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Hilbert symbols as maps of functors

by

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1. Introduction. The Hilbert symbol on a local field, F, gives a homomorphism $H_F : K_2F \to \mu(F)$. It is natural to ask how this behaves under extension of fields; that is, if E/F is a finite extension of local fields, what map $\mu(F) \to \mu(E)$ (if any) makes the diagram

$$\begin{tikzcd} K_2E \arrow[r] & \mu(E) \\ K_2F \arrow[r] \arrow[u] & \mu(F) \arrow[u] \end{tikzcd}$$

commute? The first thing to observe is that the inclusion map $\mu(F) \to \mu(E)$ does not usually work. In other words if we interpret the assignment $F \to \mu(F)$ as a functor in the obvious way, the Hilbert symbol is not a morphism of functors. In fact, it is more naturally a map of *contravariant* functors, in the sense that the diagram

$$\begin{tikzcd} K_2E \arrow[r] \arrow[d, "\text{transfer}"] & \mu(E) \arrow[d] \\ K_2F \arrow[r] & \mu(F) \end{tikzcd}$$

will commute, where the right hand vertical arrow is the appropriate surjective power map; see [1, Proposition 2] and [3]. (Indeed, if we use this fact in conjunction with properties of the K-theory transfer, we can arrive at an answer to the question above. This gives us, in an ad hoc manner, a formula for a map $\mu(F) \to \mu(E)$ which will make the diagram commute; see [8, Lemma 1.3.3] or [5]. But we are seeking a more conceptual answer to the question in this paper.)

Of course it is well known to K-theorists and number theorists that when the nth roots of unity are contained in a local field F, then the Hilbert

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symbol of order n on F is essentially equal to the Galois symbol

$$\Gamma_{F,n} : K_2F \to H^2(G_F, \mu_n^{\otimes 2}).$$

The primary reference for this last fact is [9, Chapter XIV, 2, Proposition 5]. The Galois symbols, $\Gamma_{F,n}$, are naturally maps of functors on the category of finite extensions of a given local field. Our question however involves comparing Hilbert symbols of (possibly) different orders. The Galois symbol is less obviously a map of functors of the second variable "n" (that is, the underlying category is the set of natural numbers partially ordered by divisibility). In order to make this work, the domain functor is $K_2F/n = K_2F \otimes \mathbb{Z}/n\mathbb{Z}$ and, when n divides m, the map $H^2(G_F, \mu_n^{\otimes 2}) \to H^2(G_E, \mu_m^{\otimes 2})$ is induced by a homomorphism of coefficient modules $\mu_n \otimes \mu_n \to \mu_m \otimes \mu_m$, but not by the natural map obtained by tensoring the inclusion $\mu_n \to \mu_m$ with itself.

In this note, we give a natural generalization of the Hilbert symbol of order n to the situation where the nth roots of unity do not belong to the local field F. The target functor can then be identified as the group, $(\mu_n)_{G_F}$, of G_F-coinvariants of μ_n, which is naturally a functor of both F and n. This (slightly) generalized Hilbert symbol $H_F : K_2F/n \to (\mu_n)_{G_F}$ is a map of functors. In fact it is a map of functors with transfers. In order to prove this, we show that our generalized Hilbert symbol is essentially equal to the Galois symbol, via Tate duality. The subtlety of this result derives from the fact that the Hilbert symbols are naturally defined in terms of Tate cohomology, which behaves poorly as a functor, while the Galois symbols involve Galois cohomology. This having been proven, our original question has a natural answer (see Corollary 2.3).

In a final section, as an application of the functorial behaviour of Hilbert symbols, we identify the quotient $WK_2F/K_2^{\infty}F$, where WK_2F is the wild kernel of the number field F and $K_2^{\infty}F = \bigcap_n (K_2F)^n$, as the well known Tate–Shafarevich group $\text{Ш}^1(F, \mu_N)$ (for N sufficiently large).

This note arose from the desire to give a short conceptual answer to the question: why is the wild kernel of a number field a functor? Recall that if F is a number field then each place v of F yields a Hilbert symbol $H_v : K_2F \to \mu(F_v)$ and the wild kernel is the intersection of the kernels of all these symbols. One would like to say that each H_v is a map of functors (so that $\ker(H_v)$ and hence $\bigcap_v \ker(H_v)$ is also a functor). Although this statement is not true as it stands, we have shown how to modify it to make it correct.

2. Hilbert symbols as maps of functors. Our goal is to interpret the classical Hilbert symbol of order n on a local field F as a map of functors:

$$K_2F/n \to (\mu_n)_{G_F}.$$

We make this statement precise as follows. Fix a field k. The domain category for these functors is the category $\mathcal{C}_k$ whose objects are pairs (F, n), where F is a finite field extension of k, and n is a positive integer which is relatively prime to $\operatorname{char} k$ if $\operatorname{char} k > 0$. There are no morphisms $(F, n) \to (E, m)$ if $n \nmid m$; otherwise the morphisms $(F, n) \to (E, m)$ are the k-algebra homomorphisms $F \to E$. The functors $(F, n) \to K_2F/n$ and $(F, n) \to (\mu_n)_{G_F}$ are *functors with transfers* on this category with values in the category of finite abelian groups; thus, given $\sigma : (F, n) \to (E, m)$ (where n divides m) there are maps

$$\frac{m}{n} K_2(\sigma) : K_2F/n \to K_2E/m, \qquad N_{E/F} : (\mu_n)_{G_F} \to (\mu_m)_{G_E},$$

and maps

$$K_2E/m \to K_2F/n, \;\; x \mapsto \operatorname{tr}_{E/F}(x), \qquad (\mu_m)_{G_E} \to (\mu_n)_{G_F}, \;\; \zeta \mapsto \widetilde{\sigma}^{-1}(\zeta^{m/n}),$$

which make these into covariant and contravariant functors. (Here $\widetilde{\sigma}$ is any extension of σ to a field isomorphism $\widetilde{\sigma} : F_{\mathrm{sep}} \to E_{\mathrm{sep}}$. Moreover, $\widetilde{\sigma}$ induces an embedding $G_E \to G_F$, $\tau \mapsto \widetilde{\sigma}^{-1}\tau\widetilde{\sigma}$, with respect to which $N_{E/F}$ is defined. For simplicity we will always assume that σ is an inclusion and $\widetilde{\sigma}$ is the identity.) Furthermore the composites $K_2F/n \to K_2E/m \to K_2F/n$ and $(\mu_n)_{G_F} \to (\mu_m)_{G_E} \to (\mu_n)_{G_F}$ are just multiplication by $(m/n)[E : F]$.

Our main result is the following:

Theorem 2.1. *For k a local field and $(F, n) \in \operatorname{Obj} \mathcal{C}_k$, there is a group homomorphism*

$$H_{F,n} : K_2F/n \to (\mu_n)_{G_F}$$

such that

(1) *if $\mu_n \subset F$, the composite*

$$F^* \times F^* \to K_2F/n \to \mu_n = (\mu_n)_{G_F}$$

is the classical Hilbert symbol,

(2) *$H_{F,n}$ is a map of functors with transfers.*

Theorem 2.1(1) is a consequence of Lemma 4.1 below, while Theorem 2.1(2) follows from Theorem 6.5, Lemma 7.1 and Lemma 7.2 below.

Corollary 2.2. *Let k be a non-archimedean local field and $(F, m) \in \operatorname{Obj} \mathcal{C}_k$, suppose that $\mu_m(F) = \mu_n$ and suppose that $r = m/n$. Then $(\mu_m)_{G_F} \cong \mu_n$ by $\zeta \mapsto \zeta^r$. The composite*

$$F^* \times F^* \to K_2F/m \xrightarrow{H_{F,m}} (\mu_m)_{G_F} \xrightarrow{\sim} \mu_n$$

is the Hilbert symbol of order n.

