ACTA ARITHMETICA
164.2 (2014)

The mean square of the divisor function

by

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1. Introduction. Let $d(n)$ be the divisor function. In 1916, S. Ramanujan [9] stated without proof that

$$d^2(1)+d^2(2)+d^2(3)+\cdots+d^2(n) = An(\log n)^3+Bn(\log n)^2+Cn\log n+Dn+O(n^{3/5+\varepsilon}); \tag{1.1}$$

here

$$A=\frac{1}{\pi^2},\quad B=\frac{12\gamma-3}{\pi^2}-\frac{36}{\pi^4}\zeta'(2),$$

where γ is Euler's constant, C, D are more complicated constants, ε is a sufficiently small positive constant. S. Ramanujan [9] also stated that, assuming the Riemann Hypothesis (RH), the error term in (1.1) can be improved to $O(n^{1/2+\varepsilon})$.

Write

$$E(x)=\sum_{n\le x}d^2(n)-xP(\log x), \tag{1.2}$$

where

$$P(x)=Ax^3+Bx^2+Cx+D.$$

Then the statement of Ramanujan is that

$$E(x)=O(x^{3/5+\varepsilon}), \tag{1.3}$$

and assuming the RH,

$$E(x)=O(x^{1/2+\varepsilon}). \tag{1.4}$$

In 1922, B. M. Wilson [13] proved (1.4) unconditionally. By a general theorem of M. Kühleitner and W. G. Nowak (see [5, 5.4]), we know

$$E(x)=\Omega(x^{3/8}). \tag{1.5}$$

2010 *Mathematics Subject Classification*: Primary 11M; Secondary 11M06.
Key words and phrases: divisor function, Riemann zeta-function, mean value.

DOI: 10.4064/aa164-2-7

Let $d_4(n)$ be the general divisor function which is the number of representations of $n = d_1 d_2 d_3 d_4$. In 1973, assuming

$$\sum_{n\le x} d_4(n) = \frac{1}{6}x(\log x)^3 + \left(2\gamma - \frac{1}{2}\right)x(\log x)^2 + ax\log x + bx + O(x^\alpha),$$

where γ is Euler's constant, a, b are constants, and α is a constant strictly less than 1/2, D. Suryanarayana and R. Sitaramachandra Rao [10] proved

$$E(x) = O(x^{1/2}\exp(-c(\log x)^{3/5}(\log\log x)^{-1/5})), \tag{1.6}$$

where c is a positive constant.

By Vinogradov's estimate, if $T/2 \le t \le T$, then

$$\frac{1}{\zeta(1+2it)} \ll (\log T)^{2/3}(\log\log T)^{1/3}.$$

So it is not difficult to prove

$$E(x) = O(x^{1/2}(\log x)^{17/3}(\log\log x)^{1/3}). \tag{1.7}$$

The direct application of the RH (or even the quasi-RH) would produce

$$E(x) = O(x^{1/2}(\log x)^5 \log\log x). \tag{1.8}$$

In 2003, K. Ramachandra and A. Sankaranarayanan [8] proved (1.8) without any assumption and put forward the following conjecture.

CONJECTURE (Ramachandra–Sankaranarayanan). *Assuming the RH, we have*

$$E(x) = O(x^{1/2}). \tag{1.9}$$

For the average situation, in 2005, H. Maier and A. Sankaranarayanan [7] proved

$$\frac{1}{X}\int_X^{2X} E^2(x)\,dx \ll X\exp(-c(\log X)^{3/5}(\log\log X)^{-1/5}), \tag{1.10}$$

where c is a positive constant.

In this paper, we prove the following theorem.

THEOREM. *If $E(x)$ is defined in* (1.2), *then unconditionally*

$$E(x) = O(x^{1/2}(\log x)^5). \tag{1.11}$$

Throughout this paper, we assume that ε is a sufficiently small positive constant and that T is sufficiently large.

2. Some lemmas

LEMMA 1 (Borel–Carathéodory; see [11, Section 5.5]). *Suppose that $f(z)$ is holomorphic in the disk $|z - z_0| \le R$ and that in the circle $z = z_0 + Re^{i\theta}$ $(0 \le \theta \le 2\pi)$,*

$$\mathrm{Re}(f(z)) \le M.$$

Then in the disk $|z - z_0| \le r$ $(< R)$, *we have*

$$|f(z)| \le \frac{2r}{R-r} M + \frac{R+r}{R-r} |f(z_0)|.$$

LEMMA 2 (Hadamard; see [11, Section 5.3]). *Suppose that* $f(z)$ *is holomorphic in the disk* $|z - z_0| \le R_3$, *and* $R_1 < R_2 < R_3$. *Write*

$$M_j = \max_{|z-z_0|=R_j} |f(z)|, \qquad j = 1, 2, 3.$$

Then

$$\log M_2 \le \frac{\log\left(\frac{R_3}{R_2}\right)}{\log\left(\frac{R_3}{R_1}\right)} \cdot \log M_1 + \frac{\log\left(\frac{R_2}{R_1}\right)}{\log\left(\frac{R_3}{R_1}\right)} \cdot \log M_3.$$

LEMMA 3 ([12, (2.15.2), p. 33]). *For* $\alpha > 0$ *and* $x > 0$, *we have*

$$\frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \Gamma(s) x^{-s} \, ds = e^{-x}.$$

