal{D}}(Q(\mathbf{m}))Q^{-2\sigma}(\mathbf{m})\Big| = \Sigma_{21}^1 + \Sigma_{21}^2 + \Sigma_{21}^3, \end{aligned} \tag{5.15}$$

say. We have

$$\Sigma_{21}^1 \ll T(e^{1/T}-1)\sum_{Q(\mathbf{m})\le X} Q^{1-2\sigma}(\mathbf{m}) \ll X^{2-2\sigma}. \tag{5.16}$$

Let

$$F(t)=\sum_{Q(\mathbf{m})\le t} P_{\mathcal{D}}(e^{1/T}Q(\mathbf{m})).$$

Then by Lemma 5.1 we have

$$F(t)\ll t^{87/68}\log^2 t.$$

Thus we have

$$\Sigma_{21}^2 \ll T\int_1^X \frac{dF(t)}{t^{2\sigma}} \ll T\log^2 T. \tag{5.17}$$

Similarly we have

$$\Sigma_{21}^3 \ll T\log^2 T. \tag{5.18}$$

Now we estimate Σ_{22}. Let

$$\delta_j = 2^j Q(\mathbf{m})T^{-1}, \qquad j=0,1,\ldots,J_0=[\log T/\log 2].$$

Then

$$\begin{aligned}
\Sigma_{22} &\ll \sum_{Q(\mathbf{m})\le X} Q^{-2\sigma}(\mathbf{m}) \sum_{e^{1/T}Q(\mathbf{m})<Q(\mathbf{n})\le 2Q(\mathbf{m})} \frac{1}{\log\frac{Q(\mathbf{n})}{Q(\mathbf{m})}} \\
&\ll \sum_{Q(\mathbf{m})\le X} Q^{1-2\sigma}(\mathbf{m}) \sum_{e^{1/T}Q(\mathbf{m})<Q(\mathbf{n})\le 2Q(\mathbf{m})} \frac{1}{Q(\mathbf{n})-Q(\mathbf{m})} \\
&\ll \sum_{Q(\mathbf{m})\le X} Q^{1-2\sigma}(\mathbf{m}) \sum_{j=0}^{J_0} \sum_{\delta_j<Q(\mathbf{n})-Q(\mathbf{m})\le\delta_{j+1}} \frac{1}{Q(\mathbf{n})-Q(\mathbf{m})} \\
&\ll \sum_{Q(\mathbf{m})\le X} Q^{1-2\sigma}(\mathbf{m}) \sum_{j=0}^{J_0} \frac{1}{\delta_j} \sum_{Q(\mathbf{m})+\delta_j<Q(\mathbf{n})\le Q(\mathbf{m})+\delta_{j+1}} 1 \\
&= \sum_{Q(\mathbf{m})\le X} Q^{1-2\sigma}(\mathbf{m}) \sum_{j=0}^{J_0} \frac{1}{\delta_j}\, a(\mathcal{D})(\delta_{j+1}-\delta_j) \\
&\quad + \sum_{Q(\mathbf{m})\le X} Q^{1-2\sigma}(\mathbf{m}) \\
&\quad \times \sum_{j=0}^{J_0} \frac{1}{\delta_j}(P_{\mathcal{D}}(Q(\mathbf{m})+\delta_{j+1}) - P_{\mathcal{D}}(Q(\mathbf{m})+\delta_j))
\end{aligned} \tag{5.19}$$

$$\ll \sum_{Q(\mathbf{m})\le X} Q^{1-2\sigma}(\mathbf{m}) \sum_{j=0}^{J_0} \frac{\delta_{j+1}-\delta_j}{\delta_j}$$
$$+ \Big| \sum_{Q(\mathbf{m})\le X} Q^{1-2\sigma}(\mathbf{m}) \sum_{j=0}^{J_0} \frac{P_{\mathcal{D}}(Q(\mathbf{m})+\delta_j)}{\delta_j} \Big| = \Sigma_{22}^1 + \Sigma_{22}^2,$$

if we notice $\delta_{j+1} \sim \delta_j$.

We have

$$\Sigma_{22}^1 \ll \sum_{Q(\mathbf{m})\le X} Q^{1-2\sigma}(\mathbf{m}) \log T \ll X^{2-2\sigma} \log T. \tag{5.20}$$

For Σ_{22}^2, we have

$$\begin{aligned}\Sigma_{22}^2 &\ll \Big| \sum_{Q(\mathbf{m})\le X} Q^{1-2\sigma}(\mathbf{m}) \sum_{j=0}^{J_0} \frac{T}{2^j Q(\mathbf{m})} P_{\mathcal{D}}((1+2^j T^{-1})Q(\mathbf{m}))\Big| \\ &\ll T \sum_{j=0}^{J_0} 2^{-j} \Big| \sum_{Q(\mathbf{m})\le X} Q^{-2\sigma}(\mathbf{m}) P_{\mathcal{D}}((1+2^j T^{-1})Q(\mathbf{m}))\Big| \\ &\ll T \log T,\end{aligned} \tag{5.21}$$

where we used the same argument of (5.17) with the help of Lemma 5.1.