Proof. The morphism $(F,n) \to (F,m)$ gives the following diagram of contravariant functors:

$$\begin{tikzcd}
K_2F \arrow[r] \arrow[d, equal] & K_2F/m \arrow[r] \arrow[d] & (\mu_m)_{G_F} \arrow[d, "()^r"] \\
K_2F \arrow[r] & K_2F/n \arrow[r] & \mu_n
\end{tikzcd}$$

hence the result. ∎

Observe, by contrast, that the diagram of *covariant* functors corresponding to the morphism $(F,n) \to (F,m)$ gives

$$\begin{tikzcd}
K_2F \arrow[r] & K_2F/m \arrow[r] & (\mu_m)_{G_F} \arrow[r, "\sim"] & \mu_n \\
K_2F \arrow[r] \arrow[u, "m/n"] & K_2F/n \arrow[r] \arrow[u, "m/n"] & \mu_n \arrow[r] \arrow[u] & \mu_n \arrow[u, "m/n"]
\end{tikzcd}$$

COROLLARY 2.3. *Let E/F be an extension of non-archimedean local fields. Let $m \in \mathbb{N}$ be such that* $\operatorname{char} E \nmid m$ *and $\mu_m \subset \mu(E)$. Suppose that $\mu_m(F) = \mu_n$ and let $r = m/n$. Then the diagram*

$$\begin{tikzcd}
K_2E \arrow[r] & \mu_m \\
K_2F \arrow[r] \arrow[u] & \mu_n \arrow[u, "\mathfrak{N}"]
\end{tikzcd}$$

commutes, *where the horizontal arrows are the Hilbert symbols of the appropriate order and $\mathfrak{N}(\zeta) = N_{E/F}(\zeta^{1/r})$.*

Proof. The following diagram commutes:

$$\begin{tikzcd}
K_2E \arrow[r] & K_2E/m \arrow[r] & \mu_m = (\mu_m)_{G_E} \\
K_2F \arrow[r] \arrow[u] \arrow[d, equal] & K_2F/m \arrow[r] \arrow[u] \arrow[d] & (\mu_m)_{G_F} \arrow[u, "N_{E/F}"] \arrow[d, "()^r"] \\
K_2F \arrow[r] & K_2F/n \arrow[r] & \mu_n
\end{tikzcd}$$

hence the claim. ∎

3. Some background. Let F be a local field, let F_{sep} denote the separable closure of F and let $G_F = \operatorname{Gal}(F_{\text{sep}}/F)$ be the absolute Galois group of F. In this situation:

(i) There is a canonical isomorphism

$$\operatorname{inv}_F : H^2(G_F, F_{\text{sep}}^*) \to I(F) \subset \mathbb{Q}/\mathbb{Z},$$

where

$$I(F) = \begin{cases} \mathbb{Q}/\mathbb{Z}, & F \text{ a } p\text{-adic local field,} \\ \mathbb{Q}/\mathbb{Z}(p)' = \bigcup_{n \nmid p} \frac{1}{n}\mathbb{Z}/\mathbb{Z}, & F \text{ a local field of characteristic } p > 0, \\ \frac{1}{2}\mathbb{Z}/\mathbb{Z}, & F = \mathbb{R}, \\ 0, & F = \mathbb{C}. \end{cases}$$

(ii) If L/F is a finite Galois extension of degree n, where n is relatively prime to $\operatorname{char} F$ if $\operatorname{char} F > 0$, then the inflation homomorphism

$$\inf_{L/F} : H^2(\mathrm{Gal}(L/F), L^*) \to H^2(G_F, F^*_{\mathrm{sep}})$$

is injective and $\mathrm{inv}_{L/F} = \mathrm{inv}_F \circ \inf_{L/F}$ maps $H^2(\mathrm{Gal}(L/F), L^*)$ isomorphically onto the subgroup $(1/n)\mathbb{Z}/\mathbb{Z}$ of $I(F)$.

(iii) Thus there is a canonical element $u_{L/F} \in H^2(\mathrm{Gal}(L/F), L^*)$ such that $\mathrm{inv}_{L/F}(u_{L/F}) = 1/n$. Cup-product with $u_{L/F}$ induces isomorphisms of Tate cohomology groups

$$\widehat{H}^i(\mathrm{Gal}(L/F), \mathbb{Z}) \xrightarrow{\sim} \widehat{H}^{i+2}(\mathrm{Gal}(L/F), L^*)$$

for all $i \in \mathbb{Z}$. In particular, when $i = -2$ this gives an isomorphism

$$\mathrm{Gal}(L/F)^{\mathrm{ab}} = \widehat{H}^{-2}(\mathrm{Gal}(L/F), \mathbb{Z}) \xrightarrow{\sim} \widehat{H}^0(\mathrm{Gal}(L/F), L^*).$$

The inverse isomorphism

$$\varrho_{L/F} : F^*/N_{L/F}(L^*) = \widehat{H}^0(\mathrm{Gal}(L/F), L^*) \to \mathrm{Gal}(L/F)^{\mathrm{ab}}$$

is the reciprocity isomorphism.

4. The Hilbert symbol. Let k be a local field and let $(F, n) \in \mathrm{Obj}\, \mathcal{C}_k$ be such that F contains the group, μ_n, of all nth roots of unity for some integer $n > 1$. Let $F_n = F(\sqrt[n]{F^*})$. By Kummer theory, F_n/F is the maximal abelian extension of F of exponent n. Let $G_n = \mathrm{Gal}(F_n/F)$. Associated to the short exact sequence of G_n-modules

$$1 \to \mu_n \to F_n^* \xrightarrow{(\,)^n} (F_n^*)^n \to 1$$

there is a long exact cohomology sequence with connecting homomorphisms denoted by $\delta_{F,n}$ or simply δ_n:

$$\cdots \to H^0(G_n, F_n^*) \to H^0(G_n, (F_n^*)^n) \xrightarrow{\delta_n} H^1(G_n, \mu_n) \to H^1(G_n, F_n^*) \to \cdots.$$

Since $H^0(G_n, F_n^*) = H^0(G_n, (F_n^*)^n) = F^*$ and since $H^1(G_n, F_n^*) = \{1\}$, by Hilbert's Theorem 90, there is a surjective homomorphism

$$\delta_n : F^* \to H^1(G_n, \mu_n) = \mathrm{Hom}(G_n, \mu_n)$$

with kernel $(F^*)^n$. Let $\varrho_n = \varrho_{F_n/F}$ denote the reciprocity map

$$\varrho_n : F^* \to \widehat{H}^{-2}(G_n, \mathbb{Z}) = G_n^{\mathrm{ab}} = G_n.$$

The *Hilbert symbol* of order n on F is the composite

$$\lambda_{F,n} = \lambda_n : F^* \times F^* \xrightarrow{\varrho_n \times \delta_n} G_n \times \mathrm{Hom}(G_n, \mu_n) \to \mu_n,$$

that is,

$$\lambda_n(a,b) = \delta_n(b)(\varrho_n(a)) \in \mu_n.$$

This pairing can be interpreted as a cup-product of Tate cohomology groups and thus generalized to all $(F,n) \in \mathrm{Obj}\, \mathcal{C}_k$ as follows. Let $F_n = F(\sqrt[n]{F^*})$. If $\mu_n \not\subset F$ then F_n/F is a finite Galois extension but no longer necessarily abelian. A surjective homomorphism $\delta_n : F^* \to \widehat{H}^1(G_n, \mu_n)$ can be constructed as above, although $\widehat{H}^1(G_n, \mu_n)$ can no longer be identified with $\mathrm{Hom}(G_n, \mu_n)$. Furthermore, there is a reciprocity map $\varrho_n : F^* \to G_n^{\mathrm{ab}}$ but it is not in general true that $G_n = G_n^{\mathrm{ab}}$.

Thus, for any $(F,n) \in \mathrm{Obj}\, \mathcal{C}_k$ we define a pairing

$$\lambda'_{F,n} = \lambda'_n : F^* \times F^* \to \widehat{H}^{-1}(G_n, \mu_n), \qquad \lambda'_n(a,b) \mapsto \varrho_n(a) \cup \delta_n(b).$$

Observe that $\widehat{H}^{-1}(G_n, \mu_n) = \widehat{H}_0(G_n, \mu_n) = (\mu_n)_{G_n} = (\mu_n)_{G_F}$.