LEMMA 4 ([12, (4.12.2), p. 78]). *For* $-1 \le \sigma \le 2$ *and* $|t| \ge 1$, *we have*

$$\Gamma(\sigma + it) \ll |t|^{\sigma - 1/2} e^{-\pi |t|/2}.$$

LEMMA 5 ([12, Lemma 3.12, p. 60]). *For* $\mathrm{Re}(s) > 1$, *let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

where $a(n) = O(\psi(n))$, $\psi(n)$ *is non-decreasing, and as* $\sigma \to 1^+$,

$$\sum_{n=1}^{\infty} \frac{|a(n)|}{n^\sigma} = O\left(\frac{1}{(\sigma - 1)^\alpha}\right).$$

If $c > 1$, x *is not an integer, and* N *is the integer nearest to* x, *then*

$$\sum_{n<x} a(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s) \frac{x^s}{s} \, ds + O\left(\frac{x^c}{T(c-1)^\alpha}\right) + O\left(\frac{\psi(2x) x \log x}{T}\right) + O\left(\frac{\psi(N) x}{T|x - N|}\right).$$

LEMMA 6 ([12, (1.2.10), p. 5]). *For* $\mathrm{Re}(s) > 1$, *we have*

$$\sum_{n=1}^{\infty} \frac{d^2(n)}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}.$$

LEMMA 7 ([12, (2.12.2), p. 29]). *For* $\mathrm{Re}(s) \ge 1/2$ *and* $|s - 1| > 1$, *we have*

$$\zeta(s) = O(|s|).$$

LEMMA 8 ([12, (3.11.8), p. 60]). *For $\sigma \geq 1$ and $t \geq 1$, we have*

$$\frac{1}{\zeta(\sigma+it)} = O(\log t).$$

LEMMA 9 ([12, Theorem 5.5, p. 99]). *For $t \geq 1$, we have*

$$\zeta(1/2+it) = O(t^{1/6+\varepsilon}).$$

REMARK. The bounds stated in Lemmas 8 and 9 suffice for our purpose, though better upper bounds are known.

LEMMA 10. *For $1/2 \leq \sigma \leq 1+\varepsilon$ and $t \geq 1$, we have*

$$\zeta(\sigma+it) = O(t^{\frac{1}{3}(1-\sigma)+\varepsilon}).$$

Proof. This follows from Lemma 9 and the explanation in Chapter 5 of [12].

LEMMA 11 ([12, (7.6.1), p. 147]). *We have*

$$\int_1^T |\zeta(1/2+it)|^4 dt = O(T(\log T)^4).$$

LEMMA 12 (Huxley [3]). *For $\sigma \geq 1/2$, let $N(\sigma, T, 2T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ which satisfy $\beta \geq \sigma$ and $T \leq \gamma \leq 2T$. Then*

$$N(\sigma, T, 2T) \ll T^{\frac{12}{5}(1-\sigma)+\varepsilon}.$$

LEMMA 13 ([4, Lemma 1]). *For $\mathrm{Re}(z) > 0$, we have*

$$\int_0^\infty e^{-zt}|\zeta(1/2+it)|^2\,dt = 2\pi e^{iz/2}\sum_{l=1}^\infty d(l)\exp(2\pi i l e^{iz}) + f(z),$$

where $f(z)$ is holomorphic in $|z| < 4\varepsilon$.

Define

$$D\left(s; \frac{h}{k}\right) = \sum_{l=1}^\infty \frac{d(l)}{l^s} e\left(l\frac{h}{k}\right). \tag{2.1}$$

LEMMA 14 (Estermann [1, (21), (34), (32), (29) and (19)]). *Suppose that $(h,k) = 1$. The function $D(s; h/k)$ is meromorphic in the whole plane with only one pole of order 2 at $s = 1$. In the neighborhood of $s = 1$,*

$$D\left(s; \frac{h}{k}\right) = \frac{1}{k}\cdot\frac{1}{(s-1)^2} + \frac{2}{k}(\gamma - \log k)\cdot\frac{1}{(s-1)} + \cdots,$$

where γ is Euler's constant. At $s = 0$, we have

$$D\left(0; \frac{h}{k}\right) = \frac{1}{4} - \frac{1}{\pi i}\sum_{a=1}^k \beta(a,k)\sum_{0<b<k/2}\eta(b,k)e\left(ab\frac{\overline{h}}{k}\right),$$

where $\overline{h}h \equiv 1 \pmod{k}$,

$$\beta(a,\, k) = \begin{cases} \dfrac{1}{1-e(-a/k)} & \text{if } 1 \le a < k, \\ 1/2 & \text{if } a = k, \end{cases}$$

and when $0 < b < k/2$,

$$0 < \eta(b,k) < 1/b.$$

Moreover, $D(s; h/k)$ *satisfies the functional equation*

$$D\left(s; \frac{h}{k}\right) = 2G^2(s)k^{1-2s}\left(D\left(1-s; \frac{\overline{h}}{k}\right) - \cos(\pi s)D\left(1-s; -\frac{\overline{h}}{k}\right)\right),$$

where

$$G(s) = (2\pi)^{s-1}\Gamma(1-s).$$

Lemma 15. *If* $(m_1, m_2) = (n_1, n_2) = 1$, *then*

$$(m_1 n_1^2, m_2 n_2^2) = (m_1, n_2^2)(m_2, n_1^2).$$

Proof. We have

$$(m_1 n_1^2, m_2 n_2^2) = (m_1, n_2^2)\left(\frac{m_1}{(m_1, n_2^2)} n_1^2, m_2 \frac{n_2^2}{(m_1, n_2^2)}\right).$$

