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# Linear mod one transformations and the distribution of fractional parts $\{\xi(p/q)^n\}$

by

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**1. Introduction.** It is well known (see e.g. [7, Chapter 1, Corollary 4.2]) that for almost all real numbers $\theta \geq 1$ the sequence $\{\theta^n\}$ is uniformly distributed in $[0,1]$. Here and in what follows, $\{\cdot\}$ denotes the fractional part. However, very few results are known for specific values of $\theta$, and the distribution of $\{(p/q)^n\}$ for coprime positive integers $p > q \geq 2$ remains an unsolved problem. Vijayaraghavan [10] showed that this sequence has infinitely many limit points, but we are unable to decide whether

$$\limsup_{n\to\infty}\left\{\left(\frac{p}{q}\right)^n\right\} - \liminf_{n\to\infty}\left\{\left(\frac{p}{q}\right)^n\right\} > \frac{1}{2}.$$

A striking progress has recently been made by Flatto, Lagarias & Pollington [5], who proved that, for all positive real numbers $\xi$, we have

$$\limsup_{n\to\infty}\left\{\xi\left(\frac{p}{q}\right)^n\right\} - \liminf_{n\to\infty}\left\{\xi\left(\frac{p}{q}\right)^n\right\} \geq \frac{1}{p}. \tag{1}$$

They were inspired by a paper of Mahler [8], who studied the hypothetical existence of so-called $Z$-numbers, i.e. positive real numbers $\xi$ such that $0 \leq \{\xi(3/2)^n\} < 1/2$ for all integers $n \geq 0$. Extending this definition, Flatto *et al.* introduce, for an interval $[s, s+t[$ included in $[0,1[$, the set

$$Z_{p/q}(s, s+t) := \left\{\xi \in \mathbb{R} : s \leq \left\{\xi\left(\frac{p}{q}\right)^n\right\} < s+t \text{ for all } n \geq 0\right\}.$$

To prove (1), they show that the set of $s$ such that $Z_{p/q}(s, s+1/p)$ is empty is dense in $[0, 1-1/p]$. Their argument uses Mahler's method, as explained in a preliminary work by Flatto [4] (for more bibliographical references, we refer the reader to [4] and [5]), and also relies on a careful study of contracting linear transformations which are very close to those investigated in [2] and [3].

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The purpose of the present work is to show how the methods used in [2] and [3] apply to the transformations considered in [5], and to derive some interesting consequences. For instance, we prove that $Z_{p/q}(s, s+1/p)$ is empty for almost all (in the sense of Lebesgue measure) real numbers $s$ in $[0, 1-1/p]$ and we answer both questions posed at the end of [5].

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**2. Statement of the results.** Before stating our results, we introduce some notation, which will be used throughout. Let $\tau$ be a real number with $0 \le \tau < 1$. For any integer $k$, set

$$\varepsilon_k(\tau) = [k\tau] - [(k-1)\tau],$$

where $[\cdot]$ denotes the integer part. The sequence $(\varepsilon_k(\tau))_{k\in\mathbb{Z}}$ only takes values 0 and 1 and, for $\tau$ irrational, it is usually called the *characteristic Sturmian sequence* associated to $\tau$. For any nonzero rational $a/b$, with $a$ and $b$ coprime, the sequence $(\varepsilon_k(a/b))_{k\in\mathbb{Z}}$ is periodic with period $b$.

Our first result concerns the sets $Z_{p/q}(s, s+1/p)$ and complements a result of Flatto *et al.* who proved in [5] that the set of $s$ in $[0, 1-1/p]$ for which $Z_{p/q}(s, s+1/p)$ is empty is a dense set.