Lemma 4.1. *If $\mu_n \subset F$ then $\widehat{H}^{-1}(G_n, \mu_n)$ is naturally identified with μ_n. With this identification, $\lambda'_n = \lambda_n$.*

Proof. In general, for a finite group G and a G-module M, $\widehat{H}^{-i}(G, M) = \widehat{H}_{i-1}(G, M)$ for $i \geq 1$ (cf. [2, VI, 4]). Thus $\widehat{H}^{-1}(G_n, \mu_n) = \widehat{H}_0(G_n, \mu_n) = \mu_n$ and $\widehat{H}^{-2}(G_n, \mathbb{Z}) = \widehat{H}_1(G_n, \mathbb{Z}) = G_n^{\mathrm{ab}}$. With these identifications, the cup-product

$$\widehat{H}^{-2}(G_n, \mathbb{Z}) \times \widehat{H}^1(G_n, \mu_n) \to \widehat{H}^{-1}(G_n, \mu_n)$$

corresponds to the cap-product

$$H_1(G_n, \mathbb{Z}) \times H^1(G_n, \mu_n) \to \mu_n,$$

which is just the natural evaluation map [2, V, 3.10]. ∎

Thus we may call λ'_n the Hilbert symbol of order n on F for any $(F,n) \in \mathrm{Obj}\, \mathcal{C}_k$ and we will denote it λ_n.

The generalized Hilbert symbol can be described in a similar way to the classical symbol (cf. [6, V, 3.1]).

Lemma 4.2. *Let $a, b \in F^*$. Let $\widetilde{\varrho}_n(a)$ denote any lifting of $\varrho_n(a)$ to* $\mathrm{Gal}(F_n/F) = G_n$. *Suppose that*

$$\widetilde{\varrho}_n(a)(\sqrt[n]{b}) = \zeta \sqrt[n]{b} \quad \textit{for } \zeta \in \mu_n.$$

Then $\lambda_n(a,b) = \overline{\zeta} \in (\mu_n)_{G_F}$.

Proof. $\delta_n(b) \in H^1(G_n, \mu_n)$ is represented by the cocycle

$$f_b(\sigma) = \frac{\sigma(\sqrt[n]{b})}{\sqrt[n]{b}}.$$

By [2, V, 3.10], the cap-product $H_1(G_n, \mathbb{Z}) \times H^1(G_n, \mu_n) \to H_0(G_n, \mu_n)$ is induced by the evaluation of cocyles. ∎

EXAMPLE 4.3. Let $F = \mathbb{Q}_3$ and $n = 4$. Take $a = 3$, $b = -1$. Then $F(\sqrt[4]{b}) = \mathbb{Q}_3(\zeta_8)$, which is an unramified quadratic extension (since, e.g., $\sqrt{-2} \in \mathbb{Q}_3$). The non-trivial element of the Galois group $\mathrm{Gal}(\mathbb{Q}_3(\zeta_8)/\mathbb{Q}_3)$ sends ζ_8 to $\zeta_8^3 = \zeta_4\zeta_8$. Thus,

$$\lambda_4(3, -1) = \widetilde{\varrho}_4(3)(\zeta_8)\zeta_8^{-1} = \varrho_4(3)(\zeta_8)\zeta_8^{-1} = \zeta_8^3\zeta_8^{-1} = \zeta_4 \in (\mu_4)_{G_{\mathbb{Q}_3}}$$

(compare with Corollary 2.2 above).

Other standard properties of the Hilbert symbol can easily be deduced from Lemma 4.2 by adapting the classical arguments [6, V, 3]:

COROLLARY 4.4. (i) *Let $m = |\mu_n(F)|$. Then $\lambda_n(a, b) = 1$ if and only if a is a norm from $F(\sqrt[m]{b})/F$.*
(ii) *$\lambda_n(a, 1 - a) = 1$ for all $a \in F^* - \{1\}$.*

5. The Galois symbol. In order to prove Theorem 2.1(2) we first introduce another symbol which can be defined on any field. Let k be any field and let $(F, n) \in \mathrm{Obj}\,\mathcal{C}_k$. There is a short exact sequence of G_F-modules

$$1 \to \mu_n \to F^*_{\mathrm{sep}} \xrightarrow{(\,)^n} F^*_{\mathrm{sep}} \to 1$$

and we denote by $d_{F,n}$, or just d_n, the associated surjective connecting homomorphism

$$d_n : H^0(G_F, F^*_{\mathrm{sep}}) \to H^1(G_F, \mu_n)$$

with kernel $(F^*)^n$. The *Galois symbol* of order n on F, $\gamma_{F,n} = \gamma_n$, is the composite,

$$F^* \times F^* \xrightarrow{d_n \times d_n} H^1(G_F, \mu_n) \times H^1(G_F, \mu_n) \xrightarrow{\cup} H^2(G_F, \mu_n^{\otimes 2}).$$

That is $\gamma_n(a, b) = d_n(a) \cup d_n(b)$. It can be shown (see [11, Theorem 3.1]) that $\gamma_n(a, 1 - a) = 1$ for $a \neq 0, 1$ and, since γ_n is clearly bimultiplicative, it is a Steinberg symbol on F. Thus γ_n induces a map on K_2F/n which we will denote by $\Gamma_{F,n}$ or Γ_n, that is,

$$\Gamma_n : K_2F/n \to H^2(G_F, \mu_n^{\otimes 2}).$$

We will also refer to Γ_n as the Galois symbol of order n on F.

We prove Theorem 2.1(2) by proving that when F is a local field then the Galois symbol is a map of functors with transfers and furthermore that the Hilbert symbol can be essentially identified with the Galois symbol.

6. Comparing the symbols. We wish to compare the Hilbert symbol $\lambda_{F,n}$ and the Galois symbol $\gamma_{F,n}$ when F is a local field.

REMARK 6.1. If $F=\mathbb{C}$, or $F=\mathbb{R}$ and n is odd, the Hilbert symbol and the Galois symbol are both trivial. If $F=\mathbb{R}$ and n is even, a straightforward calculation shows that $\gamma_{\mathbb{R},n}:\mathbb{R}^*\times\mathbb{R}^*\to H^2(G_F,\mu_n^{\otimes 2})$ is equivalent to the Hilbert symbol $\lambda_{\mathbb{R},2}$ which is given by

$$\lambda_{\mathbb{R},2}(a,b)=(-1)^{((\mathrm{sign}(a)-1)/2)((\mathrm{sign}(b)-1)/2)}.$$

For a (non-archimedean) local field k, and $(F,n)\in\mathrm{Obj}\,\mathcal{C}_k$, Tate duality identifies $H^2(G_F,\mu_n^{\otimes 2})$, the target of λ_n, with $H_0(G_F,\mu_n)$, the target of γ_n. We will show that with this identification the two symbols agree up to sign.

Let μ be the group of all roots of unity in F^*_{sep}. Then

$$H^2(G_F,\mu)=H^2(G_F,F^*_{\mathrm{sep}})_{\mathrm{tors}}=H^2(G_F,F^*_{\mathrm{sep}})$$

so that inv_F induces an isomorphism $H^2(G_F,\mu)\to\mathbb{Q}/\mathbb{Z}$.

For a finite abelian group A let $A^{\#}=\mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z})$ be the Pontryagin dual. If A has exponent n then $A^{\#}=\mathrm{Hom}(A,(1/n)\mathbb{Z}/\mathbb{Z})$. If A is a G-module for some group G then so is $A^{\#}$ via $(\sigma(\chi))(a)=\chi(\sigma^{-1}(a))$ for all $a\in A$, $\chi\in A^{\#}$ and $\sigma\in G$.

Likewise for a finite abelian group A let A' denote the group $\mathrm{Hom}(A,\mu)$. If A has exponent n then $A'=\mathrm{Hom}(A,\mu_n)$ and if A is a G-module then A' is a G-module via $(\sigma(\chi))(a)=\sigma(\chi(\sigma^{-1}(a)))$ for all $a\in A$, $\chi\in A'$ and $\sigma\in G$. For an abelian group A and $n\in\mathbb{N}$ let $A[n]=\{a\in A\mid na=0\}$.

THEOREM 6.2 (Tate). *If F is a local field and if A is a finite G_F-module, then for $0\le i\le 2$, cup-product induces a natural duality pairing*

$$H^i(G_F,A')\times H^{2-i}(G_F,A)\to H^2(G_F,\mu)\to I(F)\subset\mathbb{Q}/\mathbb{Z},$$
$$(f,g)\mapsto\mathrm{inv}_F(f\cup g).$$

Thus, $H^i(G_F,A')\cong H^{2-i}(G_F,A)^{\#}$.