Since

$$\left(\frac{m_1}{(m_1, n_2^2)}, m_2\right) = 1, \qquad \left(\frac{m_1}{(m_1, n_2^2)}, \frac{n_2^2}{(m_1, n_2^2)}\right) = 1,$$

we have

$$\left(\frac{m_1}{(m_1, n_2^2)} n_1^2, m_2 \frac{n_2^2}{(m_1, n_2^2)}\right) = \left(n_1^2, m_2 \frac{n_2^2}{(m_1, n_2^2)}\right) = (n_1^2, m_2).$$

Thus, the conclusion of Lemma 15 follows.

Lemma 16. *If* a *is a positive integer, then*

$$\sum_{M < m \le 2^\varepsilon M} (m, a) \ll M d(a).$$

Proof. We have

$$\sum_{M < m \le 2^\varepsilon M} (m,a) = \sum_{d \mid a} d \sum_{\substack{M < m \le 2^\varepsilon M \\ (m,a) = d}} 1 = \sum_{d \mid a} d \sum_{\substack{M/d < m_1 \le 2^\varepsilon M/d \\ (m_1, a/d) = 1}} 1$$

$$\le \sum_{d \mid a} d \sum_{M/d < m_1 \le 2^\varepsilon M/d} 1 \ll \sum_{d \mid a} d \cdot \frac{M}{d} = M d(a).$$

Lemma 17. *Suppose that $0 < A < B < 2q$ and b is a positive integer. Then*

$$\sum_{\substack{A<a\le B\\(a,q)=1\\(a,b)=1}} e\left(l\frac{\overline{a}}{q}\right) \ll (l,q)^{1/2}q^{1/2+\varepsilon}b^{\varepsilon}.$$

Here $\overline{a}$ is the integer such that $a\overline{a} \equiv 1 \pmod q$.

Proof. By Lemma 3 of [4], for $0 < A < B < 2q$, we have

$$\sum_{\substack{A<a\le B\\(a,q)=1}} e\left(l\frac{\overline{a}}{q}\right) \ll (l,q)^{1/2}q^{1/2+\varepsilon}.$$

Hence,

$$\begin{aligned}\sum_{\substack{A<a\le B\\(a,q)=1\\(a,b)=1}} e\left(l\frac{\overline{a}}{q}\right) &= \sum_{\substack{A<a\le B\\(a,q)=1}} \Big(\sum_{d\mid(a,b)} \mu(d)\Big) e\left(l\frac{\overline{a}}{q}\right) = \sum_{d\mid b}\mu(d) \sum_{\substack{A<a\le B\\(a,q)=1\\d\mid a}} e\left(l\frac{\overline{a}}{q}\right)\\
&= \sum_{d\mid b}\mu(d) \sum_{\substack{A/d<t\le B/d\\(dt,q)=1}} e\left(l\frac{\overline{dt}}{q}\right)\\
&= \sum_{\substack{d\mid b\\(d,q)=1}} \mu(d) \sum_{\substack{A/d<t\le B/d\\(t,q)=1}} e\left(l\overline{d}\frac{\overline{t}}{q}\right)\\
&\ll \sum_{\substack{d\mid b\\(d,q)=1}} |\mu(d)|(l\overline{d},q)^{1/2}q^{1/2+\varepsilon}\\
&\ll (l,q)^{1/2}q^{1/2+\varepsilon}\sum_{d\mid b} 1 \ll (l,q)^{1/2}q^{1/2+\varepsilon}b^{\varepsilon}.\end{aligned}$$

Thus, Lemma 17 is proved.

3. An asymptotic expression of $\zeta(1+it)$. Let

$$\rho_1 = \beta_1 + i\gamma_1, \quad \rho_2 = \beta_2 + i\gamma_2, \quad \ldots, \quad \rho_J = \beta_J + i\gamma_J$$

be all zeros of $\zeta(s)$ which satisfy $\beta \ge 1 - 4\varepsilon, T \le \gamma \le 2T$. By Lemma 12,

$$J = N(1-4\varepsilon, T, 2T) \ll T^{11\varepsilon}. \tag{3.1}$$

We write

$$D = \{s = \sigma + it : 1 - 4\varepsilon \le \sigma,\ T \le t \le 2T\}$$

and

$$
\begin{aligned}
U_1 &= \bigcup_{j=1}^{J} (\gamma_j - (\log T)^{10}, \gamma_j + (\log T)^{10}),\\
U_2 &= \bigcup_{j=1}^{J} (\gamma_j - 2(\log T)^{10}, \gamma_j + 2(\log T)^{10}),\\
U_3 &= \bigcup_{j=1}^{J} (\gamma_j - 3(\log T)^{10}, \gamma_j + 3(\log T)^{10}),\\
U_4 &= \bigcup_{j=1}^{J} (\gamma_j - 4(\log T)^{10}, \gamma_j + 4(\log T)^{10}).
\end{aligned} \tag{3.2}
$$