From (5.11)–(5.21), we get

$$W_1 \ll X^{2-2\sigma} \log X + T \log^2 T. \tag{5.22}$$

For W_2, we have the estimate

$$W_2 \log^{-3} X \ll X^{-2\sigma+\gamma+1/4}, \tag{5.23}$$

where γ denotes the smallest α such that $P_{\mathcal{D}}(x) \ll x^{\alpha}$ holds. This estimate (5.23) is formula (18) of Müller [12]. Note that X^{ε} therein can be replaced by $\log^3 X$.

Now Lemma 5.2 follows from (5.10), (5.22) and (5.23) by taking $\sigma = 749/1168$, $X \sim T^{584/419}$ and $\gamma = 23/73$. ∎

In order to estimate S_2, we need the following

LEMMA 5.3. *Suppose $\mathcal{D} \in \mathcal{S}$. If RH is true and*

$$\int_T^{2T} |Z_{\mathcal{D}}(\sigma+it)|^2\,dt \ll T^{1+\varepsilon}$$

for some $\sigma \ge 1/2$, then

$$S_2 = a(\mathcal{D})x \sum_{m>y} \frac{\mu(m)}{m^2} + x^{1/3+\varepsilon} + x^{\sigma+\varepsilon} y^{1/2-2\sigma}.$$

The proof of this lemma is contained in Huxley and Nowak [7].

From Lemmas 5.2 and 5.3 we immediately get

PROPOSITION 5.1. *Suppose $\mathcal{D} \in \mathcal{S}$. If RH is true, then*

$$S_2 = a(\mathcal{D})x \sum_{m>y} \frac{\mu(m)}{m^2} + x^{1/3+\varepsilon} + x^{749/1168+\varepsilon} y^{-914/1168}.$$

Now we prove Theorem 2. By (4.16) and Proposition 5.1, the estimate

$$\begin{aligned} x^{-\varepsilon} E_{\mathcal{D}}(x) \ll\ & y + x^{1/3} y^{1/9} + x^{4/15} y^{5/12} + x^{1/4} y^{11/24} \\ & + \delta^*(x,y) + x^{5/13} + x^{749/1168} y^{-914/1168} \end{aligned} \tag{5.24}$$

holds for any $x^\varepsilon \ll y \ll x^{1/2-\varepsilon}$. Now Theorem 2 follows from (5.24) by choosing a best y via Lemma 3.3.

6. Proof of Theorem 4. Suppose $U = x^\theta,\ 0 < \theta < 1$. We have

$$\begin{aligned} B_{\mathcal{D}}(x + x^\theta) - B_{\mathcal{D}}(x) &= \sum_{\substack{x < Q(m,n) \le x+U \\ \gcd(m,n)=1}} 1 = \sum_{x < d^2 Q(m,n) \le x+U} \mu(d) \\ &= \sum_{d \le x^\varepsilon} + \sum_{d > x^\varepsilon}, \end{aligned} \tag{6.1}$$

where $\varepsilon > 0$ is a sufficiently small fixed real number. By (1.1) we have

$$\begin{aligned} \sum_{d \le x^\varepsilon} &= \sum_{d \le x^\varepsilon} \mu(d) \left(A_{\mathcal{D}}\left(\frac{x+U}{d^2}\right) - A_{\mathcal{D}}\left(\frac{x}{d^2}\right) \right) \\ &= \frac{6}{\pi^2} a(\mathcal{D}) U + O(U x^{-\varepsilon} + x^{23/73+\varepsilon}). \end{aligned} \tag{6.2}$$

For $\sum_{d > x^\varepsilon}$, we have

$$\begin{aligned} \Big| \sum_{d > x^\varepsilon} \Big| &\le \sum_{\substack{x < d^2 Q(m,n) \le x+U \\ d > x^\varepsilon}} 1 = \sum_{\substack{x < d^2 Q(m,n) \le x+U \\ x^\varepsilon < d \le y_1}} 1 + \sum_{\substack{x < d^2 Q(m,n) \le x+U \\ d > y_1}} 1 \\ &= \Sigma_1 + \Sigma_2, \end{aligned} \tag{6.3}$$

where $x^\varepsilon \ll y_1 \ll x^{1/2}$ is a parameter to be determined.

