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Local-global principle for certain biquadratic normic bundles

by

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1. Introduction. Let k be a number field and Ω be the set of places of k. We denote by k_v the completion of k at $v \in \Omega$. We will discuss the local-global principle for rational points and for 0-cycles on algebraic varieties X which are proper smooth and geometrically integral over k. We write simply $X_v = X \times_k k_v$ for all $v \in \Omega$ and we denote by $\mathrm{Br}(X)$ the cohomological Brauer group of X. For any abelian group M, denote $\mathrm{Coker}(M \stackrel{n}{\to} M)$ by M/n.

The so-called Brauer–Manin obstruction to the local-global principle for rational points on X was defined by Manin in the 1970's using the Brauer group $\mathrm{Br}(X)$, and it is conjectured to be the only obstruction for geometrically rational varieties (or even a larger family of varieties) [CTS77]. The local-global principle for 0-cycles is obstructed similarly. After some reformulations, the following sequence (E) is conjectured to be exact for all proper smooth varieties:

$$\text{(E)}\qquad \varprojlim_n \mathrm{CH}_0(X)/n \to \prod_{v\in\Omega} \varprojlim_n \mathrm{CH}'_0(X_v)/n \to \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z}),$$

which means that the Brauer group gives the only obstruction to the local-global principle for 0-cycles (cf. [CTS81, KS86, CT95] for more information about the conjecture, and [Wit12] for more details on the sequence). Instead of giving a very long list of references of contributions to this question, we mention some recent papers in which more historical details are presented: [BHB12, DSW] for rational points, [Wit12] for 0-cycles, and [Pey] for older results.

In this paper, we restrict ourselves to a very concrete situation. Let K/k be a finite extension of degree n and $P(t_1, \dots, t_m)$ be a polynomial (or even

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a rational function). Then the equation

$$N_{K/k}(\mathbf{x}) = P(t_1, \ldots, t_m)$$

defines in $R_{K/k}\mathbb{A}^1 \times \mathbb{A}^m = \mathbb{A}^{n+m}$ a closed subvariety fibered over $\mathbb{A}^m$ via the parametric variables $t_1, \ldots, t_m$. We consider proper smooth models of such varieties.

Question. *Is the Brauer–Manin obstruction the only obstruction to the local-global principle for rational points and for* 0*-cycles on this family of varieties?*

We consider the case where K/k is a biquadratic Galois extension—a "simple" case mentioned in particular in [CTSSD98, Rem. 1.5]. In such a case, theoretical difficulties come from not only the degeneracy of the fibers but also the non-triviality of the Brauer groups of the fibers. Under various assumptions on $P(t_1, \ldots, t_m)$ and/or on k, for the question on rational points one finds different related results [HBS02, CTHS03, SJ13, BHB12, DSW, SS, BM]; they apply to the biquadratic extension K/k even without assuming Schinzel's hypothesis.

We state the main result of the present paper as follows.

Theorem 1.1. *Let k be a number field and $Q(t_1, \ldots, t_m) \in k(t_1, \ldots, t_m)$ be a non-zero rational function. Let X be an arbitrary proper smooth model of the variety defined by the equation*

$$N_{K/k}(\mathbf{x}) = Q(t_1, \ldots, t_m)^2,$$

where K/k is a biquadratic extension, i.e. a Galois extension whose Galois group is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then

(1) *the sequence* (E) *is exact for X;*
(2) *assuming Schinzel's hypothesis, the Brauer–Manin obstruction is the only obstruction to the Hasse principle and to weak approximation for rational points on X.*

Our proof is based on a geometric observation which permits us to reduce the question to a case that can be deduced easily from existing results. In the end, we consider naturally expected generalizations and explain why they cannot be proved in the same manner.

2. Proof of the theorem

2.1. Several preliminaries

2.1.1. *Birational invariance of the question.* Let X and X' be proper smooth and geometrically integral k-varieties. Suppose that they are birationally equivalent, i.e. they have the same function field $k(X) = k(X')$. Then they have isomorphic Brauer groups. It follows from Lang–Nishimura's theorem that the statement "Brauer–Manin obstruction is the only obstruction to the Hasse principle and to weak approximation for rational points" is valid for X' as long as it is valid for X. Moreover the Chow group $\mathrm{CH}_0(-)$ of 0-cycles is also a birational invariant [CTC79, Prop. 6.3], whence so is the exactness of the sequence (E).

2.1.2. *Algebraic tori associated to the equations.* Let F be a field of characteristic 0. Any biquadratic extension E of F can be written in the form $E = F(\sqrt{a}, \sqrt{b})$ with $a, b \in F^* \setminus F^{*2}$. It has three different non-trivial subfields $F(\sqrt{a})$, $F(\sqrt{b})$ and $F(\sqrt{ab})$ each of degree 2 over F.