After removing all domains of the form $\{s = \sigma + it : 1 - 4\varepsilon \le \sigma < 1, t \in U_1\}$ from D, we denote the remaining domain as D_1. Then D_1 is a connected domain in which $\zeta(s) \ne 0$ so that we can define a holomorphic function $\log \zeta(s)$ in D_1. For $\mathrm{Re}(s) > 1$, Euler's product formula produces

$$
\log \zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^s}\right) = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} = \sum_{n=2}^{\infty} \frac{\Lambda_1(n)}{n^s}, \tag{3.3}
$$

where

$$
\Lambda_1(n) = \frac{\Lambda(n)}{\log n}.
$$

After removing all domains of the form $\{s = \sigma + it : 1 - 4\varepsilon \le \sigma, t \in U_2\}$ from D, we denote the remaining domain as D_2. Now Lemma 1 can be applied. Take $f(z) = \log \zeta(z)$. For $s = \sigma + it \in D_2$, $1 - 2\varepsilon \le \sigma \le 2$, let the center of the circles be $z_0 = 2 + it$, the radius of the bigger circle be $R = 2 - (1 - 4\varepsilon) = 1 + 4\varepsilon$, the radius of the smaller circle be $r = 2 - (1 - 2\varepsilon) = 1 + 2\varepsilon$. On the bigger circle, by Lemma 7,

$$
\mathrm{Re}(\log \zeta(z)) = \log |\zeta(z)| \le C \log T,
$$

where C is a positive constant. Thus, for s in the smaller circle, Lemma 1 yields

$$
|\log \zeta(s)| \le \frac{2r}{R - r} \cdot C \log T + \frac{R + r}{R - r} \cdot |\log \zeta(2 + it)| \ll \log T.
$$

For $\mathrm{Re}(s) \ge 2$, it is easy to see that

$$
\log \zeta(s) = O(1).
$$

Hence, for $s = \sigma + it \in D_2$, $\sigma \ge 1 - 2\varepsilon$, we have

$$
|\log \zeta(s)| \ll \log T. \tag{3.4}
$$

After removing all domains of the form $\{s = \sigma + it : 1 - 4\varepsilon \le \sigma,\ t \in U_3\}$ from D, then limiting $\sigma \ge 1-2\varepsilon$, we denote the resulting domain as D_3. Now Lemma 2 can be applied. Take $f(z) = \log\zeta(z)$. For $s = \sigma + it \in D_3, 1-\varepsilon \le \sigma \le 1+\varepsilon$, let the center of the circles be $z_0 = 2+it$, $R_3 = 2-(1-2\varepsilon) = 1+2\varepsilon$, $R_2 = 2-(1-\varepsilon) = 1+\varepsilon$, $R_1 = 2-(1+\varepsilon) = 1-\varepsilon$. By (3.4), $M_3 \ll \log T$. It is obvious that $M_1 = O(1)$. Lemma 2 yields

$$\begin{aligned}\log M_2 &\le \frac{\log\left(\frac{1+2\varepsilon}{1+\varepsilon}\right)}{\log\left(\frac{1+2\varepsilon}{1-\varepsilon}\right)} \cdot \log M_1 + \frac{\log\left(\frac{1+\varepsilon}{1-\varepsilon}\right)}{\log\left(\frac{1+2\varepsilon}{1-\varepsilon}\right)} \cdot \log M_3 \\ &\le O(1) + \frac{2\varepsilon + O(\varepsilon^2)}{3\varepsilon + O(\varepsilon^2)} \log\log T = O(1) + \left(\frac{2}{3} + O(\varepsilon)\right) \log\log T \\ &\le \frac{3}{4}\log\log T.\end{aligned}$$

Hence, for $s = \sigma + it \in D_3, 1-\varepsilon \le \sigma \le 1+\varepsilon$, we have

$$|\log\zeta(s)| \le (\log T)^{3/4}.$$

For $\mathrm{Re}(s) \ge 1+\varepsilon$, it is obvious that $1/\zeta(s) = O_\varepsilon(1)$. Thus, for $s = \sigma + it \in D_3$, $\sigma \ge 1-\varepsilon$, we have

$$\frac{1}{\zeta(s)} \ll \exp((\log T)^{3/4}). \tag{3.5}$$

After removing all domains of the form $\{s = \sigma + it : 1 - 4\varepsilon \le \sigma,\ t \in U_4\}$ from D, then limiting $\sigma \ge 1-\varepsilon$, we denote the resulting domain as D_4. For $s \in D_4$, $u \ge 0$, $|v| \le (\log T)^3$, we have

$$\frac{1}{\zeta(s+u+iv)} \ll \exp((\log T)^{3/4}). \tag{3.6}$$

For $s = 1+it \in D_4$, $w = u+iv$, $X > 1$, we have

$$\begin{aligned}&\frac{1}{2\pi i} \int\limits_{u=\varepsilon,\,|v|\le(\log T)^3} \frac{1}{\zeta(s+w)} \cdot \Gamma(w) X^w\,dw \\ &\qquad = \frac{1}{2\pi i} \int\limits_{u=\varepsilon,\,|v|\le(\log T)^3} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s+w}} \cdot \Gamma(w) X^w\,dw \\ &\qquad = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \cdot \frac{1}{2\pi i} \int\limits_{u=\varepsilon,\,|v|\le(\log T)^3} \Gamma(w)\left(\frac{X}{n}\right)^w dw.\end{aligned}$$

