ACTA ARITHMETICA
164.2 (2014)

On the behaviour close to the unit circle of the power series with Möbius function coefficients

by

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1. Notations and introduction. We study the series

$$\mathfrak{M}(z) = \sum_{n=1}^{\infty} \mu(n) z^n,$$

where μ is the Möbius function. We will use the following notations:

$$e^{2\pi i\theta} = e(\theta); \qquad M(x,\theta) = \sum_{n\le x} \mu(n) e(n\theta); \qquad \tau(\chi, l) = \sum_{k=1}^{q} \chi(k) e(lk/q)$$

for a character χ modulo q; $\bar{\chi}$ is the character conjugate to χ; $\tau(\chi) = \tau(\chi, 1)$; GRH stands for the generalized Riemann hypothesis.

Let $g(x) \ge 0$. Then $f(x) = \Omega(g(x))$ as $x \to a$ means that there is an infinite sequence $t_k \to a$ such that $|f(t_k)| > \delta g(t_k)$ for some $\delta > 0$. Let $f(x)$ be real, $g(x) \ge 0$. Then $f(x) = \Omega_{\pm}(g(x))$ as $x \to a$ means that there are infinite sequences $t_k \to a$, $u_k \to a$ such that $f(t_k) > \delta g(t_k)$, $f(u_k) < -\delta g(u_k)$ for some $\delta > 0$.

From the Szegő theorem [S] it easily follows that the unit circle is the natural boundary of $\mathfrak{M}(z)$. In 1967 I. Katai [Ka] proved that

$$\mathfrak{M}(r) = \Omega_{\pm}((1-r)^{-1/2}), \qquad r \to 1- .$$

In 2000 this result was reproved by Delange [D]. While Katai used complicated integral inequalities, Delange [D] applied E. Landau's theorem on Dirichlet series with non-negative coefficients. In 2010 S. Gerhold [G] proved the following estimate: if z tends to 1 in an arbitrary sector of the form

2010 *Mathematics Subject Classification*: Primary 11N37; Secondary 30B30.
Key words and phrases: power series, Möbius function, omega-estimates, arithmetic functions.

$|\arg(1-z)| \le \frac{1}{2}\pi - \delta$, $\delta > 0$, then

$$\mathfrak{M}(z) = O\left(\frac{1}{t}\exp\left(-\frac{0.0203\log\frac{1}{t}}{\left(\log\log\frac{1}{t}\right)^{2/3}\left(\log\log\log\frac{1}{t}\right)^{1/3}}\right)\right), \qquad t = -\log z.$$

In 1991 Baker and Harman [BH] proved that if GRH is true, then

$$M(x,\theta) \ll x^{3/4+\varepsilon}, \qquad x \to +\infty,$$

for any real number θ. From their result it easily follows that under GRH,

$$\mathfrak{M}(e(\theta)r) \ll (1-r)^{-3/4-\varepsilon}, \qquad r \to 1- .$$

In this paper we obtain unconditional Ω-results.

THEOREM 1.1. *For each $\beta \in \mathbb{Q}$ there exists $a > 0$ such that*

$$\mathfrak{M}(e(\beta)r) = \Omega((1-r)^{-a}), \qquad r \to 1-, \tag{1.1}$$

$$M(x,\beta) = \Omega(x^{a}), \qquad x \to +\infty. \tag{1.2}$$

This theorem is proved in Sections 2–5.

In Section 6 we study the behaviour of these functions for $\beta \in \mathbb{Q}$ with denominators $q \le 100$ and obtain the following results that are stronger than those for arbitrary β.

THEOREM 1.2. *If $q \le 100$ and $\beta = l/q$, then*

$$\mathfrak{M}(e(\beta)r) = \Omega((1-r)^{-1/2}), \qquad r \to 1-, \tag{1.3}$$

$$M(x,\beta) = \Omega(x^{1/2}), \qquad x \to +\infty. \tag{1.4}$$

2. Preliminary results. Let $\alpha(n)$ be a function of a natural variable. We define

$$\mathfrak{A}(z) = \sum_{n=1}^{\infty} \alpha(n) z^n \quad \text{and} \quad F(s) = \sum_{n=1}^{\infty} \alpha(n) n^{-s}.$$

For a Dirichlet character χ we define

$$F(s,\chi) = \sum_{n=1}^{\infty} \alpha(n)\chi(n) n^{-s}.$$

LEMMA 2.1. *Let $\alpha(n)$ be an arbitrary sequence of complex numbers, and $l \in \mathbb{Z}$. Suppose that the Dirichlet series $F(s) = \sum_{n=1}^{\infty} \alpha(n) e(ln/q) n^{-s}$ is convergent for $\sigma = \Re s > \sigma_0 > 0$. Then*

$$\Gamma(s)\sum_{n=1}^{\infty} \alpha(n) e(ln/q) n^{-s} = \int_0^{\infty} t^{s-1}\mathfrak{A}(e(l/q)e^{-t})\,dt.$$

Proof. This follows from the results of [H]. ∎

LEMMA 2.2. *Let $\alpha(n)$ be an arbitrary sequence of complex numbers, and $q > 1$. Suppose that the Dirichlet series $\sum_{n=1}^{\infty} \alpha(n)n^{-s}$ is absolutely convergent for $\Re s > \sigma_0$. Then for any $l \in \mathbb{Z}$ and s with $\Re s > \sigma_0$,*

$$\sum_{(n,q)=1} \frac{\alpha(n)e(ln/q)}{n^s} = \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \tau(\bar{\chi}, l) F(s, \chi).$$

Proof. We have

$$\sum_{(n,q)=1} \frac{\alpha(n)e(ln/q)}{n^s} = \sum_{n=1}^{\infty} \alpha(n) \frac{u(n)}{n^s}, \tag{2.1}$$

where $u(n) = e(ln/q)$ if $(n,q) = 1$, and $u(n) = 0$ if $(n,q) \neq 1$. Since $u(n) = \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \tau(\bar{\chi}, l)\chi(n)$, we obtain the conclusion. ∎

For convenience, define

$$F[\beta](s) = \sum_{n=1}^{\infty} \alpha(n)e(\beta n)n^{-s} \quad \text{for } \beta \in \mathbb{R}.$$

The following lemma relates $F[l/q](s)$ to $F(s,\chi)$.