For the proof see [7, VII, 7.2.6].

In the particular case $A=\mu_n^{\otimes 2}$, there is a natural pairing of G_F-modules

$$\mu_n^{\#}\times\mu_n^{\otimes 2}\to\mu_n,\qquad(\chi,\zeta\otimes\eta)\mapsto\eta^{n\chi(\zeta)}=\zeta^{n\chi(\eta)}.$$

This pairing identifies $\mu_n^{\#}$ with $(\mu_n^{\otimes 2})'$ as G_F-modules. Since $H^2(G_F,\mu_n)=H^2(G_F,F^*_{\mathrm{sep}})[n]$, inv_F induces an isomorphism $H^2(G_F,\mu_n)\cong(1/n)\mathbb{Z}/\mathbb{Z}$. So Tate's duality theorem implies:

LEMMA 6.3. *For k a local field and $(F,n)\in\mathrm{Obj}\,\mathcal{C}_k$ there is a natural duality pairing*

$$H^2(G_F,\mu_n^{\otimes 2})\times H^0(G_F,\mu_n^{\#})\to\frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

given by $(x,\chi) \mapsto \mathrm{inv}_F(x \cup \chi)$ *where the cup-product is defined with respect to the pairing* $\mu_n^{\otimes 2} \times \mu_n^{\#} \to \mu_n$ *described above.*

Note that $H^0(G_F, \mu_n^{\#}) = \mathrm{Hom}_{G_F}(\mu_n, \mathbb{Q}/\mathbb{Z})$ is also naturally dual to $H_0(G_F, \mu_n) = (\mu_n)_{G_F}$ via the evaluation map

$$H_0(G_F, \mu_n) \otimes H^0(G_F, \mu_n^{\#}) \to \mathbb{Q}/\mathbb{Z}, \qquad \zeta \otimes \chi \mapsto \chi(\zeta).$$

Putting all this together we get:

LEMMA 6.4. *For k a local field and $(F,n) \in \mathrm{Obj}\,\mathcal{C}_k$, there is a unique isomorphism of groups*

$$\theta_{F,n} = \theta_n : H^2(G_F, \mu_n^{\otimes 2}) \to (\mu_n)_{G_F},$$

determined by

$$\mathrm{inv}_F(x \cup \chi) = \chi(\theta_n(x))$$

for all $x \in H^2(G_F, \mu_n^{\otimes 2})$ and $\chi \in H^0(G_F, \mu_n^{\#})$.

We will eventually prove:

THEOREM 6.5. *For k a non-archimedean local field and $(F,n) \in \mathrm{Obj}\,\mathcal{C}_k$, $\theta_{F,n} \circ \gamma_{F,n} = -\lambda_{F,n}$ as maps $F^* \times F^* \to (\mu_n)_{G_F}$.*

Remark 6.1 and Theorem 6.5 imply that for a local field k and for $(F,n) \in \mathrm{Obj}\,\mathcal{C}_k$, $\lambda_{F,n}(a, 1-a) = 1$ for $a \neq 0, 1$ and thus $\lambda_{F,n}$ is a Steinberg symbol on F. We denote by $H_{F,n}$ or H_n the map induced by $\lambda_{F,n}$ on K_2F/n, that is,

$$H_n : K_2F/n \to (\mu_n)_{G_F}$$

and we will also refer to H_n as the Hilbert symbol of order n on F. Note that we have proved Theorem 2.1(1).

In view of Remark 6.1 and Theorem 6.5, it only remains to show that the maps $\Gamma_{F,n}$ and $\theta_{F,n}$ are maps of functors with transfers to complete the proof of Theorem 2.1. We shall do this in the next section; the proof of Theorem 6.5 will be given in Sections 8 and 9.

7. $H^2(G_F, \mu_n^{\otimes 2})$ as a functor with transfers. If $n \mid m$, let $\pi_n = \pi_{n,m} : \mu_m \to \mu_n$ be given by $\zeta \mapsto \zeta^{m/n}$ and $j = j_{n,m} : \mu_n \to \mu_m$ be the natural inclusion. Thus $\pi \otimes \mathrm{id} : \mu_m \otimes \mu_n \to \mu_n \otimes \mu_n$ is an isomorphism. Let $J : \mu_n \otimes \mu_n \to \mu_m \otimes \mu_m$ be the map $J = (\mathrm{id} \otimes j) \circ (\pi \otimes \mathrm{id})^{-1}$, so that $J(\zeta \otimes \eta) = \zeta \otimes \eta'$ where $\eta' \in \mu_m$ satisfies $(\eta')^{n/m} = \eta$.

Whenever $(F,n) \to (E,m)$ is a morphism, then there are homomorphisms

$$\mathrm{res} \circ J_* : H^2(G_F, \mu_n^{\otimes 2}) \to H^2(G_E, \mu_m^{\otimes 2})$$

and

$$(\pi \otimes \pi)_* \circ \mathrm{cores} : H^2(G_E, \mu_m^{\otimes 2}) \to H^2(G_F, \mu_n^{\otimes 2})$$

which make $H^2(G_F, \mu_n^{\otimes 2})$ into a covariant and contravariant functor. Furthermore, the composite of these two homomorphisms is

$$(\pi\otimes\pi)_*\circ \mathrm{cores}\circ \mathrm{res}\circ J_* = ((\pi\otimes\pi)\circ J)_*[E:F] = \frac{m}{n}\,[E:F]$$

since $(\pi\otimes\pi)\circ J$ is the map $\zeta\otimes\eta\mapsto(\zeta\otimes\eta)^{m/n}$. Thus $H^2(G_F,\mu_n^{\otimes 2})$ is a functor with transfers.

Lemma 7.1. *For any field k and $(F,n)\in \mathrm{Obj}\,\mathcal{C}_k$, $\Gamma_{F,n} : K_2F/n \to H^2(G_F,\mu_n^{\otimes 2})$ is a map of functors with transfers.*

Proof. Suppose that n divides m. Given a morphism $(F,n)\to(E,m)$ we can factor it as $(F,n)\to(F,m)\to(E,m)$ and we prove the result for the morphisms $(F,n)\to(F,m)$ and $(F,m)\to(E,m)$ separately.

Note that the following:

$$\begin{tikzcd}
1 \arrow[r] & \mu_m \arrow[r] \arrow[d, shift right, "\pi"'] & F_{\mathrm{sep}}^* \arrow[r, "m"] \arrow[d, shift right, "\pi"'] & F_{\mathrm{sep}}^* \arrow[r] \arrow[d, shift right, "\mathrm{id}"'] & 1 \\
1 \arrow[r] & \mu_n \arrow[r] \arrow[u, shift right, "j"'] & F_{\mathrm{sep}}^* \arrow[r, "n"] \arrow[u, shift right, "\mathrm{id}"'] & F_{\mathrm{sep}}^* \arrow[r] \arrow[u, shift right, "m/n"'] & 1
\end{tikzcd}$$

is a commutative diagram of G_F-modules. Hence $j_*(d_n(x)) = d_m(x^{m/n})$ and $\pi_*(d_m(y)) = d_n(y)$ for x and $y\in F^*$. Then the diagram

$$\begin{tikzcd}
K_2F/m \arrow[r, "\Gamma_{F,m}"] & H^2(G_F,\mu_m^{\otimes 2}) \\
K_2F/n \arrow[r, "\Gamma_{F,n}"] \arrow[u, "m/n"] & H^2(G_F,\mu_n^{\otimes 2}) \arrow[u, "J_*"']
\end{tikzcd}$$

commutes. For,

$$(\pi\otimes\mathrm{id})_*(d_m(a)\cup d_n(b)) = d_n(a)\cup d_n(b)$$

and so

$$\begin{aligned}
J_*(\Gamma_{F,n}(\{a,b\})) &= (\mathrm{id}\otimes j)_*\circ(\pi\otimes\mathrm{id})_*^{-1}(d_n(a)\cup d_n(b))\\
&= (\mathrm{id}\otimes j)_*(d_m(a)\cup d_n(b)) = d_m(a)\cup d_m(b^{m/n}) = \Gamma_{F,m}(\{a,b\})^{m/n}.
\end{aligned}$$