By Lemma 4, if $|v| \ge 1$, then on the vertical line $u = \epsilon$, we have

$$\Gamma(w) \ll |v|^{\varepsilon-1/2} e^{-\pi|v|/2}.$$

Hence,

$$\frac{1}{2\pi i}\int\limits_{u=\varepsilon,\,|v|>(\log T)^3}\Gamma(w)\Big(\frac{X}{n}\Big)^w\,dw$$
$$\ll\Big(\frac{X}{n}\Big)^\varepsilon\int\limits_{u=\varepsilon,\,|v|>(\log T)^3}|\Gamma(w)|\,|dw|\ll\Big(\frac{X}{n}\Big)^\varepsilon\int\limits_{|v|>(\log T)^3}|v|^{\varepsilon-1/2}e^{-\pi|v|/2}\,dv$$
$$\ll\Big(\frac{X}{n}\Big)^\varepsilon\int\limits_{(\log T)^3}^{\infty}e^{-\pi v/2}\,dv\ll\Big(\frac{X}{n}\Big)^\varepsilon\exp\Big(-\frac{\pi}{2}(\log T)^3\Big).$$

By Lemma 3,

$$\frac{1}{2\pi i}\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty}\Gamma(w)\Big(\frac{X}{n}\Big)^w\,dw=e^{-n/X}.$$

Therefore

$$\frac{1}{2\pi i}\int\limits_{u=\varepsilon,\,|v|\le(\log T)^3}\frac{1}{\zeta(s+w)}\cdot\Gamma(w)X^w\,dw$$
$$=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\Big(e^{-n/X}+O\Big(\Big(\frac{X}{n}\Big)^\varepsilon\exp\Big(-\frac{\pi}{2}(\log T)^3\Big)\Big)\Big)$$
$$=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}e^{-n/X}+O\Big(X^\varepsilon\exp\Big(-\frac{\pi}{2}(\log T)^3\Big)\Big).$$

We move the line of integration to $\mathrm{Re}(w)=-\varepsilon$. At $w=0$, $\Gamma(w)$ has a pole of order 1 with residue 1. Hence, the residue of $\frac{1}{\zeta(s+w)}\cdot\Gamma(w)X^w$ at $w=0$ is $1/\zeta(s)$. On two horizontal lines, by (3.6),

$$\frac{1}{2\pi i}\int\limits_{-\varepsilon\le u\le\varepsilon,\,|v|=(\log T)^3}\frac{1}{\zeta(s+w)}\cdot\Gamma(w)X^w\,dw$$
$$\ll X^\varepsilon\exp((\log T)^{3/4})\int\limits_{-\varepsilon}^{\varepsilon}e^{-\frac{\pi}{2}(\log T)^3}\,du\ll X^\varepsilon\exp(-(\log T)^3).$$

Integration on $\mathrm{Re}(w)=-\varepsilon$ yields

$$\frac{1}{2\pi i}\int\limits_{u=-\varepsilon,\,|v|\le(\log T)^3}\frac{1}{\zeta(s+w)}\Gamma(w)X^w\,dw$$
$$\ll X^{-\varepsilon}\exp((\log T)^{3/4})\int\limits_{u=-\varepsilon,\,|v|\le(\log T)^3}|\Gamma(w)|\,|dw|$$

$$\ll X^{-\varepsilon}\exp((\log T)^{3/4})\Big(\int\limits_{u=-\varepsilon,\,|v|\le 1}|\Gamma(w)|\,|dw| + \int\limits_{u=-\varepsilon,\,1\le|v|\le(\log T)^3}|\Gamma(w)|\,|dw|\Big)$$

$$\ll X^{-\varepsilon}\exp((\log T)^{3/4})\bigg(\int\limits_{u=-\varepsilon,\,|v|\le 1}\frac{|dw|}{|w|} + \int\limits_{1\le|v|\le(\log T)^3}|v|^{-\varepsilon-1/2}e^{-\pi|v|/2}\,dv\bigg)$$

$$\ll_\varepsilon X^{-\varepsilon}\exp((\log T)^{3/4}).$$

Combining all of the above, we get (with $s = 1+it$)

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}e^{-n/X} + O\big(X^{\varepsilon}\exp(-(\log T)^3)\big) + O\big(X^{-\varepsilon}\exp((\log T)^{3/4})\big). \tag{3.7}$$

Therefore we obtain an asymptotic expression of $\zeta(1+it)$ as follows.

Proposition 1. *Suppose that $T\le t\le 2T$, $t\notin U_4$ and*

$$X = \exp\bigg(\frac{2}{\varepsilon}(\log T)^{3/4}\bigg). \tag{3.8}$$

Then

$$\frac{1}{\zeta(1+it)} = \sum_{n\le X}\frac{\mu(n)}{n^{1+it}}e^{-n/X} + O(1). \tag{3.9}$$

4. A mean value estimate for $\zeta(s)$. In this section we prove

Proposition 2. *If k is any given positive number, then*

$$\int\limits_1^T \frac{|\zeta(1/2+it)|^4}{|\zeta(1+2it)|^k}\,dt \ll_k T(\log T)^4.$$

Firstly we prove Proposition 3 below. We use essentially the method of Iwaniec [4] but with some modification and refinement.