Let T be the algebraic torus defined by the exact sequence of F-tori induced by the norm map

$$1 \to T \to R_{E/F}\mathbb{G}_{\mathrm{m}} \xrightarrow{N_{E/F}} \mathbb{G}_{\mathrm{m}} \to 1,$$

where $R_{E/F}$ is the Weil restriction of scalars. Similarly, the multiplication of norm maps defines a torus S fixed into the exact sequence

$$1 \to S \to R_{F(\sqrt{a})/F}\mathbb{G}_{\mathrm{m}} \times R_{F(\sqrt{b})/F}\mathbb{G}_{\mathrm{m}} \times R_{F(\sqrt{ab})/F}\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}} \to 1.$$

More explicitly, the tori T and S are defined respectively by

$$N_{E/F}(\mathbf{x}) = 1$$

and

$$N_{F(\sqrt{a})/F}(\mathbf{u}) \cdot N_{F(\sqrt{b})/F}(\mathbf{v}) \cdot N_{F(\sqrt{ab})/F}(\mathbf{w}) = 1.$$

The multiplication $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \mapsto \mathbf{u} \cdot \mathbf{v} \cdot \mathbf{w}$ in the field E induces the middle vertical morphism of tori in the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & S \arrow[r] \arrow[d, "\alpha"] & R_{F(\sqrt{a})/F}\mathbb{G}_{\mathrm{m}} \times R_{F(\sqrt{b})/F}\mathbb{G}_{\mathrm{m}} \times R_{F(\sqrt{ab})/F}\mathbb{G}_{\mathrm{m}} \arrow[r, "\lambda"] \arrow[d] & \mathbb{G}_{\mathrm{m}} \arrow[r] \arrow[d, "2"] & 1 \\
1 \arrow[r] & T \arrow[r] & R_{E/F}\mathbb{G}_{\mathrm{m}} \arrow[r, "\mu"] & \mathbb{G}_{\mathrm{m}} \arrow[r] & 1
\end{tikzcd}$$

Note that $N_{E/F}(\mathbf{u} \cdot \mathbf{v} \cdot \mathbf{w}) = [N_{F(\sqrt{a})/F}(\mathbf{u}) \cdot N_{F(\sqrt{b})/F}(\mathbf{v}) \cdot N_{F(\sqrt{a})/F}(\mathbf{w})]^2$, the square on the right hand side is commutative, and so the left vertical morphism α is induced.

Lemma 2.1 ([CT, Prop. 3.1(b)]). *The morphism α is an epimorphism whose kernel S_0 is isomorphic to $\mathbb{G}_{\mathrm{m}}^2$.*

Proof. One may prove the lemma by an argument on corresponding Galois modules as in [CT, Prop. 3.1(b)]; we give an alternative proof here.

On the level of rational points, for any extension L of F we need to show that $S_0(L)$ is (functorially) isomorphic to $\mathbb{G}_{\mathrm{m}}^2(L)$ as abelian groups.

By definition $S_0(L)$ is given by the triples

$$(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in F(\sqrt{a}) \otimes_F L \times F(\sqrt{b}) \otimes_F L \times F(\sqrt{ab}) \otimes_F L$$

satisfying

$$(\star)\qquad \begin{cases} \mathbf{u}\cdot\mathbf{v}\cdot\mathbf{w}=1,\\ N_{F(\sqrt{a})/F}(\mathbf{u})\cdot N_{F(\sqrt{b})/F}(\mathbf{v})\cdot N_{F(\sqrt{ab})/F}(\mathbf{w})=1. \end{cases}$$

Fix an L-linear base of $F(\sqrt{a})\otimes_F L$ (resp. $F(\sqrt{b})\otimes_F L$, $F(\sqrt{ab})\otimes_F L$), and write $\mathbf{u}=u_1+u_2\sqrt{a}$ (resp. $\mathbf{v}=v_1+v_2\sqrt{b}$, $\mathbf{w}=w_1+w_2\sqrt{ab}$) with $u_1,u_2,v_1,v_2,w_1,w_2\in L$. Easy calculation shows that $(\star)$ is equivalent to

$$(\star\star)\qquad \begin{cases} 1=u_1v_1w_1,\\ 0=u_2v_2w_2,\\ 0=u_1v_2w_2b+u_2v_1w_1,\\ 0=u_1v_2w_1+u_2v_1w_2a,\\ 0=u_1v_1w_2+u_2v_2w_1. \end{cases}$$

Hence one of u_2,v_2,w_2 must be 0, and no matter which one equals 0, the last three equalities imply that the other two are also 0. Then $(\star\star)$ is equivalent to

$$(\star\star\star)\qquad \begin{cases} 1=u_1v_1w_1,\\ 0=u_2=v_2=w_2, \end{cases}$$

which by definition is exactly $\mathbb{G}_{\mathrm{m}}^2(L)$. ∎

In the following proof of Theorem 1.1, F will be the function field $k(t_1,\dots,t_m)$, the F-tori T and S will be isotrivial, i.e. $a,b\in k^*$ and $E=F(\sqrt{a},\sqrt{b})=k(\sqrt{a},\sqrt{b})(t_1,\dots,t_m)$. By abuse of notation, we also denote by T the k-torus $N_{k(\sqrt{a},\sqrt{b})/k}(\mathbf{x})=1$, and similarly for S.