THEOREM 1. *Let $p > q \ge 2$ be coprime integers. Then the set $Z_{p/q}(s, s+1/p)$ is empty for a set of $s$ of full Lebesgue measure in $[0, 1-1/p]$ . More precisely, this set is empty when there exists a rational number $a/b$, with $b > a \ge 1$, such that*

$$\frac{\sum_{k=1}^{b-2} \varepsilon_{-k}(a/b)\left(\frac{q}{p}\right)^k + \left(\frac{q}{p}\right)^b}{1 + \frac{q}{p} + \cdots + \left(\frac{q}{p}\right)^{b-1}} \le \{(p-q)s\} \le \frac{\sum_{k=1}^{b-2} \varepsilon_{-k}(a/b)\left(\frac{q}{p}\right)^k + \left(\frac{q}{p}\right)^{b-1}}{1 + \frac{q}{p} + \cdots + \left(\frac{q}{p}\right)^{b-1}}.$$

*Further, if for some $s$ in $[0, 1-1/p]$ the set $Z_{p/q}(s, s+1/p)$ is nonempty, then there exists an irrational number $\tau$ in $]0, 1[$ such that*

$$\{(p-q)s\} = \frac{p-q}{p} \sum_{k=1}^{\infty} \varepsilon_{-k}(\tau)\left(\frac{q}{p}\right)^k. \tag{2}$$

The proof of Theorem 1 is given in Section 3 and relies upon the main result of [2]. It also allows us to answer (in Theorem 3) a problem posed by Flatto *et al.* at the end of [5].

As an immediate application, we considerably improve Corollary 1.4a of [5].

COROLLARY 1. *The set* $Z_{3/2}(s, s+1/3)$ *is empty if*

$$s \in \{0\} \cup [8/57, 4/19] \cup [4/15, 2/5] \cup [26/57, 10/19] \cup \{2/3\}.$$

To prove Corollary 1, we check that the three intervals are given by Theorem 1 applied to the rationals 1/3, 1/2 and 2/3. Further, since for any irrational $\tau$ in $]0,1[$ there exist $k \geq 1$ and $l \geq 1$ such that $\varepsilon_{-k}(\tau) = 1$ and $\varepsilon_{-l}(\tau) = 0$, we get $\frac{1}{3}(\sum_{k=1}^{\infty} \varepsilon_{-k}(\tau)(2/3)^k) \neq 0, 2/3$, and hence, by the last part of Theorem 1, the sets $Z_{3/2}(0, 1/3)$ and $Z_{3/2}(2/3, 1)$ are empty (a fact already proved in [5]).

We have not been able to determine whether $Z_{p/q}(s, s+1/p)$ is empty for *all* values of $s$. However, we obtain some additional information concerning the sets $Z_{p/q}(s, s+1/p)$ for exceptional values of $s$.

THEOREM 2. *Let* $p > q \geq 2$ *be coprime integers. Let* $s$ *in* $[0, 1-1/p]$ *satisfy* (2) *for some irrational* $\tau$. *Then*

$$\mathrm{Card}\{\xi : 0 \leq \xi \leq x \text{ and } \xi \in Z_{p/q}(s, s+1/p)\} = O((\log_q x)^3).$$

Theorem 2 considerably improves Theorem 1.1 of [5] for $t = 1/p$, where the estimate $O(x^{\gamma})$ is obtained with $\gamma = \log_q \min\{2, p/q\}$. Its proof is given in Section 4, where we get strong conditions on $[\xi]$ for $\xi$ belonging to $Z_{p/q}(s, s+1/p)$.

**3. Proof of Theorem 1.** In all what follows, for a map $F$ and an integer $n \geq 0$, we denote by $F^n$ the map $F \circ \cdots \circ F$, composed $n$ times.

Keeping the notation of Section 3 of [5], we define for any real numbers $\beta > 1$ and $0 \leq \alpha < 1$ the map $f_{\beta,\alpha}$ by

$$f_{\beta,\alpha}(x) = \{\beta x + \alpha\} \quad \text{for } x \in [0,1[.$$

We set

$$S_{\beta,\alpha} := \{x \in [0,1[ \,:\, 0 \leq f^n_{\beta,\alpha}(x) < 1/\beta \text{ for all } n \geq 0\}.$$

Theorem 3.4 of [5] asserts that $S_{\beta,\alpha}$ is finite as soon as $0 \notin S_{\beta,\alpha}$, which is the case for a dense set of values of $\alpha$ in $[0,1]$ (see Theorem 3.5 of [5]). The problem of the existence of values of $\alpha$ such that $S_{\beta,\alpha}$ is infinite is left open in [5]. Indeed, setting

$$E_\beta := \{\alpha \in [0,1[ \,:\, S_{\beta,\alpha} \text{ is an infinite set}\},$$

Flatto *et al.* conjecture that, for all $\beta > 1$, the set $E_\beta$ is nonempty and perfect, and has Lebesgue measure zero. The present section is concerned with the study of this problem, which we solve in Theorem 3 below.