In the other direction,

$$\begin{tikzcd}
K_2F/m \arrow[r, "\Gamma_{F,m}"] \arrow[d] & H^2(G_F,\mu_m^{\otimes 2}) \arrow[d, "(\pi\otimes\pi)_*"] \\
K_2F/n \arrow[r, "\Gamma_{F,n}"] & H^2(G_F,\mu_n^{\otimes 2})
\end{tikzcd}$$

commutes. For, given $x=\{a,b\}\in K_2F/m$,

$$
\begin{aligned}
(\pi\otimes\pi)_*(\Gamma_{F,m}(\{a,b\})) &= (\pi\otimes\pi)_*(d_m(a)\cup d_m(b))\\
&= \pi_*(d_m(a))\cup\pi_*(d_m(b)) = d_n(a)\cup d_n(b) = \Gamma_n(\{a,b\}).
\end{aligned}
$$

Now $(F,m)\to(E,m)$ is a morphism. Then

$$
\begin{array}{ccc}
K_2E/m & \xrightarrow{\Gamma_{E,m}} & H^2(G_E,\mu_m^{\otimes 2})\\
\uparrow & & \uparrow{\scriptstyle \mathrm{res}}\\
K_2F/m & \xrightarrow{\Gamma_{F,m}} & H^2(G_F,\mu_m^{\otimes 2})
\end{array}
$$

commutes, since

$$
\mathrm{res}(d_{F,m}(a)\cup d_{F,m}(b)) = (\mathrm{res}(d_{F,m}(a))\cup\mathrm{res}(d_{F,m}(b))) = d_{E,m}(a)\cup d_{E,m}(b).
$$

Finally the diagram

$$
\begin{array}{ccc}
K_2E/m & \xrightarrow{\Gamma_{E,m}} & H^2(G_E,\mu_m^{\otimes 2})\\
{\scriptstyle \mathrm{tr}_{E/F}}\downarrow & & \downarrow{\scriptstyle \mathrm{cores}}\\
K_2F/m & \xrightarrow{\Gamma_{F,n}} & H^2(G_F,\mu_m^{\otimes 2})
\end{array}
$$

commutes (see for example [10, Chapter 8, Lemma 8.7]). ■

LEMMA 7.2. *For any local field k and $(F,n)\in\mathrm{Obj}\,\mathcal{C}_k$ the map $\theta_{F,n}$: $H^2(G_F,\mu_n^{\otimes 2})\to H_0(G_F,\mu_n)$ is a map of functors with transfers.*

Proof. Suppose there is a morphism $(F,n)\to(E,m)$. We begin by showing that the following diagram commutes:

$$
\begin{array}{ccc}
H^2(G_F,\mu_n^{\otimes 2}) & \xrightarrow{\theta_{F,n}} & H_0(G_F,\mu_n)\\
{\scriptstyle J_*}\downarrow & & \downarrow{\scriptstyle j_*}\\
H_2(G_F,\mu_m^{\otimes 2}) & \xrightarrow{\theta_{F,m}} & H_0(G_F,\mu_m)
\end{array}
$$

By definition of θ this amounts to showing, for all $x\in H^2(G_F,\mu_n^{\otimes 2})$ and $\chi\in\mathrm{Hom}_{G_F}(\mu_m,\mathbb{Q}/\mathbb{Z})$, that $\mathrm{inv}_F(J_*x\cup\chi)=\mathrm{inv}_F(x\cup(\chi|_{\mu_n}))$, that is, $J_*x\cup\chi = x\cup(\chi|_{\mu_n})$. But $J_*x\cup\chi = x\cup\chi\circ J$ (by definition of J_*). Now for $\zeta\otimes\eta\in\mu_n\otimes\mu_n$, $\chi(J(\zeta\otimes\eta))=\chi(\zeta\otimes\eta')=\zeta^{m\chi(\eta')}=\zeta^{n\chi(\eta)}=\chi|_{\mu_n}(\zeta\otimes\eta)$.

Next, the diagram

$$
\begin{array}{ccc}
H^2(G_F,\mu_m^{\otimes 2}) & \xrightarrow{\theta_{F,m}} & H_0(G_F,\mu_m)\\
{\scriptstyle \mathrm{res}}\downarrow & & \downarrow{\scriptstyle N_{E/F}}\\
H^2(G_E,\mu_m^{\otimes 2}) & \xrightarrow{\theta_{E,m}} & H_0(G_E,\mu_m)
\end{array}
$$

commutes. For, given $x \in H^2(G_F, \mu^{\otimes 2})$ and $\chi \in H^0(G_E, \mu_m)$,

$$\begin{aligned}\chi(\theta_{E,m}(\mathrm{res}(x))) &= \mathrm{inv}_E(\mathrm{res}(x) \cup \chi) = \mathrm{inv}_E(\mathrm{res}(x \cup \mathrm{cores}(\chi)))\\ &= \mathrm{inv}_F(x \cup \mathrm{cores}(\chi)) \quad (\text{since } \mathrm{inv}_E \circ \mathrm{res} = \mathrm{inv}_F)\\ &= \mathrm{cores}(\chi)(\theta_{F,m}(x)) = \chi(N_{E/F}(\theta_{F,m}(x))).\end{aligned}$$

These two diagrams together show that $\theta_{F,n}$ is a map of covariant functors.

In the other direction, the diagram

$$\begin{tikzcd} H^2(G_E, \mu_m^{\otimes 2}) \arrow[r, "\theta_{E,m}"] \arrow[d, "\mathrm{cores}"'] & H_0(G_E, \mu_m) \arrow[d] \\ H^2(G_F, \mu_m^{\otimes 2}) \arrow[r, "\theta_{F,m}"] & H_0(G_F, \mu_m) \end{tikzcd}$$

commutes. For, given $x \in H^2(G_E, \mu_m^{\otimes 2})$ and $\chi \in H^0(G_F, \mu_m)$ we have

$$\begin{aligned}\chi(\theta_{F,m}(\mathrm{cores}(x))) &= \mathrm{inv}_F(\mathrm{cores}(x) \cup \chi) = \mathrm{inv}_E(\mathrm{res}(\mathrm{cores}(x) \cup \chi))\\ &= \mathrm{inv}_E(x \cup \mathrm{res}(\chi)) = \mathrm{res}(\chi)(\theta_{E,m}) = \chi(\theta_{E,m}(x))\end{aligned}$$

and hence $\theta_{F,m}(\mathrm{cores}(x)) = \theta_{E,m}(x)$.

Finally the diagram

$$\begin{tikzcd} H^2(G_F, \mu_m^{\otimes 2}) \arrow[r, "\theta_{E,m}"] \arrow[d, "(\pi\otimes\pi)_*"'] & H_0(G_F, \mu_m) \arrow[d, "m/n"] \\ H^2(G_F, \mu_n^{\otimes 2}) \arrow[r, "\theta_{F,n}"] & H_0(G_F, \mu_n) \end{tikzcd}$$

commutes: for $x \in H^2(G_F, \mu_m^{\otimes 2})$ and $\chi \in H^0(G_F, \mu_n)$ it is enough to prove that $(\pi \otimes \pi)_*(x) \cup \chi = x \cup (\chi \circ (m/n))$. This is true since $(\pi \otimes \pi)_*(x) \cup \chi = x \cup \chi \circ (\pi \otimes \pi)$ and for $\zeta \otimes \eta \in \mu_m \otimes \mu_m$,

$$\begin{aligned}\chi((\pi \otimes \pi)(\zeta \otimes \eta)) &= \chi(\zeta^{m/n} \otimes \eta^{m/n}) = (\zeta^{m/n})^{n\chi(\eta^{m/n})}\\ &= \zeta^{m\chi(\eta^{m/n})} = \chi \circ (m/n)(\zeta \otimes \eta). \ \blacksquare\end{aligned}$$

Thus we have proven Theorem 2.1(2). If $F = \mathbb{R}$ or $F = \mathbb{C}$ then the result follows from Remark 6.1 and Lemma 7.1. For F a non-archimedean local field it is a consequence of Lemmas 7.1 and 7.2 together with Theorem 6.5.