We first estimate Σ_1. We have

$$\begin{aligned} \Sigma_1 &= \sum_{x^\varepsilon < d \le y_1} \left(A_{\mathcal{D}}\left(\frac{x+U}{d^2}\right) - A_{\mathcal{D}}\left(\frac{x}{d^2}\right) \right) \\ &\ll U x^{-\varepsilon} + \max_{x \le x_0 \le x+U} \Big| \sum_{x^\varepsilon < d \le y_1} P_{\mathcal{D}}\left(\frac{x_0}{d^2}\right) \Big| \\ &\ll U x^{-\varepsilon} + \max_{x \le x_0 \le x+U} \Big| \sum_{d \sim D_0} P_{\mathcal{D}}\left(\frac{x_0}{d^2}\right) \Big| \log x \end{aligned} \tag{6.4}$$

for some $x^\varepsilon \ll D_0 \ll y_1$. It suffices to estimate

$$S_1^*(D_0,x)=\sum_{d\sim D_0} P_{\mathcal{D}}\Big(\frac{x}{d^2}\Big).$$

Since the procedure is the same as that for $S_1(D_0,x)$ in Section 4 if we replace $\mu(d)$ by 1, we only give the final estimate. The error term of the B-process will produce the contribution $\ll D_0\log x$ if $\mathcal{G}_{\mathcal{D}}\subset\mathbb{Q}$ and the contribution $x^{\alpha(\mathcal{D})/4+\varepsilon}D_0^{1-\alpha(\mathcal{D})/2}$ if $\mathcal{G}_{\mathcal{D}}\cap\overline{\mathbb{Q}}$ is nonempty.

The B-process will also produce the exponential sum

$$\Sigma_{11}^{**}(D_0,H,x)=\sum_{d\sim D_0}\frac{D_0^{1/2}}{d^{1/2}}\sum_{(m,h)\in\mathcal{T}}\kappa^*(m,h)e\bigg(\frac{\sqrt{x}\,G(m,h)}{d}\bigg).$$

We use the exponent pair $(2/18, 13/18)$ to estimate the sum over d and then choose a best H_0.

We finally get

$$x^{-\varepsilon}\Sigma_1\ll x^{11/29}+y_1+\delta^*(x,y_1), \tag{6.5}$$

where $\delta^*(x,y)$ was defined in last section.

For Σ_2, we trivially have

$$\begin{aligned}\Sigma_2 &\ll \sum_{Q(m,n)\ll x/y_1^2}\bigg(\bigg[\Big(\frac{x+U}{Q(m,n)}\Big)^{1/2}\bigg]-\bigg[\Big(\frac{x}{Q(m,n)}\Big)^{1/2}\bigg]\bigg)\\ &\ll \sum_{Q(m,n)\ll x/y_1^2}\bigg(\frac{Ux^{-1/2}}{Q^{1/2}(m,n)}+1\bigg)\ll Uy_1^{-1}+xy_1^{-2}.\end{aligned} \tag{6.6}$$

Now Theorem 4 follows from (6.1)–(6.6) by taking $y_1=x^{\frac{4-\alpha(\mathcal{D})}{12-2\alpha(\mathcal{D})}}$.

7. Proof of Theorem 5. We use the notations in the last section. If

$$A_{\mathcal{D}}(x)=a(\mathcal{D})x+O(x^{\theta_0+\varepsilon}),$$

then

$$\sum_{d\le x^\varepsilon}=\frac{6}{\pi^2}\,a(\mathcal{D})U+O(Ux^{-\varepsilon}+x^{\theta_0+\varepsilon}). \tag{7.1}$$

For $\sum_{d>x^\varepsilon}$, we have

$$\Big|\sum_{d>x^\varepsilon}\Big|\le\sum_{\substack{x<d^2Q(m,n)\le x+U\\ d>x^\varepsilon}}1=\sum_{\substack{x<d^2n\le x+U\\ d>x^\varepsilon}} r_Q(n), \tag{7.2}$$

where

$$r_Q(n)=\sum_{n=Q(m,l)}1.$$

By the estimate $r_Q(n) \ll n^{\varepsilon^2}$ we get

$$\Big| \sum_{d>x^\varepsilon} \Big| \ll x^{\varepsilon^2} \sum_{\substack{x<d^2n\le x+U \\ d>x^\varepsilon}} 1. \tag{7.3}$$

Now the problem is reduced to estimating the sum on the right side of (7.3). For this sum, we have the following Lemma 7.1, which is contained in the proof of Theorem 1 of Filaseta and Trifonov [2].

LEMMA 7.1. *We have*

$$\sum_{\substack{x<d^2n\le x+U \\ d>x^\varepsilon}} 1 \ll Ux^{-\varepsilon} + x^{1/5+\varepsilon}.$$

Lemma 7.1 implies that

$$\Big| \sum_{d>x^\varepsilon} \Big| \le Ux^{-\varepsilon/2} + x^{1/5+2\varepsilon}. \tag{7.4}$$

Now Theorem 5 follows from (7.1) and (7.4).

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