In the rest of this section, $f$ stands for $f_{\beta,\alpha}$.

LEMMA 1. *Assume that there exists an integer* $N \geq 1$ *with* $f^N(0) = 0$ *and such that* $f^k(0) \notin \{0\} \cup [1/\beta, 1[$ *for any integer* $1 \leq k \leq N-1$. *Then* $S_{\beta,\alpha}$ *is the finite set* $\{0, f(0), \ldots, f^{N-1}(0)\}$.

*Proof.* We use the notation of Lemmas 3.1–3.3 of [5], and we only point out which slight changes should be made in their proofs to get our lemma. A continuity argument shows that $\alpha$ is the right endpoint of the interval $I_{N-1} = [\cdot, \alpha[$. It follows that $R_{-1}$ is empty, whence all the $R_k$'s, $k \geq 0$, are empty. Arguing as in [5], we denote by $p_n$ the left endpoint of $L_n$ for $n \geq 0$, and we set

$$q_k := \lim_{j \to \infty} p_{k+jN}, \qquad 0 \leq k \leq N-1.$$

Each $q_k$ coincides with a right endpoint of some $I_j$, $0 \leq j \leq N-1$. It follows that $S_{\beta,\alpha} = \{q_0, \ldots, q_{N-1}\} = \{0, f(0), \ldots, f^{N-1}(0)\}$, as claimed. ■

LEMMA 2. *Let $\beta > 1$. Then, for each $\alpha$ in $[0,1[$, the set $S_{\beta,\alpha}$ is infinite if, and only if, $f^N_{\beta,\alpha}(0) \notin \{0\} \cup [1/\beta, 1[$ for all integers $N \geq 1$.*

*Proof.* The "only if" part follows from Lemma 1 and Theorem 3.4 of [5], which states that $f^N(0) \geq 1/\beta$ implies that $S_{\beta,\alpha}$ is a finite set. To prove the "if" part, we assume that

$$0 < f^N(0) < 1/\beta \quad \text{for all integers } N \geq 1, \tag{3}$$

and we show by contradiction that the $f^k(0)$'s, $k \geq 1$, are distinct. To this end, assume that $1 \leq k < l$ are minimal with $f^k(0) = f^l(0)$. Then there is an integer $j$ with $0 \leq j \leq [\beta]$ such that

$$f^{k-1}(0) = f^{l-1}(0) + j/\beta.$$

By (3), we must have $j = 0$, which, by minimality of $k$, yields $k = 1$. It follows that $f^{l-1}(0) = 0$, a contradiction with (3). ■

LEMMA 3. *Let $\beta > 1$ and put $\gamma = 1/\beta$. Set $J_1^1(\gamma) = [\gamma, 1[$ and, for coprime integers $b > a \geq 1$,*

$$J_b^a(\gamma) = \left[ \frac{\sum_{k=1}^{b-1} \varepsilon_{-k}(a/b)\gamma^k + \gamma^b}{1 + \gamma + \cdots + \gamma^{b-1}}, \frac{\sum_{k=1}^{b-1} \varepsilon_{-k}(a/b)\gamma^k + \gamma^{b-1}}{1 + \gamma + \cdots + \gamma^{b-1}} \right].$$

*Then the intervals $J_b^a(\gamma)$ are disjoint for different choices of coprime integers $b > a \geq 1$. Further, the set $S_{\beta,\alpha}$ is finite if, and only if, there exist coprime integers $b \geq a \geq 1$ such that $\alpha \in J_b^a(\gamma)$. Moreover, $S_{\beta,\alpha}$ is empty if, and only if, $\alpha$ is the left endpoint of some $J_b^a(\gamma)$, and otherwise $S_{\beta,\alpha}$ has exactly $b$ elements, which are cyclically permuted under the action of $f$ if $\alpha$ is in $J_b^a(\gamma)$ but is not its left endpoint. Finally, $S_{\beta,\alpha}$ is infinite if, and only if, there exists some irrational number $\tau$ in $]0,1[$ such that*