8. Main results. Finally we prove Theorem 6.5. We need a few preliminary results. Fix $n \in \mathbb{N}$. Let

$$D_n : \widehat{H}^i\Big(G_n, \frac{1}{n}\mathbb{Z}/\mathbb{Z}\Big) \to \widehat{H}^{i+1}(G_n, \mathbb{Z})$$

be the connecting homomorphism associated to the exact sequence of G_n-modules

$$0 \to \mathbb{Z} \to \frac{1}{n}\mathbb{Z} \to \frac{1}{n}\mathbb{Z}/\mathbb{Z} \to 0.$$

The crucial part of the comparison of the symbols $\lambda_{F,n}$ and $\gamma_{F,n}$ is to relate the connecting homomorphisms D_n and δ_n. The following lemma shows us how to do that.

LEMMA 8.1. *Let $a \in F^*$, $\chi \in H^1(G_n, (1/n)\mathbb{Z}/\mathbb{Z})$. Let $\beta : \mu_n \otimes (1/n)\mathbb{Z}/\mathbb{Z} \to F_n^*$ be the map $\zeta \otimes (r/n) \mapsto \zeta^r$. Then*

$$\beta_*(\delta_n(a) \cup \chi) = -a \cup D_n\chi$$

in $H^2(G_n, F_n^*)$.

The lemma is a special case of the following technical lemma.

LEMMA 8.2. *Let G be a finite group and A a G-module. Let $n \in \mathbb{N}$. Let D and d, respectively, be the connecting homomorphisms associated to the sequences of G-modules*

$$0 \to \mathbb{Z} \to \frac{1}{n}\mathbb{Z} \to \frac{1}{n}\mathbb{Z}/\mathbb{Z} \to 0 \quad \textit{and} \quad 0 \to A[n] \to A \to nA \to 0.$$

Let $x \in H^i(G, nA)$, $y \in H^j(G, (1/n)\mathbb{Z}/\mathbb{Z})$. Let α be the inclusion $nA \to A$ and let β be the map $A[n] \otimes (1/n)\mathbb{Z}/\mathbb{Z} \to A$, $a \otimes (r/n) \mapsto ra$. Then

$$(-1)^i(\alpha_* x \cup Dy) + \beta_*(dx \cup y) = 0 \quad \textit{in } H^{i+j+1}(G, A).$$

COROLLARY 8.3. *Let a, β and χ be as in Lemma 8.1. Then*

$$D_n(\varrho_n(a) \cup \chi) = -I_n(\beta_*(\delta_n(a) \cup \chi)) \quad \textit{in } \widehat{H}^0(G_n, \mathbb{Z}),$$

where $I_n : \widehat{H}^2(G_n, F_n^*) \to \widehat{H}^0(G_n, \mathbb{Z})$ is the inverse of the isomorphism induced by the cup-product with $u_n = u_{F_n/F}$.

Proof.

$$\begin{aligned} u_n \cup (D_n(\varrho_n(a) \cup \chi)) &= u_n \cup (\varrho_n(a) \cup D_n(\chi)) = (u_n \cup \varrho_n(a)) \cup D_n\chi \\ &= \overline{a} \cup D_n\chi \quad \text{(by definition of } \varrho_n) \\ &= a \cup D_n\chi \\ &= -\beta_*(\delta_n(a) \cup \chi) \quad \text{(by Lemma 8.1).} \ \blacksquare \end{aligned}$$

For any G_n-module A and any $i \geq 0$ let res denote the natural map

$$\mathrm{res} : H^i(G_n, A) \to H^i(G_F, A).$$

COROLLARY 8.4. *For F a local field and n relatively prime to* char F *the following diagram anti-commutes for any $n \in \mathbb{N}$:*

$$\begin{tikzcd}
\widehat{H}^{-2}(G_n,\mathbb{Z}) \times \widehat{H}^1(G_n,(1/n)\mathbb{Z}/\mathbb{Z}) \arrow[r, "\cup"] \arrow[dd, "\mathrm{res}"] & \widehat{H}^{-1}(G_n,(1/n)\mathbb{Z}/\mathbb{Z}) \arrow[d, equal] \\
F^* \arrow[u, "\varrho_n"] \arrow[d, "d_n"'] & (1/n)\mathbb{Z}/\mathbb{Z} \\
H^1(G_F,\mu_n) \times \widehat{H}^1(G_F,(1/n)\mathbb{Z}/\mathbb{Z}) \arrow[r, "\cup"] & H^2(G_F,\mu_n) \arrow[u, "\mathrm{inv}_F"']
\end{tikzcd}$$

where the cup-product at the bottom is taken with respect to the map

$$\mu_n \otimes \frac{1}{n}\mathbb{Z}/\mathbb{Z} \to \mu_n, \qquad \zeta \otimes \frac{r}{n} \mapsto \zeta^r.$$

Proof. Let $N = [F_n : F] = |G_n|$ and $\mathrm{inv}_n = \mathrm{inv}_F \circ \inf_{F_n/F} : H^2(G_n, F_n^*) \to \mathbb{Q}/\mathbb{Z}$. Observe that the connecting homomorphism

$$D_N : \frac{1}{N}\mathbb{Z}/\mathbb{Z} = \widehat{H}^{-1}\left(G_n, \frac{1}{N}\mathbb{Z}/\mathbb{Z}\right) \to \widehat{H}^0(G_n,\mathbb{Z}) \cong \mathbb{Z}/|G_n|\mathbb{Z} = \mathbb{Z}/N\mathbb{Z}$$

is an isomorphism and $\mathrm{inv}_n(x) = D_N^{-1}(I_n(x))$ for all $x \in \widehat{H}^2(G_n, F_n^*)$. (For, $D_N(1/N) = 1$, so $D_N^{-1}(I_n(u_n)) = D_N^{-1}(I_n(u_n \cup 1)) = D_N^{-1}(1) = 1/N = \mathrm{inv}_n(u_n)$.)

Hence for any $a \in F^*$, and $\chi \in \widehat{H}^1(G_n,(1/n)\mathbb{Z}/\mathbb{Z})$, we have

$$\begin{aligned}
\mathrm{inv}_F(d_n(a) \cup \mathrm{res}(\chi)) &= \mathrm{inv}_F(\beta_*(d_n(a) \cup \mathrm{res}(\chi))) \\
&= \mathrm{inv}_F(\beta_*(\mathrm{res}(\delta_n(a)) \cup \mathrm{res}(\chi))) \\
&= \mathrm{inv}_F(\beta_*(\mathrm{res}(\delta_n(a) \cup \chi))) \\
&= \mathrm{inv}_F(\inf_{F_n/F}(\beta_*(\delta_n(a) \cup \chi))) \\
&= D_N^{-1}(I_n(\beta_*(\delta_n(a) \cup \chi))) \\
&= -D_N^{-1}(D_n(\varrho_n(a) \cup \chi)) \quad \text{(by Corollary 8.3)} \\
&= -\varrho_n(a) \cup \chi
\end{aligned}$$

since $D_N^{-1} \circ D_n : (1/n)\mathbb{Z}/\mathbb{Z} \to (1/N)\mathbb{Z}/\mathbb{Z}$ is the natural inclusion. ∎

REMARK 8.5. A closely related diagram for the case $\mu_n \subset F$ occurs in [7, VII, Prop. 7.2.13], but a proof is not given there. The essential point is to relate the reciprocity map ϱ_n to the connecting homomorphism δ_n; that is, our Corollary 8.3.

We are now in a position to prove Theorem 6.5. For, suppose that F is a non-archimedean local field and let $a, b \in F^*$. For some $n \geq 1$ let

$\chi \in H^0(G_F, \mu_n^\#) = \widehat{H}^0(G_n, \mu_n^\#)$. Then

$$\begin{aligned}\mathrm{inv}_F(\gamma_n(a,b) \cup \chi) &= \mathrm{inv}_F((d_n(a) \cup d_n(b)) \cup \chi) \\ &= \mathrm{inv}_F(\beta_*(d_n(a) \cup (d_n(b) \cup \chi))) \\ &= \mathrm{inv}_F(\beta_*(d_n(a) \cup \mathrm{res}(\delta_n(b) \cup \chi))) \\ &= -\varrho_n(a) \cup (\delta_n(b) \cup \chi) \quad \text{(by Corollary 8.4)} \\ &= \chi(-\lambda_n(a,b))\end{aligned}$$

since the map $\widehat{H}^{-1}(G_F, \mu_n) \times H^0(G_F, \mu_n^\#) \to H^0(G_F, \mathbb{Q}/\mathbb{Z})$ induced by the cup-product is identical to the evaluation map $H_0(G_F, \mu_n) \times H^0(G_F, \mu_n^\#) \to \mathbb{Q}/\mathbb{Z}$. Since this equation holds for all $\chi \in H^0(G_F, \mu_n^\#)$, it follows that

$$-\lambda_n(\{a,b\}) = \theta_n(\gamma_n(\{a,b\}))$$

by definition of θ_n. ∎

9. Proof of Lemma 8.2. Finally we give the proof of Lemma 8.2. This proof was suggested by Serre's proof of [9, Chapter XIV, 2, Proposition 5].