LEMMA 2.3. *Let $\alpha(n)$ be a multiplicative function with $\alpha(n) = O(n^{\sigma_0})$, where $\sigma_0 > 0$, and $\alpha(n) = 0$ if there is a prime p with $p^2 \mid n$. If $\Re s > \sigma_0 + 1$ then for all integers $q > 1$ and l with $(l,q) = 1$,*

$$F[l/q](s) = \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \sum_{d \mid q} \frac{\alpha(d)}{d^s} \tau(\bar{\chi}, ld) F(s, \chi), \tag{2.2}$$

$$\int_0^{\infty} t^{s-1} \mathfrak{A}(e(l/q)e^{-t})\, dt = \Gamma(s) \bigg(\frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \sum_{d \mid q} \frac{\alpha(d)}{d^s} \tau(\bar{\chi}, ld) F(s, \chi) \bigg). \tag{2.3}$$

Proof. Let D be the set of square-free numbers. Note that every $n \in D$ has a unique representation

$$n = dm, \quad d \mid q,\ (m,q) = 1. \tag{2.4}$$

Set $B_d = \{n : n = dm,\ (m,q) = 1\}$ for each $d \mid q$. These sets do not intersect. From (2.4) it follows that $D \subseteq \bigcup_{d \mid q} B_d$, hence

$$D = \bigcup_{d \mid q} (B_d \cap D). \tag{2.5}$$

Let

$$S_d = \sum_{n \in B_d} \frac{\alpha(n)e(ln/q)}{n^s}.$$

Taking into account (2.5) we obtain

$$\sum_{n=1}^{\infty} \frac{\alpha(n)e(ln/q)}{n^s} = \sum_{d|q} S_d. \tag{2.6}$$

If $d \mid q$, then, by Lemma 2.2,

$$\begin{aligned} S_d &= \sum_{(n,q)=1} \frac{\alpha(nd)}{(nd)^s} e(ldn/q) = \frac{\alpha(d)}{d^s} \sum_{(n,q)=1} \frac{\alpha(n)}{n^s} e(ldn/q) \\ &= \frac{\alpha(d)}{d^s} \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \tau(\bar{\chi}, ld) F(s, \chi). \end{aligned}$$

Hence taking into account (2.6),

$$\sum_{n=1}^{\infty} \frac{\alpha(n)e(ln/q)}{n^s} = \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \sum_{d|q} \frac{\alpha(d)}{d^s} \tau(\bar{\chi}, ld) F(s, \chi).$$

Thus we have derived (2.2). Using Lemma 2.1 we obtain (2.3). ∎

The following theorem relates the behaviour of $\mathfrak{A}(e(l/q)r)$ as $r \to 1-$ to the behaviour of $F[l/q](s)$.

Theorem 2.4. *Let $\alpha(n)$ be an arbitrary sequence of complex numbers, and $q \in \mathbb{N}$, $q > 1$, $(l, q) = 1$. Suppose that $F[l/q](s)$ is meromorphic in $\{\Re s > 0\}$ and has a pole at $\sigma_0 + it_0$ with $\sigma_0 > 0$. Then*

$$\mathfrak{A}(e(l/q)r) = \Omega((1-r)^{-\sigma_0}), \qquad r \to 1-. \tag{2.7}$$

Proof. By Lemma 2.1,

$$\Gamma(s)F[l/q](s) = \int_0^{\infty} t^{s-1} \mathfrak{A}(e(l/q)e^{-t})\, dt.$$

Assume that for any $c > 0$ there exists a T such that $|\mathfrak{A}(e(l/q)e^{-t})| < ct^{-\sigma_0}$ when $0 < t < T$. Then for $s = \sigma_0 + it_0 + x$, $x \to 0+$,

$$\Gamma(s)F[l/q](s) = \int_0^1 t^{s-1} \mathfrak{A}(e(l/q)e^{-t})\, dt + O(1).$$

Hence

$$|\Gamma(s)F[l/q](s)| \le c\int_0^1 t^{-\sigma_0} t^{\sigma_0 + x - 1}\, dt + O(1) = c\int_0^1 t^{x-1}\, dt + O(1) \le 2cx^{-1}.$$

This inequality contradicts the fact that $\sigma_0 + it_0$ is a pole of $F[l/q](s)$. ∎

3. Estimates of $1/\zeta(s), 1/L(s, \chi)$ on lines $\Re s = -0.5 - N$. In Sections 3–5 we will use following notations. We denote by q a fixed positive integer. Let B be a positive function and A be an arbitrary function. If $|A| \le CB$, where $C > 0$, and the constant C depends only on q, we will write $A \ll B$ or $A = O(B)$. Let A and B be some positive functions. If $C_1|B| \le A \le C_2|B|$ where $C_1, C_2 > 0$, we will write $A \asymp B$. As usual $t = \Im s$, $\sigma = \Re s$.

In this section we deduce some inequalities involving $L(s,\chi)$, where χ are characters modulo $k \le q$.