$$\alpha = (1-\gamma) \sum_{k=1}^{\infty} \varepsilon_{-k}(\tau)\gamma^k.$$

The proof of Lemma 3 depends heavily on results obtained in [2] (see also [3]) concerning the function

$$T_{\gamma,\alpha} : x \mapsto \{\gamma x + \alpha\},$$

where $0 < \gamma \le 1$ and $0 \le \alpha < 1$ are real numbers such that $\gamma + \alpha > 1$. The map $T_{\gamma,\alpha}$ is piecewise linear, contracting and is continuous except on the left at $\theta := (1-\alpha)/\gamma$. For $n \ge 1$, we put

$$T^n_{\gamma,\alpha}(1) = T^{n-1}_{\gamma,\alpha}(\gamma + \alpha - 1) = T^{n-1}_{\gamma,\alpha}(\lim_{x\to 1_-} T_{\gamma,\alpha}(x)). \tag{4}$$

In [2] we have obtained a precise description of the dynamics of $T_{\gamma,\alpha}$.

PROPOSITION 1. *Let $\alpha$ and $\gamma$ be real numbers with $0 < \gamma < 1$ and $0 \le \alpha < 1$. Let $a$ and $b$ be coprime integers with $b \ge a \ge 1$ and define the interval $I^a_b(\gamma)$ by*

$$I^a_b(\gamma) = \left[\frac{P^a_b(\gamma)}{1+\gamma+\cdots+\gamma^{b-1}}, \frac{P^a_b(\gamma)+\gamma^{b-1}-\gamma^b}{1+\gamma+\cdots+\gamma^{b-1}}\right],$$

*where $P^a_b$ is the polynomial*

$$P^a_b(\gamma) = \sum_{k=0}^{b-1} \varepsilon_{-k}(a/b)\gamma^k.$$

*Then the map $T_{\gamma,\alpha}$ has an attractive periodic orbit with the same dynamics as the rotation $T_{1,a/b}$ if, and only if, $\alpha \in I^a_b(\gamma)$.*

*Proof.* This is Théorème 1.1 of [3]. Observe that in the case $b = a = 1$ of the proposition, the attractive orbit of $T_{\gamma,\alpha}$ is equal to $\{0\}$ when $\gamma+\alpha \le 1$. ∎

REMARK 1. Let $\gamma$ be a real number with $0 < \gamma < 1$. It follows from Proposition 1 that the intervals $I^a_b(\gamma)$ are disjoint for different choices of $a, b$. Indeed, for distinct rational numbers $a/b$ and $a'/b'$, the rotations $T_{1,a/b}$ and $T_{1,a'/b'}$ have different dynamics.

REMARK 2. In [2] and [3], we gave two different proofs of Proposition 1: one dynamical (see [2]) and one algebraic (see [3]). The dynamical proof rests on the study of the position of the critical point $\theta := (1-\alpha)/\gamma$ of $T_{\gamma,\alpha}$, which lies in $[0, 1[$, since $\gamma + \alpha > 1$. We assumed that $\theta \notin T^n_{\gamma,\alpha}([0, 1[)$ for some integer $n \ge 1$, and we set

$$b := \inf\{n : \theta \notin T^n_{\gamma,\alpha}([0, 1[)\} + 1.$$

Since $\theta$ is the only discontinuity of $T_{\gamma,\alpha}$, it is easy to see that for any $n \ge b$ the set $T^n_{\gamma,\alpha}([0, 1[)$ is the union of $b$ disjoint intervals, whose lengths tend to zero when $k$ goes to infinity. To give a more precise result, it is convenient to introduce some notation.

NOTATION. For any real numbers $x < y$ in $[0, 1]$, we write $\langle x, y\rangle$ for the closed interval $[x, y]$ if $y < 1$ and $x > 0$; also, we write $\langle x, 1\rangle$ for $\{0\} \cup [x, 1[$

and $\langle 0,x\rangle$ for $[0,x]$. Moreover, for any increasing function $f$ on $]x,1[$, we write $f(\langle x,1\rangle)$ for $\langle f(x),f(1)\rangle$.