Proposition 3. *Suppose that $N\ll T^{1/16-\varepsilon}$ and that $a(n) = O(N^{-1+\varepsilon})$ for $N<n\le 2^{\varepsilon}N$. Then*

$$\int\limits_{T/2}^{T}|\zeta(1/2+it)|^4\bigg|\sum_{N<n\le 2^{\varepsilon}N}\frac{a(n)}{n^{2it}}\bigg|^2\,dt \ll \frac{T(\log T)^4}{N^{1-8\varepsilon}}.$$

Proof. By the discussion in Section 2 of [4], we estimate

$$\log T \sum_{r\le \frac{1}{2\varepsilon\log 2}\log T+O(1)} \int_0^\infty e^{-t/T}|\zeta(1/2+it)|^2 \times \Big|\sum_{2^{\varepsilon r}<m\le 2^\varepsilon\cdot 2^{\varepsilon r}} \frac{1}{m^{1/2+it}}\Big|^2 \Big|\sum_{N<n\le 2^\varepsilon N}\frac{a(n)}{n^{2it}}\Big|^2\,dt.$$

Write

$$\Big|\Big(\sum_{M<m\le 2^\varepsilon M}\frac{1}{m^{1/2+it}}\Big)\Big(\sum_{N<n\le 2^\varepsilon N}\frac{a(n)}{n^{2it}}\Big)\Big|^2 = \Big|\sum_{K<k\le 8^\varepsilon K}\frac{b(k)}{k^{it}}\Big|^2 = \sum_{K<k,h\le 8^\varepsilon K} b(k)\overline{b(h)}(h/k)^{it},$$

where $M=2^{\varepsilon r}$, $M\ll T^{1/2}$, $K=MN^2$, and

$$b(k)=\sum_{\substack{mn^2=k\\ M<m\le 2^\varepsilon M\\ N<n\le 2^\varepsilon N}}\frac{a(n)}{m^{1/2}}.$$

In the following we shall estimate

$$\int_0^\infty e^{-t/T}|\zeta(1/2+it)|^2\Big|\sum_{K<k\le 8^\varepsilon K}\frac{b(k)}{k^{it}}\Big|^2\,dt = \sum_{K<k,h\le 8^\varepsilon K} b(k)\overline{b(h)}\int_0^\infty e^{-(1/T-i\log(h/k))t}|\zeta(1/2+it)|^2\,dt. \tag{4.1}$$

Let

$$z=\frac{1}{T}-i\log\Big(\frac{h}{k}\Big) \tag{4.2}$$

and note that

$$|z|\le \frac{1}{T}+\Big|\log\Big(\frac{h}{k}\Big)\Big|<4\varepsilon$$

for $K<k,h\le 8^\varepsilon K$. By Lemma 13,

$$\int_0^\infty e^{-zt}|\zeta(1/2+it)|^2\,dt = 2\pi e^{i/(2T)}(h/k)^{1/2}\sum_{l=1}^\infty d(l)\exp(2\pi i l(h/k)e^{i/T})+O(1) \tag{4.3}$$

$$= 2\pi e^{i/(2T)}(h/k)^{1/2}\sum_{l=1}^{\infty} d(l)e\left(l\frac{h}{k}\right)\exp(2\pi il(h/k)(e^{i/T}-1)) + O(1)$$
$$= 2\pi e^{i/(2T)}(h/k)^{1/2}\sum_{l=1}^{\infty} d(l)e\left(l\frac{h}{k}\right)\exp(2\pi ilx) + O(1),$$

where

$$x = \frac{h}{k}(e^{i/T}-1). \tag{4.4}$$

The contribution of the term $O(1)$ to (4.1) is

$$O\Big(\sum_{K<k,h\le 8^{\varepsilon}K} |b(k)b(h)|\Big)$$
$$\ll \sum_{M<m_1\le 2^{\varepsilon}M}\frac{1}{m_1^{1/2}}\sum_{M<m_2\le 2^{\varepsilon}M}\frac{1}{m_2^{1/2}}\sum_{N<n_1\le 2^{\varepsilon}N}|a(n_1)|\sum_{N<n_2\le 2^{\varepsilon}N}|a(n_2)|$$
$$\ll M\sum_{N<n_1\le 2^{\varepsilon}N}\frac{1}{N^{1-\varepsilon}}\sum_{N<n_2\le 2^{\varepsilon}N}\frac{1}{N^{1-\varepsilon}} \ll MN^{2\varepsilon} \ll \frac{T}{N^{1-8\varepsilon}}.$$

Let

$$S\left(x;\frac{h}{k}\right) = \sum_{l=1}^{\infty} d(l)e\left(l\frac{h}{k}\right)\exp(2\pi ilx). \tag{4.5}$$

Write

$$\mathfrak{z} = -2\pi ix = 4\pi(h/k)\sin(1/(2T))e^{i/(2T)}. \tag{4.6}$$

By the discussion in Section 3 of [4], we know

$$S\left(x;\frac{h}{k}\right) = \frac{1}{2\pi i}\int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} D\left(s;\frac{h}{k}\right)\Gamma(s)\mathfrak{z}^{-s}\,ds, \tag{4.7}$$

where

$$D\left(s;\frac{h}{k}\right) = \sum_{l=1}^{\infty}\frac{d(l)}{l^s}e\left(l\frac{h}{k}\right).$$