We begin by reformulating the functional equations for ζ- and L-functions. First,

$$\zeta(1-s) = h(s)\zeta(s), \tag{3.1}$$

where

$$h(s) = \pi^{1/2-s}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}. \tag{3.2}$$

If χ is an even primitive character modulo k, then

$$L(1-s,\chi) = h_{\bar\chi}(s)L(s,\bar\chi), \tag{3.3}$$

where

$$h_\chi(s) = \left(\frac{\pi}{k}\right)^{1/2-s}\frac{\sqrt{k}}{\tau(\chi)}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}. \tag{3.4}$$

If χ is an odd primitive character modulo k, then

$$L(1-s,\chi) = h_{\bar\chi}(s)L(s,\bar\chi),$$

where

$$h_\chi(s) = \left(\frac{\pi}{k}\right)^{1/2-s} i\frac{\sqrt{k}}{\tau(\chi)}\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)}. \tag{3.5}$$

From the formula $\Gamma(s+1) = s\Gamma(s)$ it is easy to deduce the identities

$$h(s+2) = -(2\pi)^{-2}s(s+1)h(s), \tag{3.6}$$

$$h_\chi(s+2) = -\left(2\frac{\pi}{k}\right)^{-2}s(s+1)h_\chi(s). \tag{3.7}$$

From the estimate $|\Gamma(\sigma+it)| \asymp |t|^{\sigma-0.5}e^{\pi|t|/2}$ as $|t| \to \infty$ [MV, p. 523] it is easy to obtain the asymptotic formulas

$$|h(0.5+it)|^{-1} \ll 1, \tag{3.8}$$

$$|h_\chi(0.5+it)|^{-1} \ll 1, \tag{3.9}$$

$$|h(1.5+it)|^{-1} \ll 1, \tag{3.10}$$

$$|h_\chi(1.5+it)|^{-1} \ll 1. \tag{3.11}$$

LEMMA 3.1. *The following asymptotic formulas are true for $M \in \mathbb{N}$:*

$$\frac{1}{\zeta(0.5-M+it)} \ll \frac{(2\pi)^M}{\Gamma(M+0.5)}, \tag{3.12}$$

and if χ *is a primitive character modulo* k, *then*

$$\frac{1}{L(0.5-M+it,\chi)} \ll \frac{(2\pi/k)^M}{\Gamma(M+0.5)}. \tag{3.13}$$

Proof. From (3.6), (3.7) we get

$$h(2N+s) = s\cdots(s+2N-1)(-2\pi)^{-2N}h(s),$$
$$h_\chi(2N+s) = s\cdots(s+2N-1)(-2\pi/k)^{-2N}h_\chi(s).$$

Using these relation and estimates (3.8)–(3.11) we obtain, for $N\in\mathbb{N}$,

$$\frac{1}{\zeta(0.5-M+it)} \le C_1\frac{(2\pi)^{2N}\Gamma(0.5)}{\Gamma(0.5+2N)} \le C_2\frac{(2\pi)^{2N}}{\Gamma(0.5+2N)}, \tag{3.14}$$

$$\frac{1}{L(0.5-M+it,\chi)} \le C_1\frac{(2\pi)^{2N}\Gamma(1.5)}{\Gamma(1.5+2N)} \le C_2\frac{(2\pi)^{2N}}{\Gamma(1.5+2N)}. \tag{3.15}$$

From (3.14) and (3.15) we obtain the estimate (3.12) of the theorem. Similarly, we deduce (3.13). ■

4. Simultaneous estimation of $1/\zeta(s)$, $1/L(s,\chi)$ for χ primitive on horizontal lines. We remind the reader that the integer q is fixed. We will consider the functions $L(s,\chi)$, where χ are primitive characters modulo $q_1 \mid q$. Let us evaluate the functions $\frac{d}{ds}\ln\zeta(s)$ and $\frac{d}{ds}\ln L(s,\chi)$.

LEMMA 4.1 ([K, p. 40]). *Let* $\rho_n=\beta_n+i\gamma_n$ *be the zeros of the* ζ*-function, and* $-1\le\sigma\le 2$, $|t|\ge 2$. *Then*

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|t-\gamma_n|\le 1}\frac{1}{s-\rho_n} + O(\ln|t|),$$

where the summation is over those zeros ρ_n *such that* $|t-\Im\rho_n|\le 1$.

LEMMA 4.2 ([K, p. 111]). *Let* $\rho_n=\beta_n+i\gamma_n$ *be the zeros of the function* $L(s,\chi)$, *where* χ *is a primitive character, and* $-1\le\sigma\le 2$, $|t|\ge 2$. *Then*

$$\frac{L'(s,\chi)}{L(s,\chi)} = \sum_{|t-\gamma_n|\le 1}\frac{1}{s-\rho_n} + O(\ln|t|),$$

where the summation is over those zeros ρ_n *such that* $|t-\Im\rho_n|\le 1$.

The following lemma gives estimates of $\zeta'(s)/\zeta(s)$, $L'(s,\chi)/L(s,\chi)$, uniform with respect to $\sigma\in[-0.5,1.5]$, on a sequence of horizontal segments.

LEMMA 4.3. *For each* $N\in\mathbb{Z}$ *with* $|N|>3$ *we can find a real number* t_N *with* $N<t_N<N+1$ *such that*

$$\frac{\zeta'(s)}{\zeta(s)} = O(\ln^2|N|), \qquad \frac{L'(s,\chi)}{L(s,\chi)} = O(\ln^2|N|),$$

where $s=\sigma+it_N$.

Proof. Let us consider the function

$$\zeta(s)\prod_{\chi} L(s,\chi),$$

where the product is taken over all primitive characters modulo $q_1 \mid q$. Denote by $\mathfrak{P}$ the multiset of zeros $\rho = \beta + i\gamma$ of the above function counted with multiplicity, and by $\mathfrak{G}$ the multiset of imaginary parts of elements of $\mathfrak{P}$.

Note that the number of zeros $\rho_1 = \beta_1 + i\gamma_1$ of $\zeta(s)$, where $\gamma_1 \in [N, N+1]$, does not exceed $c_0 \log|N|$ (see [K, p. 40]), and the number of zeros $\rho(\chi) = \beta(\chi) + i\gamma(\chi)$ of the functions $L(s,\chi)$, where $\gamma(\chi) \in [N, N+1]$, does not exceed $c_\chi \log|N|$ (see [K, p. 111]). Hence

$$\#\{\rho \in \mathfrak{P} : |\gamma| \in [N, N+1]\} \le c \ln|N|, \tag{4.1}$$

where # denotes the cardinaty of a multiset.