With these notations, we have

$$T_{\gamma,\alpha}^{b-1}([0,1[) = [0,1[ \setminus \bigcup_{k=1}^{b-1} \langle T_{\gamma,\alpha}^{k}(1), T_{\gamma,\alpha}^{k}(0)\rangle$$

and $\theta \in \langle T_{\gamma,\alpha}^{b-1}(1), T_{\gamma,\alpha}^{b-1}(0)\rangle$. As shown in [2], the critical point $\theta$ is in $\langle T_{\gamma,\alpha}^{b-1}(1), T_{\gamma,\alpha}^{b-1}(0)\rangle$ if, and only if, there exists a positive integer $a < b$, coprime with $b$, such that $\alpha$ is in $I_b^a(\gamma)$.

We now have all the tools to prove Lemma 3.

*Proof of Lemma 3.* Let $\beta > 1$ and $0 \le \alpha < 1$. Put $\gamma = 1/\beta$. We readily verify that the conclusion of the lemma holds when $\alpha$ is in $J_1^1(\gamma)$, by Lemma 2. Thus, we now assume that $0 \le \alpha < \gamma$. We recall that $f$ stands for $f_{\beta,\alpha}$. We observe that $f$ is a bijection from $[0,1/\beta[$ onto $[0,1[$, and we denote by $g := g_{\beta,\alpha}$ the inverse of this restriction, i.e. for $x \in [0,1[$,

$$g_{\beta,\alpha}(x) = \begin{cases} \dfrac{1}{\beta}x + \dfrac{1-\alpha}{\beta}, & 0 \le x < \alpha, \\[2mm] \dfrac{1}{\beta}x - \dfrac{\alpha}{\beta}, & \alpha \le x < 1. \end{cases} \tag{5}$$

Thus $g$ is piecewise linear, contracting and continuous, except on the left at $\alpha$.

The maps $T_{\gamma,1-\alpha}$ and $g_{1/\gamma,\alpha}$ are closely related. Namely, for any $x$ in $[0,1[$, we have

$$\{T_{\gamma,1-\alpha}(x) + \alpha\} = g_{1/\gamma,\alpha}(\{x+\alpha\}) = \gamma x. \tag{6}$$

Assume now that the set $S_{\beta,\alpha}$ is finite. In view of Lemma 2, there exists a positive integer $N$ such that $f_{\beta,\alpha}^N(0) \in \langle 1/\beta, 1\rangle$. Denote by $b$ the smallest positive integer with this property. Then we have $0 \in g^b(\langle\gamma,1\rangle)$, or, equivalently, $\alpha \in g^{b-1}(\langle\gamma,1\rangle)$. Since $\alpha < \gamma$, we have $b \ge 2$. According to Lemma 3.1 of [5], the sets $g^k(\langle\gamma,1\rangle)$ for $0 \le k \le b-1$ are nonempty intervals. It follows from (6) and (4) that

$$\begin{aligned} g^{b-1}(\langle\gamma,1\rangle) &= \langle T_{\gamma,1-\alpha}^{b-1}(\gamma-\alpha)+\alpha, T_{\gamma,1-\alpha}^{b-1}(1-\alpha)+\alpha\rangle \\ &= \langle T_{\gamma,1-\alpha}^{b}(1)+\alpha, T_{\gamma,1-\alpha}^{b}(0)+\alpha\rangle. \end{aligned}$$

Consequently, $\alpha$ belongs to $g^{b-1}(\langle\gamma,1\rangle)$ if, and only if, 0 is in $\langle T_{\gamma,1-\alpha}^{b}(1)+\alpha$, $T_{\gamma,1-\alpha}^{b}(0)+\alpha\rangle$, which, since $\alpha<\gamma$, is equivalent to $\alpha/\gamma$ belongs to $\langle T_{\gamma,1-\alpha}^{b-1}(1)$, $T_{\gamma,1-\alpha}^{b-1}(0)\rangle$. Setting $u := 1-\alpha$, we have shown that

$$\alpha \in g^{b-1}(\langle\gamma,1\rangle)$$

if, and only if,

$$\frac{1-u}{\gamma} \in \langle T_{\gamma,u}^{b-1}(1), T_{\gamma,u}^{b-1}(0)\rangle. \tag{7}$$