We begin by recalling the formula for the cup-product of cocycles in terms of the bar resolution: if $x \in H^i(G,M)$ and $y \in H^j(G,N)$ are represented by $f \in C^i(G,M)$ and $g \in C^j(G,N)$ then $x \cup y$ is represented by the cocycle $f \cup g \in C^{i+j}(G, M \otimes N)$ given by

$$f \cup g(\sigma_1, \ldots, \sigma_{i+j}) = f(\sigma_1, \ldots, \sigma_j) \otimes \sigma_1 \cdots \sigma_j g(\sigma_{j+1}, \ldots, \sigma_{i+j}).$$

Next recall that if

$$0 \to X \to Y \to Z \to 0$$

is a short exact sequence of G-modules then the associated connecting homomorphism, δ, is described as follows: if $z \in H^k(G,Z)$ is represented by the cocycle $h \in C^k(G,Z)$ then $\delta z \in H^{k+1}(G,X)$ is represented by the cocycle $\delta h \in C^{k+1}(G,X)$ defined by

$$\begin{aligned}&\delta h(\sigma_1, \ldots, \sigma_{k+1}) = \sigma_1 l(h(\sigma_2, \ldots, \sigma_{k+1})) \\ &\quad + \sum_{r=1}^{k} (-1)^r l(h(\sigma_1, \ldots, \sigma_r\sigma_{r+1}, \ldots, \sigma_{k+1})) + (-1)^{k+1} l(h(\sigma_1, \ldots, \sigma_k))\end{aligned}$$

where $l : Z \to Y$ is any set-theoretic section of the surjective map $Y \to Z$.

Thus, let $t : (1/n)\mathbb{Z}/\mathbb{Z} \to (1/n)\mathbb{Z}$ and $s : nA \to A$ be fixed set-theoretic sections of the maps $(1/n)\mathbb{Z} \to (1/n)\mathbb{Z}/\mathbb{Z}$ and $A \to nA$. Observe:

(1) If $a_i \in A$ are such that $a = \sum a_i \in A[n]$ and if $b \in (1/n)\mathbb{Z}/\mathbb{Z}$ then

$$\beta(a \otimes b) = a \cdot nt(b) = \sum_i a_i nt(b) \in A.$$

(2) If $a \in nA$ and $b_j \in (1/n)\mathbb{Z}$ are such that $\sum_j b_j \in \mathbb{Z}$, then $a(\sum_j b_j) = \sum_j s(a) \cdot (nb_j)$ in A.

Suppose now that the elements $x \in H^i(G, nA)$ and $y \in H^j(G, (1/n)\mathbb{Z}/\mathbb{Z})$ are represented by the cocycles $f \in C^i(G, nA)$ and $g \in C^j(G, (1/n)\mathbb{Z}/\mathbb{Z})$. Then $\alpha_* x$ is represented by $\alpha_* f = \alpha \circ f \in C^i(G, A)$. Thus $\alpha_* x \cup Dy$ is represented by $\alpha_* f \cup Dg \in C^{i+j+1}(G, A)$ where

$$\begin{aligned}
&(\alpha_* f \cup Dg)(\sigma_1, \ldots, \sigma_{i+j+1}) \\
&\quad = f(\sigma_1, \ldots, \sigma_i) \cdot \Big[t(g(\sigma_{i+2}, \ldots, \sigma_{i+j+1})) \\
&\qquad + \sum_{r=1}^{j} (-1)^r t(g(\ldots, \sigma_{i+r}\sigma_{i+r+1}, \ldots)) + (-1)^{j+1} t(g(\sigma_{i+1}, \ldots, \sigma_{i+j})) \Big] \\
&\quad = s(f(\sigma_1, \ldots, \sigma_i)) \cdot nt(g(\sigma_{i+2}, \ldots, \sigma_{i+j+1})) \\
&\qquad + s(f(\sigma_1, \ldots, \sigma_i)) \cdot \Big(\sum (-1)^r nt(g(\ldots, \sigma_{i+r}\sigma_{i+r+1}, \ldots)) \Big) \\
&\qquad + (-1)^{j+1} s(f(\sigma_1, \ldots, \sigma_i)) \cdot nt(g(\sigma_{i+1}, \ldots, \sigma_{i+j})),
\end{aligned}$$

by (2) above. Similarly, $\beta_*(dx \cup y)$ is represented by the cocycle $\beta_*(df \cup g) \in C^{i+j+1}(G, A)$ where

$$\begin{aligned}
&\beta_*(df \cup g)(\sigma_1, \ldots, \sigma_{i+j+1}) \\
&\qquad = \beta\Big(\Big[\sigma_1 s(f(\sigma_2, \ldots, \sigma_{i+1})) + \sum_{k=1}^{i} (-1)^k s(f(\ldots, \sigma_k\sigma_{k+1}, \ldots)) \\
&\qquad\quad + (-1)^{i+1} s(f(\sigma_1, \ldots, \sigma_i)) \Big] \otimes g(\sigma_{i+2}, \ldots, \sigma_{i+j+1}) \Big) \\
&\qquad = \sigma_1 s(f(\sigma_2, \ldots, \sigma_{i+1})) \cdot nt(g(\sigma_{i+1}, \ldots, \sigma_{i+j+1})) \\
&\qquad\quad + \sum_{k=1}^{i} (-1)^k s(f(\ldots, \sigma_k\sigma_{k+1}, \ldots)) \cdot nt(g(\sigma_{i+2}, \ldots, \sigma_{i+j+1})) \\
&\qquad\quad + (-1)^{i+1} s(f(\sigma_1, \ldots, \sigma_i)) \cdot nt(g(\sigma_{i+2}, \ldots, \sigma_{i+j+1})),
\end{aligned}$$

by (1) above.

Hence the term $s(f(\sigma_1, \ldots, \sigma_i)) \cdot nt(g(\sigma_{i+2}, \ldots, \sigma_{i+j+1}))$ occurs twice but with opposite signs in the expression $(-1)^j(\alpha_* f \cup Dg) + \beta_*(df \cup g)$. So this expression becomes

$$\begin{aligned}
&(-1)^j(\alpha_* f \cup Dg) + \beta_*(df \cup g) \\
&\qquad = \sigma_1 s(f(\sigma_2, \ldots, \sigma_{i+1})) \cdot nt(g(\sigma_{i+1}, \ldots, \sigma_{i+j+1})) \\
&\qquad\quad + \sum_{k=1}^{i} (-1)^k s(f(\ldots, \sigma_k\sigma_{k+1}, \ldots)) \cdot nt(g(\sigma_{i+2}, \ldots, \sigma_{i+j+1}))
\end{aligned}$$

$$+\sum_{k=i+1}^{i+j}(-1)^k s(f(\sigma_1,\ldots,\sigma_i))\cdot nt(g(\ldots,\sigma_k\sigma_{k+1},\ldots))$$
$$+(-1)^{i+j+1}s(f(\sigma_1,\ldots,\sigma_i))\cdot nt(g(\sigma_{i+1},\ldots,\sigma_{i+j})).$$

But this is just the coboundary of the cocycle $h\in C^{i+j}(G,A)$ defined by

$$h(\sigma_1,\ldots,\sigma_{i+j})=s(f(\sigma_1,\ldots,\sigma_i))\cdot nt(g(\sigma_{i+1},\ldots,\sigma_{i+j})). \ \blacksquare$$

10. An application to the wild kernel. Let F be a number field. For any $n\in\mathbb{N}$ and any (finite or infinite) place, v, of F the Hilbert symbol of order n on F_v induces a map

$$H_{v,n}:K_2F\to(\mu_n)_{G_v}.$$

LEMMA 10.1. *For any number field F and any $n\in\mathbb{N}$ there is an exact sequence*

$$K_2F/n\xrightarrow{H_n}\coprod_{\substack{v\text{ finite or}\\ \text{real infinite}}}(\mu_n)_{G_v}\xrightarrow{\Pi_n}(\mu_n)_{G_F}\to 1,$$

where $H_n=\sum_v H_{n,v}$ and Π_n is the natural product map.