From (4.1) it follows that there is a strip $\alpha \le \Im s \le \beta$, where $N \le \alpha < \beta \le N+1$, $\beta - \alpha = 1/(2c\ln|N|)$, containing no element of $\mathfrak{P}$. Let $t_N = (\beta+\alpha)/2$. Then for each $\gamma \in \mathfrak{G}$,

$$|\gamma - t_N| \ge \frac{1}{4c\ln|N|}. \tag{4.2}$$

By Lemmas 4.1 and 4.2,

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \le \sum_{|t-\gamma_1|\le 1} \frac{1}{|s-\rho_1|} + O(\ln|t|),$$

$$\left|\frac{L'(s,\chi)}{L(s,\chi)}\right| \le \sum_{|t-\gamma(\chi)|\le 1} \frac{1}{|s-\rho(\chi)|} + O(\ln|t|).$$

Since $|a+bi| \ge |b|$ for all $a, b \in \mathbb{R}$, we have $|s-\rho| \ge |t-\gamma|$. Hence

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \le \sum_{|t-\gamma_1|\le 1} \frac{1}{|t-\gamma_1|} + O(\ln|t|), \tag{4.3}$$

$$\left|\frac{L'(s,\chi)}{L(s,\chi)}\right| \le \sum_{|t-\gamma(\chi)|\le 1} \frac{1}{|t-\gamma(\chi)|} + O(\ln|t|). \tag{4.4}$$

Since the number of summands in (4.3) is $O(\ln|t|)$ (see [K, p. 40]), from (4.2) it follows that when $t = t_N$, we have

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \ll (\ln|N|)^2 + \ln|N| \ll \ln^2|N|. \tag{4.5}$$

Similarly from (4.4) we find that

$$\left|\frac{L'(s,\chi)}{L(s,\chi)}\right| \ll (\ln|N|)^2 + \ln|N| \ll \ln^2|N|. \;\blacksquare \tag{4.6}$$

Now we state estimates of $1/\zeta(s)$ and $1/L(s,\chi)$, uniform with respect to $\sigma = \Re s$, on a sequence of horizontal segments.

LEMMA 4.4. *For each integer N with $|N| > 3$ there exists a real number t_N satisfying $N < t_N < N+1$ such that if $s = \sigma + it_N$ and $\sigma \in [-0.5, 1.5]$ then*

$$\frac{1}{\zeta(s)} = O(\exp(c\ln^2|N|)), \tag{4.7}$$

$$\frac{1}{L(s,\chi)} = O(\exp(c\ln^2|N|)). \tag{4.8}$$

Proof. Let t_N be the numbers of Lemma 4.3. By the Newton–Leibniz formula,

$$\ln\zeta(\sigma+it_N) = \ln\zeta(1.5+it_N) + \int_{1.5+it_N}^{\sigma+it_N} \frac{\zeta'(s)}{\zeta(s)}\,ds. \tag{4.9}$$

Since $\ln\zeta(s)$ is represented by an absolutely convergent Dirichlet series for $\Re s > 1$, we have $\ln\zeta(1.5+it_N) \ll 1$. Using Lemma 4.3 and estimating the integral in (4.9) from above we derive

$$\left|\int_{1.5+it_N}^{\sigma+it_N} \frac{\zeta'(s)}{\zeta(s)}\,ds\right| \ll 2\ln^2 N.$$

Hence

$$|\ln\zeta(\sigma+it_N)| \ll \ln^2 N,$$

thus

$$\left|\ln\frac{1}{\zeta(\sigma+it_N)}\right| \ll \ln^2 N.$$

This yields (4.7). Similarly we obtain (4.8). ∎

Lemma 4.4 is useful in proving integral identities of the form

$$\int_{1.5-i\infty}^{1.5+i\infty} x^{-s}\Gamma(s)f(s)\,ds = \int_{-0.5-i\infty}^{-0.5+i\infty} x^{-s}\Gamma(s)f(s)\,ds,$$

where $f(s)$ is some function related to $1/\zeta(s)$ and $1/L(s,\chi)$.

5. The proof of the main theorem. Applying Lemma 2.3, we obtain

$$\begin{aligned}\int_0^\infty x^{s-1}\mathfrak{M}(e(l/q)e^{-x})dx &= \Gamma(s)F[l/q](s)\\ &= \Gamma(s)\frac{1}{\phi(q)}\biggl(\sum_{\chi\,(\mathrm{mod}\,q)}\Bigl(\sum_{d|q}\frac{\mu(d)}{d^s}\tau(\bar{\chi},ld)\Bigr)\frac{1}{L(s,\chi)}\biggr).\end{aligned} \tag{5.1}$$

Note that the functions $1/\zeta(s)$ and $1/L(s,\chi)$, where χ is a primitive character, have no zeros in the set $\{\Re s = 0\} \setminus \{0\}$.

If χ_0 is the principal character modulo q, then

$$\frac{1}{L(s,\chi_0)} = \frac{1}{\zeta(s)} \frac{1}{C_{\chi_0}(s)},$$

where $C_{\chi_0}(s) = \prod_{p|q}(1 - 1/p^s)$.

If χ is a non-principal character modulo q which is not primitive, and χ_1 is a primitive character modulo q_1 that induces χ, then

$$\frac{1}{L(s,\chi)} = \frac{1}{L(s,\chi_1)} \frac{1}{C_\chi(s)}, \tag{5.2}$$

where $C_\chi(s) = \prod_{p|q,\ p\nmid q_1}(1 - \chi_1(p)/p^s)$.