Notice that the condition $\alpha < \gamma$ is equivalent to $\gamma + u > 1$. According to Remark 2 following Proposition 1, we know that (7) holds if, and only if, there exists a positive integer $a < b$, coprime with $b$, such that

$$u \in I_b^a(\gamma). \tag{8}$$

As $u = 1 - \alpha$, Proposition 1 implies that (8) can be rewritten as

$$\frac{\sum_{k=0}^{b-1}(1-\varepsilon_{-k}(a/b))\gamma^k + \gamma^b - \gamma^{b-1}}{1+\gamma+\cdots+\gamma^{b-1}} \le \alpha \le \frac{\sum_{k=0}^{b-1}(1-\varepsilon_{-k}(a/b))\gamma^k}{1+\gamma+\cdots+\gamma^{b-1}}. \tag{9}$$

Since $\varepsilon_k(1-a/b) = 1-\varepsilon_k(a/b)$ for any integer $k$ not a multiple of $b$ and not congruent to one modulo $b$ (to see this, it suffices to note that $[-ja/b] = -[ja/b]-1$ if $b$ does not divide $j$), (9) becomes

$$\frac{\sum_{k=1}^{b-1}\varepsilon_{-k}(1-a/b)\gamma^k + \gamma^b}{1+\gamma+\cdots+\gamma^{b-1}} \le \alpha \le \frac{\sum_{k=1}^{b-1}\varepsilon_{-k}(1-a/b)\gamma^k + \gamma^{b-1}}{1+\gamma+\cdots+\gamma^{b-1}},$$

which proves that the set $S_{\beta,\alpha}$ is finite if, and only if, there exist coprime integers $b \ge a \ge 1$ such that $\alpha \in J_b^a(\gamma)$.

Further, for $\alpha$ in $J_b^a(\gamma)$, a direct calculation shows that $S_{\beta,\alpha}$ is empty if $\alpha$ is the left endpoint of $J_b^a(\gamma)$, and has exactly $b$ elements otherwise.

Moreover, we infer from Lemma 2 and (8) that $S_{\beta,\alpha}$ is infinite if, and only if,

$$1-\alpha \in [0,1[ \setminus \bigcup_{\substack{1\le a\le b\\(a,b)=1}} I_b^a(\gamma),$$

that is (see [2, Théorème 2] ([1]) or [3, p. 207]) if, and only if, there exists some irrational number $\tau$ in $]0,1[$ such that

$$1-\alpha = (1-\gamma)\sum_{k=0}^{\infty}\varepsilon_{-k}(\tau)\gamma^k,$$

and the last assertion of the lemma follows since $\varepsilon_0(\tau) = 1$ and $\varepsilon_{-k}(\tau) = 1-\varepsilon_{-k}(1-\tau)$ for any integer $k \ge 1$.

Finally, the fact that the intervals $J_b^a(\gamma)$ are disjoint follows from (8), (9), and Remark 1. ■

([1]) There is a misprint in the statement of [2, Théorème 2]: one should read $(S(\alpha))_{-k}$ instead of $(S(\alpha))_k$.

Theorem 3. *For any real number $\beta > 1$, the set*

$$E_\beta := \{\alpha \in [0,1[ \,:\, S_{\beta,\alpha} \text{ is an infinite set}\} = [0,1[ \setminus \bigcup_{\substack{1\le a\le b\\ (a,b)=1}} J_b^a(\gamma)$$

*has measure zero, is uncountable and is not closed.*

*Proof.* Denote by $\mu$ the Lebesgue measure. It follows from Lemma 3 that

$$\mu(E_\beta) = 1 - \sum_{b=1}^{\infty} \frac{\varphi(b)(\gamma^{b-1} - \gamma^b)}{1 + \gamma + \dots + \gamma^{b-1}},$$

where $\varphi$ is the Euler totient function, i.e. $\varphi(b)$ counts the number of integers $a$, $1 \le a \le b$, which are coprime to $b$. Since

$$\frac{\gamma^{b-1} - \gamma^b}{1 + \dots + \gamma^{b-1}} = (1-\gamma)^2 \, \frac{\gamma^{b-1}}{1-\gamma^b} \quad \text{for } b \ge 1,$$

and

$$\sum_{b=1}^{\infty} \varphi(b) \, \frac{\gamma^{b-1}}{1-\gamma^b} = \frac{1}{(1-\gamma)^2},$$

we infer from Theorem 309 of [6] that $\mu(E_\beta) = 0$.