Proof. The Poitou–Tate exact sequence ([7, VIII, 8.6.13]) for the number field F gives an exact sequence

$$H^2(G_F,\mu_n^{\otimes 2})\xrightarrow{\beta_n}\coprod_v H^2(G_v,\mu_n^{\otimes 2})\xrightarrow{P_n}H^0(G_F,(\mu_n^{\otimes 2})')^{\#},$$

where the map β_n is induced from the restriction maps $H^2(G_F,\mu_n^{\otimes 2})\to H^2(G_v,\mu_n^{\otimes 2})$ and P_n is induced from the duals of the natural maps

$$H^0(G_F,(\mu_n^{\otimes 2})')\to H^0(G_v,(\mu_n^{\otimes 2})')\xrightarrow{\sim}H^2(G_v,\mu_n^{\otimes 2})^{\#}.$$

It follows by Theorem 6.5 that we have a commutative (up to sign) diagram:

$$\begin{tikzcd}
K_2F/n \arrow[r, "H_n"] \arrow[d, "\gamma_{F,n}"'] & \coprod_v(\mu_n)_{G_v} \arrow[r] & (\mu_n)_{G_F} \arrow[r] & 1 \\
H^2(G_F,\mu_n^{\otimes 2}) \arrow[r] & \coprod_v H^2(G_v,\mu_n^{\otimes 2}) \arrow[r] \arrow[u, "\theta_n=\coprod\theta_{v,n}"'] & H^0(G_F,(\mu_n^{\otimes 2})')^{\#} \arrow[r] \arrow[u, "\wr"'] & 1
\end{tikzcd}$$

Here $\gamma_{F,n}$ is an isomorphism by [11] and θ_n is an isomorphism by Lemma 6.4 above. Thus the top row is exact as required. $\blacksquare$

Thus, the Hilbert symbols $H_{v,n}$ satisfy the reciprocity laws, $\prod_v H_{v,n}(a,b) = 1$ (in $(\mu_n)_{G_F}$) for all $a,b\in F$.

Now let $Ш_n^2(F)=Ш^2(F,\mu_n^{\otimes 2}):=\ker(\beta_n)$. By Poitou–Tate duality,

$$Ш_n^2(F)\cong Ш^1(F,(\mu_n^{\otimes 2})')^{\#}=Ш^1(F,\mu_n^{\#})^{\#}.$$

Now $\mu_n^{\#} \cong \mu_n$ (non-canonically) as G_F-modules. By the Hasse principle for the G_F-module μ_n,

$$Ш^1(F,\mu_n) = \begin{cases} \mathbb{Z}/2 & \text{if } F \text{ is } \textit{special} \text{ and } n \text{ is large enough,} \\ 0 & \text{otherwise.} \end{cases}$$

Here, F is *special* if $\mathrm{Gal}(F(\mu_{2^m})/F)$ is non-cyclic for all sufficiently large m (i.e. F is *exceptional*) and every dyadic prime of F decomposes in $F(\mu_{2^m})$ for any sufficiently large m. It follows that $Ш^2_n(F) = Ш^1(F,\mu_n)$.

For a number field F, let $Ш^2(F)$ denote the limiting value of $Ш^2_n(F)$ for large n ("large" means divisible by a sufficiently large power of 2).

Corollary 10.2. *For all n there is an exact sequence*

$$0 \to Ш^2_n(F) \to K_2F/n \to \coprod_v (\mu_n)_{G_v} \to (\mu_n)_{G_F} \to 1.$$

Let $\mu_v = \mu(F_v)$ for any place v and let $m_v = |\mu_v|$. Recall that the wild kernel, WK_2F, of the number field F is the kernel of the map $H : K_2F \to \coprod_v \mu_v$, $x \mapsto (H_{v,m_v}(x))_v$. Let $K_2^\infty F = \bigcap_{n\in\mathbb{N}}(K_2F)^n$ be the group of infinitely divisible elements of K_2F.

Theorem 10.3. *There is a natural short exact sequence*

$$1 \to K_2^\infty F \to WK_2F \to Ш^2(F) \to 1.$$

Proof. Let $k_n(F) = K_2F/((K_2F)^n.Ш^2_n(F))$. If $n\,|\,m$ the identity map on K_2F induces a surjection $p_{n,m} : k_m(F) \to k_n(F)$ which fits into a map of short exact sequences,

$$\begin{tikzcd}
1 \arrow[r] & k_n(F) \arrow[r, "H_n"] & \coprod_v (\mu_n)_{G_v} \arrow[r] & (\mu_n)_{G_F} \arrow[r] & 1 \\
1 \arrow[r] & k_m(F) \arrow[r] \arrow[u, "p_{n,m}"] & \coprod_v (\mu_m)_{G_v} \arrow[r] \arrow[u, "m/n"'] & (\mu_m)_{G_F} \arrow[r] \arrow[u, "m/n"'] & 1
\end{tikzcd}$$

Thus, taking a limit over the set $\mathbb{N}$ (ordered by divisibility) we obtain a short exact sequence

$$1 \to \lim k_n(F) \to \lim \coprod_v (\mu_n)_{G_v} \to \lim\,(\mu_n)_{G_F} \to 1.$$

Observe that when $K = F$ or $K = F_v$ for some v the diagram

$$\begin{tikzcd}
(\mu_n)_{G_K} \arrow[r] & \mu_n(K) := \mu_s \\
(\mu_m)_{G_K} \arrow[r] \arrow[u, "m/n"'] & \mu_m(K) := \mu_r \arrow[u, "r/s"]
\end{tikzcd}$$

commutes for any $n \mid m$. It follows that $\lim\,(\mu_n)_{G_K} \cong \mu_K$ via $(\bar{\zeta}_n)_n \to \zeta_N^{N/m_K}$ for sufficiently large N. Furthermore, by Corollary 2.2 above, the diagrams

$$\begin{tikzcd} K_2F/n \arrow[r, "H_{v,n}"] \arrow[dr, "H_{v,s}"'] & (\mu_n)_{G_v} \arrow[d, "\wr"] \\ & \mu_n(F_v) = \mu_s \end{tikzcd}$$

commute for all v and n.

Observe that the natural map

$$i : \coprod_v \mu_v \cong \coprod_v \lim\,(\mu_n)_{G_v} \to \lim \coprod_v (\mu_n)_{G_v}$$

is injective.

Putting all this together, we obtain a commutative diagram

$$\begin{tikzcd} & K_2F \arrow[r, "H"] \arrow[d, "j"'] & \coprod_v \mu_v \arrow[r, "\pi"] \arrow[d, "i"] & \mu_F \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \lim k_n(F) \arrow[r] & \lim \coprod_v (\mu_n)_{G_v} \arrow[r] & \lim\,(\mu_n)_{G_F} \arrow[r] & 1 \end{tikzcd}$$

where $\pi((\zeta_v)_v) = \prod \zeta_v^{m_v/m_F}$ (in fact, the top row is exact by Moore's reciprocity uniqueness theorem).

Since i is injective, $WK_2F = \ker(H) = \ker(j)$. Observe also that $K_2^\infty F$ is the kernel of the natural map $K_2F \to \lim K_2F/n$. The result now follows by applying the snake lemma to the diagram

$$\begin{tikzcd} & & K_2F \arrow[r, equal] \arrow[d, "J"'] & K_2F \arrow[d, "j"] & \\ 1 \arrow[r] & Ш^2(F) \arrow[r] & \lim K_2F/n \arrow[r] & \lim k_n(F) \arrow[r] & 1 \end{tikzcd}$$

Note that $Ш^2(F) \subset K_2F/n$ for sufficiently large n, and thus $Ш^2(F) \subset \mathrm{image}(J)$. ∎

Thus $K_2^\infty F = WK_2F$ when F is not special and $K_2^\infty F$ has index 2 in WK_2F otherwise.

Note that this confirms and clarifies Corollary 4.5 of [4].

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