Let $C(s) = \prod_\chi C_\chi(s)$, where the product is taken over all non-primitive characters modulo q. Define

$$f(s) = C(s) \int_0^\infty x^{s-1} \mathfrak{M}(e(nl/q)e^{-x})\, dx.$$

By Lemma 2.3 we have, for $\Re s > 1$,

$$\begin{aligned} f(s) &= \Gamma(s) C(s) \sum_{n=1}^\infty \frac{\mu(n) e(l/q)}{n^s} \\ &= \Gamma(s) \biggl(\frac{1}{\phi(q)} \sum_{d|q} \frac{\mu(d)}{d^s} \tau(\chi_0, ld) \frac{1}{L(s,\chi_0)} \\ &\quad + \frac{1}{\phi(q)} \sum_{\chi \ne \chi_0 \,(\mathrm{mod}\, q)} \sum_{d|q} \frac{\mu(d)}{d^s} \tau(\bar\chi, ld) \frac{1}{L(s,\chi)} \biggr) C(s). \end{aligned} \tag{5.3}$$

Hence

$$\begin{aligned} f(s) = \Gamma(s) \biggl(&\frac{1}{\phi(q)} \biggl(\sum_{d|q} \frac{\mu(d)}{d^s} \tau(\chi_0, ld) \biggr) \frac{1}{\zeta(s)} D_{\chi_0}(s) \\ &+ \frac{1}{\phi(q)} \sum_{\chi \ne \chi_0 \,(\mathrm{mod}\, q)} \biggl(\sum_{d|q} \frac{\mu(d)}{d^s} \tau(\bar\chi, ld) \biggr) \frac{1}{L(s,\chi_1)} D_\chi(s) \biggr), \end{aligned} \tag{5.4}$$

where $D_\chi(s) = C(s)$ if χ is a primitive character, and $D_\chi(s) = \prod_{\psi \ne \chi} C_\psi(s)$ where the product is taken over non-primitive characters if χ is a non-primitive character. In (5.4), χ_1 is a primitive character that induces χ.

Estimating $D_\chi(s)$ from above we obtain the following lemma:

LEMMA 5.1. *There are constants $C, Q, E > 0$, depending only on q, such that for each character χ modulo q,*

$$|D_\chi(s)| \le C Q^{-\sigma} \quad \textit{for each } s \textit{ with } \sigma \le 0, \tag{5.5}$$

$$|D_\chi(s)| \le E \quad \textit{for each } s \textit{ with } \sigma > 0. \tag{5.6}$$

Proof. Let $\sigma \le 0$. Note that if $|a| = 1$, then $|1 - ap^{-s}| \le 2p_0^{-\sigma}$, where p_0 is the largest prime divisor of q.

Hence, $|C_\chi(s)| \le \prod_{p|q} 2p_0^{-\sigma} = 2^{\omega(q)}(p_0^{\omega(q)})^{-\sigma}$. Thus

$$|D_\chi(s)| \le \prod_{\chi \,(\mathrm{mod}\, q)} 2^{\omega(q)}(p_0^{\omega(q)})^{-\sigma} \le 2^{\phi(q)\omega(q)}(p_0^{\omega(q)\phi(q)})^{-\sigma},$$

where $\omega(q)$ is the number of prime divisors of q, and ϕ is Euler's totient function.

Let now $\sigma > 0$. Note that if $|a| = 1$, then $|1 - ap^{-s}| < 2$, hence similarly $|C_\chi(s)| \le 2^{\omega(q)}$. Thus

$$|D_\chi(s)| \le 2^{\omega(q)\phi(q)}. \blacksquare$$

Lemma 5.2. *Let $A < \sigma \le -0.5$. Then for $|t| \to \infty$,*

$$\left|\frac{1}{\zeta(s)}\right| \le C|t|^{\sigma-0.5}, \qquad \left|\frac{1}{L(s,\chi)}\right| \le C|t|^{\sigma-0.5},$$

where C depends only on q and A.

Proof. This follows from [MV, p. 330 Corollary 10.5, p. 334 Corollary 10.10, p. 27 Corollary 1.17, and p. 350 Lemma 10.15]. ■

The following statement, proved by Szegő, will be applied to some power series related to $\mathfrak{M}(z)$. If for some rational number l/q,

$$\mathfrak{M}(e(l/q)r) = O((1-r)^{\varepsilon})$$

for each $\varepsilon > 0$, we will obtain some power series whose coefficients take a finite number of values.

Theorem 5.3 ([S]). *A power series*

$$\sum_{n=1}^{\infty} f_n z^n \tag{5.7}$$

whose coefficients take a finite number of values is either a rational function, or cannot be continued beyond the unit circle. In the case of rationality of (5.7), *the coefficients form an eventually periodic sequence.*

From (5.1) it follows that the function $g(s) = F[l/q](s)$ with rational β can be continued to a meromorphic function in $\mathbb{C}$. From (5.3) it follows that $f(s)$ is a meromorphic function in $\mathbb{C}$.

The following lemma is central to the proof of Theorem 1.1.

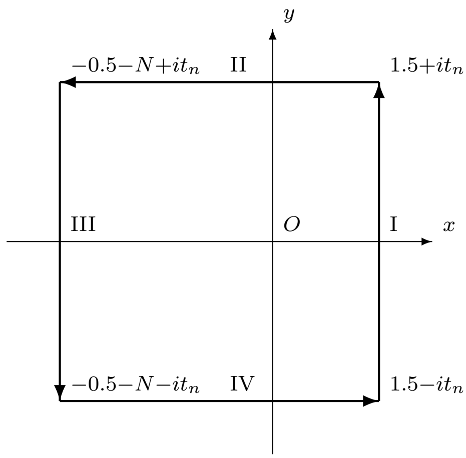
Lemma 5.4. *For $\beta \in \mathbb{Q}$ the function $g(s) = F[l/q](s)$ has a pole in the strip $\{0 < \Re s < 1\}$.*

Proof. Assume that $g(s)$ is holomorphic in $\{0 < \Re s < 1\}$. From (5.4) it follows that the poles of f are in the set $\{0, -1, -2, \ldots\}$, and their orders do not exceed 2. Let t_n be the numbers of Lemma 4.4, $|n| > 3$. Since $D_\chi(s) \ll 1$

and $\Gamma(s) \ll |t|^C e^{-\pi|t|/2}$ when $t \to \infty$ and σ is bounded, it follows from Lemma 5.2, (4.7), (4.8), (5.5) and (5.6) that there is an $\alpha > 0$ such that $f(\sigma + it_n) \ll e^{-\alpha|t_n|}$. Hence for any $A < 1.5$,

$$\int_{A+it_n}^{0.5+it_n} x^{-s} f(s)\, ds \to 0, \qquad |n| \to \infty. \tag{5.8}$$

Let Π be the rectangle with vertices $1.5 + it_{-n}$, $1.5 + it_n$, $-0.5 - N + it_n$, $-0.5 - N + it_{-n}$. Let $\mathrm{I} = [1.5 + it_{-n}, 1.5 + it_n]$, $\mathrm{II} = [1.5 + it_n, -0.5 - N + it_n]$, $\mathrm{III} = [-0.5 - N + it_n, -0.5 - N + it_{-n}]$, $\mathrm{IV} = [-0.5 - N + it_{-n}, 1.5 + it_{-n}]$.