In the following we write

$$k^* = \frac{k}{(k,h)}, \qquad h^* = \frac{h}{(k,h)}. \tag{4.8}$$

We move the line of integration from $\mathrm{Re}(s) = 1+\varepsilon$ to $\mathrm{Re}(s) = -\varepsilon$, and get

$$
\begin{aligned}
(4.9)\qquad S\Big(x;\frac{h}{k}\Big) &= \frac{1}{2\pi i}\int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} D\Big(s;\frac{h^*}{k^*}\Big)\Gamma(s)\mathfrak{z}^{-s}\,ds\\
&= \frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} D\Big(s;\frac{h^*}{k^*}\Big)\Gamma(s)\mathfrak{z}^{-s}ds + R_1(T;h,k) + R_0(T;h,k)\\
&= R(T;h,k) + R_1(T;h,k) + R_0(T;h,k),
\end{aligned}
$$

where

$$
(4.10)\qquad R(T;h,k) = \frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} D\Big(s;\frac{h^*}{k^*}\Big)\Gamma(s)\mathfrak{z}^{-s}\,ds,
$$

$R_1(T;h,k)$ and $R_0(T;h,k)$ are residues of $D(s;h^*/k^*)\Gamma(s)\mathfrak{z}^{-s}$ coming from the poles at $s=1$ and $s=0$ respectively.

By the discussion in Section 3 of [4] and by Lemma 14, we know that

$$
(4.11)\qquad R_1(T;h,k) = \frac{1}{\mathfrak{z}k^*}(\gamma - \log\mathfrak{z} - 2\log k^*) \ll \frac{T\log T}{k^*},
$$

$$
\begin{aligned}
(4.12)\qquad R_0(T;h,k) &= D\Big(0;\frac{h^*}{k^*}\Big)\\
&= \frac{1}{4} - \frac{1}{\pi i}\sum_{a=1}^{k^*}\beta(a,k^*)\sum_{0<b<k^*/2}\eta(b,k^*)e\Big(ab\frac{\overline{h^*}}{k^*}\Big).
\end{aligned}
$$

Now we consider the contribution of $R_1(T;h,k), R(T;h,k)$ and $R_0(T;h,k)$ to (4.1).

1. *The contribution of* $R_1(T;h,k)$. We note that $h/k \ll 1$ for $K < h,k \le 8^\varepsilon K$. Therefore the contribution of $R_1(T;h,k)$ is

$$
\begin{aligned}
(4.13)\qquad &\ll \sum_{K<k,h\le 8^\varepsilon K} |b(k)b(h)|\,|R_1(T;h,k)|\\
&\ll \sum_{K<k,h\le 8^\varepsilon K} |b(k)b(h)|\cdot\frac{T\log T}{k}(k,h)\\
&\ll T\log T\sum_{M<m_1\le 2^\varepsilon M}\ \sum_{M<m_2\le 2^\varepsilon M}\ \sum_{N<n_1\le 2^\varepsilon N}\ \sum_{N<n_2\le 2^\varepsilon N}\frac{|a(n_1)|}{m_1^{1/2}}\cdot\\
&\qquad\times\frac{|a(n_2)|}{m_2^{1/2}}\cdot\frac{1}{m_1n_1^2}(m_1n_1^2, m_2n_2^2)
\end{aligned}
$$

$$\ll T\log T\cdot\frac{1}{MN^{2-2\varepsilon}}$$
$$\times\frac{1}{MN^2}\sum_{M<m_1\le 2^\varepsilon M}\ \sum_{M<m_2\le 2^\varepsilon M}\ \sum_{N<n_1\le 2^\varepsilon N}\ \sum_{N<n_2\le 2^\varepsilon N}(m_1n_1^2,m_2n_2^2)$$
$$=\frac{T\log T}{M^2N^{4-2\varepsilon}}\sum_{M<m_1\le 2^\varepsilon M}\ \sum_{M<m_2\le 2^\varepsilon M}\ \sum_{N<n_1\le 2^\varepsilon N}\ \sum_{N<n_2\le 2^\varepsilon N}(m_1n_1^2,m_2n_2^2).$$