2.2. Proof of Theorem 1.1. We may write $K=k(\sqrt{a},\sqrt{b})$ with $a,b\in k^*$. The variety X that we consider is a proper smooth model of the equation $N_{K/k}(\mathbf{x})=Q(t_1,\dots,t_m)^2$. With the notation in the diagram of §2.1.2, this affine equation defines the fiber of μ over the point $Q(t_1,\dots,t_m)^2$ of $\mathbb{G}_{\mathrm{m},k(t_1,\dots,t_m)}$; we denote it by W. It is a principal homogeneous space under the torus T defined over $F=k(t_1,\dots,t_m)$.

Consider the fiber, denoted by V, of λ over the point $Q(t_1,\dots,t_m)$ of $\mathbb{G}_{\mathrm{m},k(t_1,\dots,t_m)}$. It is defined by the equation

$$N_{k(\sqrt{a})/k}(\mathbf{u})\cdot N_{k(\sqrt{b})/k}(\mathbf{v})\cdot N_{k(\sqrt{ab})/k}(\mathbf{w})=Q(t_1,\dots,t_m),$$

and is a principal homogeneous space under the F-torus S.

Associating $(\mathbf{u},\mathbf{v},\mathbf{w})$ to the product $\mathbf{x}=\mathbf{u}\cdot\mathbf{v}\cdot\mathbf{w}$, we define a $k(t_1,\dots,t_m)$-morphism $\phi: V\to W$. Then there exist

- a sufficiently small non-empty open subset U of $\mathbb{P}^m$,
- a k-torsor $W_0\to U$ under T whose generic fiber is W,

- a k-torsor $V_0 \to U$ under S whose generic fiber is V,
- a k-morphism $\Phi : V_0 \to W_0$ extending ϕ,

such that the following diagram commutes:

$$\begin{tikzcd} V_0 \arrow[rr, "\Phi"] \arrow[dr] & & W_0 \arrow[dl] \\ & U & \end{tikzcd}$$

Tautologically V_0, W_0 and Φ are given by the same equations as V, W and ϕ, but viewed as k-varieties and as a k-morphism (well-defined over a suitable open subset $U \subset \mathbb{P}^m$). As $S \to T$ is an epimorphism of tori, and torsors V and W become trivial over the algebraic closure of F, the morphism ϕ is then geometrically surjective. By shrinking U if necessary, we may assume moreover that Φ is also geometrically surjective. The morphism $\Phi : V_0 \to W_0$ is a torsor under the torus S_0. In fact, the fiber of Φ over each point $(\mathbf{x}, t_1, \ldots, t_m)$ of W_0 is defined by the equations

$$\begin{cases} \mathbf{u} \cdot \mathbf{v} \cdot \mathbf{w} = \mathbf{x}, \\ N_{F(\sqrt{a})/F}(\mathbf{u}) \cdot N_{F(\sqrt{b})/F}(\mathbf{v}) \cdot N_{F(\sqrt{ab})/F}(\mathbf{w}) = Q(t_1, \ldots, t_m), \end{cases}$$

on which the torus S_0 (defined in Lemma 2.1 explicitly by $(\star)$ as in its proof) acts freely transitively by multiplication at each coordinate component. In other words, the variety V_0 defines a class $[V_0]$ in the cohomology $H^1(W_0, S_0)$. Note that $S_0 \simeq \mathbb{G}_{\mathrm{m}}^2$ by Lemma 2.1, and by Hilbert's 90 we obtain $H^1(k(W_0), S_0) = 0$. Restricted to the generic point $\mathrm{Spec}(k(W_0))$ of W_0, the class $[V_0]$ becomes 0. Therefore the function field $k(V_0)$ is a purely transcendental extension of $k(W_0)$ of transcendental degree 2. We deduce that V_0 is birationally equivalent to $W_0 \times \mathbb{P}^2$, and the latter is birationally equivalent to $X \times \mathbb{P}^2$.