The contour Π

By the Cauchy theorem on residues,

$$\int_{\Pi} x^{-s} f(s)\, ds = 2\pi i \sum_{k=0}^{N} r_k, \tag{5.9}$$

where $r_k = \mathrm{res}_{s=-k}\, x^{-s} f(s)$. From (5.8) and (5.9) we obtain

$$\int_{1.5-i\infty}^{1.5+i\infty} x^{-s} f(s)\, ds = \int_{-0.5-N-i\infty}^{-0.5-N+i\infty} x^{-s} f(s)\, ds + 2\pi i \sum_{k=0}^{N} r_k. \tag{5.10}$$

By the inversion formula of [FGD, p. 4],

$$\frac{1}{2\pi i} \int_{1.5-i\infty}^{1.5+i\infty} x^{-s} f(s)\, ds = \sum_{n=1}^{\infty} \delta(n) e^{-nx},$$

where $\delta(n)$ are the coefficients of the Dirichlet series of the function

$$C(s) \sum_{n=1}^{\infty} \frac{\mu(n) e(ln/q)}{n^s}.$$

Note that $C(s)$ is a Dirichlet polynomial,

$$C(s) = \sum_{n=1}^{M} \frac{a(n)}{n^s},$$

and $a(1) = 1$. Hence

$$\delta(n) = \sum_{d|n,\, d\le M} a(d)\mu(n/d)e(ln/qd). \tag{5.11}$$

Thus

$$2\pi i \sum_{n=1}^{\infty} \delta(n)e^{-nx} = \int_{0.5-N-i\infty}^{0.5-N+i\infty} x^{-s} f(s)\, ds + 2\pi i \sum_{k=0}^{N-1} r_k, \tag{5.12}$$

where $r_k = \operatorname{res}_{s=-k} x^{-s} f(s)$. From (5.11) it follows that $\delta(n)$ takes finitely many values. Note that

$$\operatorname{res}_{s=-k} x^{-s} f(s) = (c_k + d_k \ln x)x^k. \tag{5.13}$$

We have the following estimates:

$$\Big|\int_{0.5-N-i\infty}^{0.5-N+i\infty} x^{-s} f(s)\, ds\Big| = \Big|\int_{-\infty}^{\infty} x^{-(0.5-N+it)} f(0.5-N+it)\, dt\Big| \le x^{N-0.5} \int_{-\infty}^{\infty} |f(0.5-N+it)|\, dt, \tag{5.14}$$

$$|d^{-s}| \le d^{-\sigma} \le q^{-\sigma}, \quad \sigma < 0, \tag{5.15}$$

$$|\tau(\chi, ld)| \le \phi(q). \tag{5.16}$$

Let us evaluate $f(0.5-N+it)$. From (3.12), (3.13), (5.5), (5.15), (5.16) we obtain the following estimates for $s = 0.5 - N + it$:

$$\Big|\frac{1}{\phi(q)}\Big(\sum_{d|q} \frac{\mu(d)}{d^s}\tau(\chi_0, ld)\frac{1}{\zeta(s)}\Big)D_{\chi_0}(s)\Big| \ll Q^{N-0.5}\tau(q)q^{N-0.5}\phi(q)\frac{(2\pi)^N}{\Gamma(N+0.5)} \ll \frac{(Qq)^{N-0.5}(2\pi)^N}{\Gamma(N+0.5)}, \tag{5.17}$$

$$\Big|\frac{1}{\phi(q)} \sum_{\chi\ne\chi_0 \,(\mathrm{mod}\, q)} \Big(\sum_{d|q} \frac{\mu(d)}{d^s}\tau(\bar\chi, ld)\frac{1}{L(s,\chi_1)}\Big)D_{\chi}(s)\Big| \ll \frac{1}{\phi(q)} \sum_{\chi\ne\chi_0 \,(\mathrm{mod}\, q)} Q^{N-0.5}\tau(q)q^{N-0.5}\phi(q)\frac{(2\pi)^N}{\Gamma(N+0.5)} \ll \frac{(Qq)^{N-0.5}(2\pi)^N}{\Gamma(N+0.5)}. \tag{5.18}$$

From inequalities (5.17), (5.18) and equality (5.4) we obtain

$$f(0.5-N+it) \ll |\Gamma(0.5-N+it)|\frac{(Qq)^{N-0.5}(2\pi)^N}{\Gamma(N+0.5)}. \tag{5.19}$$

From estimates (5.14) and (5.19) it follows that

$$(5.20)\qquad \Big|\int_{0.5-N-i\infty}^{0.5-N+i\infty} x^{-s}f(s)\,ds\Big| \ll x^{N-0.5}\frac{(Qq)^{N-0.5}(2\pi)^N}{\Gamma(N+0.5)}\int_{-\infty}^{\infty}|\Gamma(0.5-N+it)|\,dt.$$

From the recurrence relation for the Γ-function we obtain

$$\Gamma(0.5-N+it)=\frac{\Gamma(0.5+it)}{(-0.5+it)(-1.5+it)(-2.5+it)\dots(-0.5-N+1+it)},$$