Further, the last assertion of Lemma 3 combined with Théorème 2 of [2] implies that $E_\beta$ is uncountable. Moreover, if the irrational number $\tau$ tends to a rational number, then $(1-\gamma)\sum_{k=1}^{\infty} \varepsilon_{-k}(\tau)\gamma^k$ tends to an endpoint of some interval $J_b^a(\gamma)$. Hence, $E_\beta$ is not closed. ∎

Remark 3. An interesting open problem is to determine the Hausdorff dimension of the sets $E_\beta$.

To complete the proof of Theorem 1, we recall a crucial result of [5].

Proposition 2 ([5, Theorem 3.2]). *Let $p > q \ge 2$ be coprime integers, and let $s \in [0, 1-1/p]$. If the set $S_{p/q,\{(p-q)s\}}$ is finite, then $Z_{p/q}(s, s+1/p)$ is empty.* ∎

*Proof of Theorem 1.* This statement easily follows from Lemma 3 and Theorem 3, combined with Proposition 2. ∎

**4. Proof of Theorem 2.** We now investigate the behaviour of $f := f_{\beta,\alpha}$ when $\alpha$ is in $E_\beta$. Recall that $g_{\beta,\alpha}$ is defined in (5). The following lemmas answer a question posed by Flatto *et al.* at the end of [5].

Lemma 4. *Let $\beta > 1$ and $\alpha \in E_\beta$. For $n \ge 0$, put*

$$I_n := g_{\beta,\alpha}^n([1/\beta, 1[).$$

*Then*

$$S_{\beta,\alpha} := [0,1[\setminus \bigcup_{n\ge 0} I_n.$$

*In particular*, $S_{\beta,\alpha}$ *has measure zero.*

*Proof.* Arguing as in Lemma 3.1 of [5], we use the fact that $f^n(0) \notin [1/\beta,1[$ for all $n\ge 0$ to deduce that the $I_n$'s, $n\ge 0$, are mutually disjoint. If $x\in[0,1[$ is in some $I_n$ with $n\ge 0$, we infer that $f^n(x)\in[1/\beta,1[$, whence $x\notin S_{\beta,\alpha}$. Otherwise, it is clear that $x\in S_{\beta,\alpha}$. Further,

$$\mu\Big(\bigcup_{n\ge 0} I_n\Big) = \Big(1-\frac{1}{\beta}\Big)\sum_{n\ge 0}\frac{1}{\beta^n} = 0,$$

as claimed. ∎

As in [5], we associate to $f = f_{\beta,\alpha}$ the natural symbolic dynamics, which assigns to each $x$ in $[0,1[$ the integer

$$S_f(x) = [\beta x+\alpha],$$

and we call the sequence

$$a_n := S_f(f^n(x)), \qquad n\ge 0,$$

the *f-expansion* of $x$. If $x$ is in $S_{\beta,\alpha}$, then $0\le f^n_{\beta,\alpha}(x) < 1/\beta$ for all $n\ge 0$, and its $f$-expansion is uniquely composed of 0's and 1's.

LEMMA 5. *Let* $\beta>1$ *and* $\alpha\in E_\beta$. *Let* $\tau$ *in* $]0,1[$ *be defined by*

$$\alpha = \Big(1-\frac{1}{\beta}\Big)\sum_{k=1}^{\infty}\frac{\varepsilon_{-k}(\tau)}{\beta^k}.$$

*The set* $S_{\beta,\alpha}$ *is uncountable*, *not closed and*, *for any* $x$ *in* $S_{\beta,\alpha}$, *there exists* $0\le\eta<1$ *such that the* $f$*-expansion of* $x$ *is the Sturmian sequence* $(a_n)_{n\ge 0}$ *given by*

$$a_n = [(n+1)\tau+\eta] - [n\tau+\eta].$$

*Proof.* We point out that $\tau$ is irrational. We first show that $g$ and the (irrational) rotation

$$R_{1-\tau} : x\mapsto \{x+1-\tau\}, \qquad x\in[0,1[,$$

are semi-conjugate.