Recall that the generic fiber of $V_0 \to U \subset \mathbb{P}^m$ is defined by

$$N_{k(\sqrt{a})/k}(\mathbf{u}) \cdot N_{k(\sqrt{b})/k}(\mathbf{v}) \cdot N_{k(\sqrt{ab})/k}(\mathbf{w}) = Q(t_1, \ldots, t_m).$$

Birationally, the equation is the same as

$$N_{k(\sqrt{a})/k}(\mathbf{u}) = \frac{Q(t_1, \ldots, t_m)}{N_{k(\sqrt{b})/k}(\mathbf{v}) \cdot N_{k(\sqrt{ab})/k}(\mathbf{w})},$$

which can be viewed as a fibration in conics over $\mathbb{P}^{m+4}$ via the parametric variables $(t_1, \ldots, t_m, \mathbf{v}, \mathbf{w})$. For proper smooth models of V_0, the statement (1) of the theorem has been proved in [Lia13, §6]; and the statement (2) has been proved by Wittenberg [Wit07, Cor. 3.5]; an alternative proof for the case $m = 1$ is also available in a recent preprint of Wei [Wei, Thm. 3.5].

By the birational invariance of the statements (1) and (2), the following lemma will complete the proof. ∎

Lemma 2.2. *Let k be a number field and X be a proper smooth and geometrically integral k-variety.*

(1) *If the sequence* (E) *is exact for $X \times \mathbb{P}^n$, then it is also exact for X.*
(2) *If the Brauer–Manin obstruction is the only obstruction to the Hasse principle and to weak approximation for rational points on $X \times \mathbb{P}^n$, then this is also the case for X.*

Proof. Note that the projection $\pi : X \times \mathbb{P}^n \to X$ induces an isomorphism $\mathrm{Br}(X) \xrightarrow{\simeq} \mathrm{Br}(X \times \mathbb{P}^n)$. Fix a k-rational point p of $\mathbb{P}^n$; the map $x \mapsto (x, p)$ defines a section $\sigma : X \to X \times \mathbb{P}^n$ of π. Diagram chasing proves the first statement. Functoriality of the Brauer–Manin pairing and an obvious fibration argument prove the second statement. ■

2.3. Remark on "generalizations". We consider naturally expected generalizations and explain why they cannot be proved in the same manner.

Let p be a prime number and K/k be a Galois extension with Galois group $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. One may expect that the analogue of Theorem 1.1 holds for proper smooth models of the equation

$$N_{K/k}(\mathbf{x}) = Q(t_1, \ldots, t_m)^p.$$

Once an analogue of Lemma 2.1 for general p is established, i.e. the homomorphism α is an epimorphism with kernel $S_0 \simeq \mathbb{G}_{\mathrm{m}}^p$, all the remaining arguments still work well. Unfortunately, as stated in Proposition 2.3 below, for $p > 2$ the kernel S_0 is not connected anymore.

To fix the notation, let E/F be a Galois extension with Galois group $G = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$; it has $p+1$ non-trivial subextensions F_i $(i = 0, \ldots, p)$. The subgroup $G_i = \mathrm{Gal}(E/F_i)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ and $H_i = G/G_i = \mathrm{Gal}(F_i/F)$ is also isomorphic to $\mathbb{Z}/p\mathbb{Z}$. We can write down explicitly the Galois equivariant homomorphism induced by the quotients $G \to H_i$:

$$\hat{\rho} : \mathbb{Z}[G] \to \prod_{i=0}^{p} \mathbb{Z}[H_i].$$

It factorizes through quotients by the diagonally embedded $\mathbb{Z}$ and gives the homomorphism

$$\hat{\alpha} : \hat{T} = \frac{\mathbb{Z}[G]}{\mathbb{Z}} \to \hat{S} = \frac{\prod_{i=0}^{p} \mathbb{Z}[H_i]}{\mathbb{Z}}.$$

This last homomorphism between Galois modules corresponds to a morphism $\alpha : S \to T$ between algebraic tori, where S is the torus defined by $\prod_{i=0}^{p} N_{F_i/F}(\mathbf{u}_i) = 1$ and T is the torus defined by $N_{E/F}(\mathbf{x}) = 1$.

Proposition 2.3. *The morphism α is an epimorphism. The kernel $S_0 = $* $\mathrm{Ker}(\alpha)$ *is a group of multiplicative type, its identity component S_0° is iso-*

morphic to $\mathbb{G}_{\mathrm{m}}^p$*, and its group of connected components* $\pi_0(S_0)$ *is isomorphic to* $(\mathbb{Z}/p\mathbb{Z})^{p-2}$ *as an abelian group. In particular* $S_0 \simeq \mathbb{G}_{\mathrm{m}}^p$ *for* $p = 2$*.*

The proposition generalizes Lemma 2.1. It can be proved by investigating exact sequences of Galois modules; the proof is not difficult but rather long, we leave it to the reader.

If we trace the proof of Lemma 2.1, we may also see to some extent why we have the difference in connectedness between the cases $p = 2$ and $p > 2$: the equivalences $(\star)\Leftrightarrow(\star\star)\Leftrightarrow(\star\star\star)$ can be easily established in the case $p = 2$, but difficulties appear when $p > 2$.

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