Hence

$$|\Gamma(0.5-N+it)|\le\frac{|\Gamma(0.5+it)|}{\prod_{n=0}^{N-1}(0.5+n)}.$$

From the above and estimate (5.20) we have

$$\Big|\int_{0.5-N-i\infty}^{0.5-N+i\infty} x^{-s}f(s)\,ds\Big| \ll x^{N-0.5}\frac{(Qq)^{N-0.5}(2\pi)^N}{\Gamma(N+0.5)\prod_{n=0}^{N-1}(0.5+N)}\int_{-\infty}^{\infty}|\Gamma(0.5+it)|\,dt.$$

Since

$$x^{N-0.5}\frac{(Qq)^{N-0.5}(2\pi)^N}{\Gamma(N+0.5)\prod_{n=0}^{N-1}(0.5+n)}\to 0$$

as $N\to\infty$, for each $x\in(0,\infty)$, and the integral $\int_{-\infty}^{\infty}|\Gamma(0.5+it)|\,dt$ is convergent, we have

$$(5.21)\qquad \int_{0.5-N-i\infty}^{0.5-N+i\infty} x^{-s}f(s)\,ds\to 0$$

as $N\to\infty$, for each $x\in(0,\infty)$. Applying (5.12), (5.13) and (5.21), we obtain

$$(5.22)\qquad \sum_{n=1}^{\infty}\delta(n)e^{-nx}=\sum_{k=0}^{\infty}(c_k+d_k\ln x)x^k$$

when $x\in(0,\infty)$, where the series on the right-hand side is convergent for all $x\in(0,\infty)$. From this convergence we obtain the following identity for $0<x<0.5$:

$$(5.23)\qquad \sum_{n=1}^{\infty}\delta(n)e^{-nx}=f(x)+g(x)\ln x,$$

where $f(z)=\sum_{k=0}^{\infty}c_kz^k$ and $g(z)=\sum_{k=0}^{\infty}d_kz^k$ are holomorphic functions in the disk $|z|<1/2$. Thus, the power series $\sum_{n=1}^{\infty}\delta(n)w^n$ can be continued beyond the unit circle. Hence by Theorem 5.3, $\delta(n)$ is a periodic sequence from some number M on. Denote by U the period of this sequence. By Dirichlet's theorem the sequence $b_h=qUh+1$ contains infinitely many prime numbers. Suppose that $p_1=qUh_1+1$ and $p_2=qUh_2+1$ are primes with $h_2>h_1$ and $p_2>p_1>M$. Then p_1p_2 is of the form $qUh+1$ and is greater than M. By (5.11) we obtain

$$\delta(p_1)=-e(l/q),\qquad \delta(p_1p_2)=e(l/q),$$

contradicting the periodicity of $\delta(n)$. ∎

Now we can prove the main theorem 1.1.

Proof of Theorem 1.1. Assume that $|\mathfrak{M}(e(l/q)e^{-t})|\ll_\varepsilon t^{-\varepsilon}$ as $t\to 0+$, for any ε. Then the function $\int_0^\infty t^{s-1}\mathfrak{M}(e(l/q)e^{-t})dt$ is holomorphic in the half-plane $\{\Re s>0\}$. Hence the Dirichlet series $\sum_{n=1}^{\infty}\mu(n)e(l/q)n^{-s}$ can be continued to a holomorphic function in $\{\Re s>0\}$. This contradicts Lemma 5.4. Thus we obtain (1.1). Using the Abel transform, from (1.1) we obtain (1.2). ∎

6. Theorems on the behaviour of $\mathfrak{M}(z)$ under some conditions. The following lemma can be used to get effective bounds of the exponent a in (1.1), (1.2).

LEMMA 6.1. *Let ψ be a character modulo q. Let $l\in\mathbb{Z}$ and $\rho_0=\beta_0+i\gamma_0$ be a zero of $L(s,\psi)$ such that*

$$\sum_{d\mid q}\mu(d)\tau(\bar\psi,ld)d^{-\rho_0}\neq 0.$$

Suppose that $L(\rho_0,\chi)\neq 0$ for any character χ modulo q that is not equal to ψ. Then

$$\mathfrak{M}(e(l/q)r)=\Omega((1-r)^{-\beta_0}). \tag{6.1}$$

Proof. Under the conditions of the lemma the function

$$\sum_{\chi\neq\psi\,(\mathrm{mod}\,q)}\Big(\sum_{d\mid q}\tau(\bar\chi,ld)\frac{\mu(d)}{d^s}\Big)\frac{1}{L(s,\chi)}$$

is analytic at ρ_0, and the function

$$\Big(\sum_{d\mid q}\tau(\bar\psi,ld)\frac{\mu(d)}{d^s}\Big)\frac{1}{L(s,\psi)}$$

has a pole at ρ_0. Thus the function

$$\sum_{n=1}^{\infty} \frac{\mu(n)e(ln/q)}{n^s} = \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \Big(\sum_{d|q} \tau(\bar{\chi}, ld) \frac{\mu(d)}{d^s}\Big) \frac{1}{L(s,\chi)}$$

has a pole at ρ_0. Applying Theorem 2.4 to $\alpha(n) = \mu(n)$, $\sigma_0 + it_0 = \beta_0 + i\gamma_0$, we obtain (6.1). ■

Let us consider the behaviour of $\mathfrak{M}(z)$ where z tends to $e(l/q)$ with $q \le 100$ along the radius of the unit disk. Using computer we can check the conditions of Lemma 6.1.

Define

$$\xi(s,\chi) = \left(\frac{\pi}{k}\right)^{-(s+\delta)/2} \Gamma\left(\frac{s+\delta}{2}\right) L(s,\chi),$$

where χ is a primitive character modulo q, $\delta = 0$ if χ is even, and $\delta = 1$ if χ is odd. Note that in the half-plane $\{\Re s > 0\}$ the equation $\xi(s,\chi) = 0$ is equivalent to $L(s,\chi) = 0$.