We claim that the intervals $I_n$, $n\ge 0$, are ordered as the sequence $(\{n(1-\tau)\})_{n\ge 0}$. To see this, for any $\beta'>1$, we define

$$\Psi_{\beta'}(\tau) = \Big(1-\frac{1}{\beta'}\Big)\sum_{k=1}^{\infty}\frac{\varepsilon_{-k}(\tau)}{\beta'^k},$$

and we observe that, for any $n \geq 0$, the function

$$\beta' \mapsto \inf(g^n_{\beta',\Psi_{\beta'}(\tau)}([1/\beta', 1[))$$

is continuous on $]1, +\infty[$ and tends to $\{n(1-\tau)\}$ when $\beta'$ tends to 1.

We set $\Phi(0) = 0$ and, for $n \geq 1$, $\Phi(I_n) = \{n(1-\tau)\}$. The map $\Phi$ is monotone and, by Lemma 4, is defined on a dense subset of $[0, 1[$, thus, we can extend it by continuity to $[0, 1[$. Consequently, the set $S_{\beta,\alpha}$ is uncountable and not closed. For all $y \in [0, 1[$, we have

$$\Phi \circ g_{\beta,\alpha}(y) = R_{1-\tau} \circ \Phi(y),$$

hence,

$$R_\tau \circ \Phi(z) = \Phi \circ f_{\beta,\alpha}(z) \tag{10}$$

for $z \in [0, 1/\beta]$. Since $\Phi(0) = 0$ and $f((1-\alpha)/\beta) = 0$, we deduce from (10) that $\Phi((1-\alpha)/\beta) = 1 - \tau$. It follows that

$$0 \leq z < \frac{1-\alpha}{\beta} \quad \text{if and only if} \quad 0 \leq \Phi(z) < 1 - \tau.$$

By induction, (10) implies for any integer $n \geq 1$ that

$$0 \leq f^n(z) < \frac{1-\alpha}{\beta} \quad \text{if and only if} \quad 0 \leq R^n_\tau(\Phi(z)) < 1 - \tau. \tag{11}$$

Let $x \in S_{\beta,\alpha}$ and denote by $(a_n)_{n\geq 0}$ its $f$-expansion. It follows from (11) that $a_n = 0$ if, and only if, $0 \leq R^n_\tau(\Phi(z)) < 1 - \tau$. Hence,

$$a_n = [(n+1)\tau + \Phi(z)] - [n\tau + \Phi(z)],$$

and the proof of Lemma 5 is complete. ∎

We need to recall an important result of [5]. For the definition of $T$-expansion, we refer the reader to [5].

PROPOSITION 3. *Let $p > q \geq 2$ be coprime integers. Then a positive real number $\xi$ is in $Z_{p/q}(s, s + 1/p)$ if, and only if, both conditions* (C1) *and* (C2) *below hold*:

(C1) $0 \leq f^n(q(\{\xi\} - s)) < q/p$ *for all* $n \geq 0$,

(C2) *the $T$-expansion $(a_n)$ of $[\xi]$ and the $f$-expansion $(b_n)$ of $q(\{\xi\} - s)$ are related by*

$$\sigma(a_n) = b_n \quad \textit{for all } n \geq 0,$$

*where $\sigma$ is the permutation of $\{0, 1, \ldots, q-1\}$ given by*

$$\sigma(i) \equiv -pi - [(p-q)s] \pmod q.$$

*Further, the set $Z_{p/q}(s, s + 1/p)$ contains at most one element in each unit interval $[m, m + 1[$, where $m$ is a nonnegative integer.*

*Proof.* This follows from Proposition 2.1 and Theorem 1.1 of [5]. ∎

*Proof of Theorem 2.* Let $\xi$ be in $Z_{p/q}(s, s+1/p)$. By Proposition 3 and Lemma 5, the $T$-expansion of $[\xi]$ is an infinite Sturmian word. It has been shown by Mignosi [9] (see [1] for an alternative proof) that, for any integer $m \geq 1$, there are $O(m^3)$ Sturmian words of length $m$ (recall that any subword of a Sturmian sequence is called a *Sturmian word*). Since Lemma 2.2 of [5] asserts that the first $m$ terms of the $T$-expansion of an integer $g$ are uniquely determined by $g$ modulo $q^m$, we conclude that at most $O(m^3)$ integers less than $q^m$ may have a Sturmian $T$-expansion, and Theorem 2 is proved. ■

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