In [MV, Theorem 10.7, p. 332], there is an identity that in the case $s = 1/2 + i\gamma$, $z = 1$ can be written in the form

$$\xi(s,\chi) = I + \tau \bar{I}, \tag{6.2}$$

where

$$I = \sum_{n=1}^{\infty} \chi(n) \left(\frac{q}{\pi n^2}\right)^{1/4 + i\gamma/2} \Gamma\left(\frac{1}{4} + i\frac{\gamma}{2}, \frac{\pi n^2}{q}\right) \quad \text{if } \chi \text{ is even}, \tag{6.3}$$

$$I = \sum_{n=1}^{\infty} n\chi(n) \left(\frac{q}{\pi n^2}\right)^{3/4 + i\gamma/2} \Gamma\left(\frac{3}{4} + i\frac{\gamma}{2}, \frac{\pi n^2}{q}\right) \quad \text{if } \chi \text{ is odd}. \tag{6.4}$$

In (6.3), (6.4) the incomplete Γ-function

$$\Gamma(a,b) = \int_b^{\infty} t^{a-1} e^{-t}\, dt$$

is used.

Omitting some technical estimates we state the following estimate of the error term:

LEMMA 6.2. *For $\gamma \in \mathbb{R}$ the following decomposition is true:*

$$\xi\left(\frac{1}{2} + i\gamma, \chi\right) = I_N + \tau \bar{I}_N + R_N,$$

where

$$I_N = \sum_{n=1}^{N} \chi(n)\Big(\frac{q}{\pi n^2}\Big)^{1/4+i\gamma/2} \Gamma\Big(\frac{1}{4}+i\frac{\gamma}{2},\frac{\pi n^2}{q}\Big) \qquad \text{if } \chi \text{ is even,}$$

$$I_N = \sum_{n=1}^{N} \chi(n)n\Big(\frac{q}{\pi n^2}\Big)^{3/4+i\gamma/2} \Gamma\Big(\frac{3}{4}+i\frac{\gamma}{2},\frac{\pi n^2}{q}\Big) \qquad \text{if } \chi \text{ is odd,}$$

and

$$|R_N| \le \Big(\frac{q}{\pi N}\Big)^2 e^{-\frac{\pi}{q}N^2}.$$

This lemma gives us an expression of ξ by rapidly convergent series. For $q \le 100$ we need no more than 28 steps to prove that $\xi(\rho,\chi) \neq 0$ where ρ is a zero of some L-function in the proof of Theorem 1.2.

Proof of Theorem 1.2. The proof is conducted with the help of computer. For each $1 < q \le 100$ we find a character ψ modulo q and $\rho = 1/2 + i\gamma$ such that:

- ρ is a zero of $L(s,\psi)$,
- for l satisfying $(l,q) = 1$,

$$\sum_{d|q} \mu(d)\tau(\bar{\psi}, ld)d^{-\rho_0} \neq 0, \tag{6.5}$$

- $L(\rho,\chi) \neq 0$ for other characters χ modulo q.

Denote by χ_k the principal character modulo k for $k > 1$, and by χ_1 the function that equals 1 identically.

If χ is a primitive character modulo q, then for $\Re s > 0$,

$$L(\rho,\chi) \neq 0 \iff \xi(\rho,\chi) \neq 0. \tag{6.6}$$

If χ is not a primitive character modulo q, we consider a primitive character χ' modulo q_1, with $q_1 \,|\, q$, that induces χ. In this case $\chi = \chi'\chi_q$. Note that for $\Re s > 0$,

$$L(s,\chi) \neq 0 \iff L(s,\chi') \neq 0 \iff \xi(s,\chi') \neq 0.$$

Hence, examination of $L(\rho,\chi) \neq 0$ for non-primitive χ is reduced to examination of this condition for primitive χ'.

For $\Re s > 0$,

$$L(s,\chi_k) \neq 0 \iff \zeta(s) \neq 0,$$

and $\zeta(s) \neq 0$ if $|\Im s| < 14$.

If $q \le 100$, $8 \nmid q$, $27 \nmid q$, then we apply Lemma 6.1 to the character χ_q and to the zero $\rho_0 = 0.5 + i14.13472514173469\ldots$ of $\zeta(s)$.

If $q = 8k$, where $2 \nmid k$, then we apply Lemma 6.1 to the character $\chi\chi_q$, where χ is a non-principal character modulo 4, and to the zero $\rho_0 = 0.5 + i6.02094890469759\ldots$ of $L(s,\chi)$.

If $q = 16k$, where $2 \nmid k$, then we apply Lemma 6.1 to the character $\chi\chi_q$, where χ is a primitive character modulo 8, and to the zero $\rho_0 = 0.5 + i3.57615483678758\ldots$ of $L(s,\chi)$.

If $q = 32k$, where $2 \nmid k$, then we apply Lemma 6.1 to the character $\chi\chi_q$, where χ is a primitive character modulo 16 defined by $\chi(5) = i$, $\chi(15) = -1$, and to the zero $\rho_0 = 0.5 + i3.34621940663383\ldots$ of $L(s,\chi)$.

If $q = 64$, then we apply Lemma 6.1 to the character χ modulo 32 defined by $\chi(5) = e^{i\pi/4}$, $\chi(31) = -1$, and to the zero $\rho_0 = 0.5+i1.72096909693815\ldots$ of $L(s,\chi)$.

If $q = 27k$, where $3 \nmid k$, then we apply Lemma 6.1 to the character $\chi\chi_q$, where χ is a character modulo 9 defined by $\chi(2) = e^{i\pi/3}$, and to the zero $\rho_0 = 0.5 + i4.57573576242485\ldots$ of $L(s,\chi)$.

If $q = 81$, then we apply Lemma 6.1 to the character χ modulo 27 defined by $\chi(2) = e^{\pi i/9}$, and to the zero $\rho_0 = 0.5 + i2.86051675138494\ldots$ of $L(s,\chi)$.

In all cases the assumptions of Lemma 6.1 are satisfied.

By Lemma 6.1 we obtain (1.3). Using the Abel transform, from (1.3) we obtain (1.4). The theorem is proved. ∎

The computations were conducted using a program written in GNU compiler collection.

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Received on 30.4.2013
and in revised form on 16.12.2013 (7421)