By Lemmas 15 and 16,

$$\sum_{M<m_1\le 2^\varepsilon M}\ \sum_{M<m_2\le 2^\varepsilon M}\ \sum_{N<n_1\le 2^\varepsilon N}\ \sum_{N<n_2\le 2^\varepsilon N}(m_1n_1^2,m_2n_2^2)$$
$$=\sum_{d\le 2^\varepsilon M}\ \sum_{M<m_1\le 2^\varepsilon M}\ \sum_{\substack{M<m_2\le 2^\varepsilon M\\(m_1,m_2)=d}}\ \sum_{r\le 2^\varepsilon N}\ \sum_{N<n_1\le 2^\varepsilon N}\ \sum_{\substack{N<n_2\le 2^\varepsilon N\\(n_1,n_2)=r}}(m_1n_1^2,m_2n_2^2)$$
$$=\sum_{d\le 2^\varepsilon M}d\sum_{M/d<m_1'\le 2^\varepsilon M/d}\ \sum_{\substack{M/d<m_2'\le 2^\varepsilon M/d\\(m_1',m_2')=1}}\ \sum_{r\le 2^\varepsilon N}r^2$$
$$\times\sum_{N/r<n_1'\le 2^\varepsilon N/r}\ \sum_{\substack{N/r<n_2'\le 2^\varepsilon N/r\\(n_1',n_2')=1}}(m_1'n_1'^2,m_2'n_2'^2)$$
$$=\sum_{d\le 2^\varepsilon M}d\sum_{M/d<m_1'\le 2^\varepsilon M/d}\ \sum_{\substack{M/d<m_2'\le 2^\varepsilon M/d\\(m_1',m_2')=1}}\ \sum_{r\le 2^\varepsilon N}r^2$$
$$\times\sum_{N/r<n_1'\le 2^\varepsilon N/r}\ \sum_{\substack{N/r<n_2'\le 2^\varepsilon N/r\\(n_1',n_2')=1}}(m_1',n_2'^2)(m_2',n_1'^2)$$
$$\le\sum_{d\le 2^\varepsilon M}d\sum_{M/d<m_1'\le 2^\varepsilon M/d}\ \sum_{M/d<m_2'\le 2^\varepsilon M/d}\ \sum_{r\le 2^\varepsilon N}r^2$$
$$\times\sum_{N/r<n_1'\le 2^\varepsilon N/r}\ \sum_{N/r<n_2'\le 2^\varepsilon N/r}(m_1',n_2'^2)(m_2',n_1'^2)$$
$$=\sum_{d\le 2^\varepsilon M}d\sum_{r\le 2^\varepsilon N}r^2\sum_{N/r<n_2'\le 2^\varepsilon N/r}\ \sum_{M/d<m_1'\le 2^\varepsilon M/d}(m_1',n_2'^2)$$
$$\times\sum_{N/r<n_1'\le 2^\varepsilon N/r}\ \sum_{M/d<m_2'\le 2^\varepsilon M/d}(m_2',n_1'^2)$$
$$\ll\sum_{d\le 2^\varepsilon M}d\sum_{r\le 2^\varepsilon N}r^2\sum_{N/r<n_2'\le 2^\varepsilon N/r}\frac{M}{d}\cdot d(n_2'^2)\sum_{N/r<n_1'\le 2^\varepsilon N/r}\frac{M}{d}\cdot d(n_1'^2)$$

$$\ll_\varepsilon M^2N^{2\varepsilon}\sum_{d\le 2^\varepsilon M}\frac{1}{d}\sum_{r\le 2^\varepsilon N} r^2 \sum_{N/r<n_1'\le 2^\varepsilon N/r}\ \sum_{N/r<n_2'\le 2^\varepsilon N/r} 1$$
$$\ll M^2N^{2\varepsilon}\log(2M)\sum_{r\le 2^\varepsilon N} r^2(N/r)^2 \ll_\varepsilon M^2N^{3+2\varepsilon}\log(2M).$$

Hence, the contribution of $R_1(T;h,k)$ is

$$\ll \frac{T\log T}{M^2N^{4-2\varepsilon}}\cdot M^2N^{3+2\varepsilon}\log(2M) \ll \frac{T(\log T)^2}{N^{1-8\varepsilon}}.$$

2. *The contribution of $R(T;h,k)$.* By the functional equation in Lemma 14, we get

$$\begin{aligned}
(4.14)\qquad R(T;h,k) &= \frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} D\Big(s;\frac{h^*}{k^*}\Big)\Gamma(s)\mathfrak{z}^{-s}\,ds\\
&= \frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} 2G^2(s)k^{*(1-2s)}\bigg(D\Big(1-s;\frac{\overline{h^*}}{k^*}\Big)\\
&\qquad - \cos(\pi s)D\Big(1-s;-\frac{\overline{h^*}}{k^*}\Big)\bigg)\Gamma(s)\mathfrak{z}^{-s}\,ds\\
&= k^*\sum_{l=1}^{\infty}\frac{d(l)}{l}\cdot\frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} 2G^2(s)\cdot\frac{l^s}{(h^*k^*)^s}\\
&\quad\times\bigg(e\Big(l\frac{\overline{h^*}}{k^*}\Big)-\cos(\pi s)e\Big(-l\frac{\overline{h^*}}{k^*}\Big)\bigg)\Gamma(s)\bigg(4\pi\sin\Big(\frac{1}{2T}\Big)e^{i/(2T)}\bigg)^{-s}ds\\
&= k^*\sum_{l=1}^{\infty}\frac{d(l)}{l}\cdot\frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} U(s,T)\Big(\frac{l}{h^*k^*}\Big)^s\\
&\qquad\times\bigg(e\Big(l\frac{\overline{h^*}}{k^*}\Big)-\cos(\pi s)e\Big(-l\frac{\overline{h^*}}{k^*}\Big)\bigg)ds,
\end{aligned}$$

where

$$(4.15)\qquad U(s,T) = 2G^2(s)\Gamma(s)\bigg(4\pi\sin\Big(\frac{1}{2T}\Big)e^{i/(2T)}\bigg)^{-s}.$$

The contribution of $R(T;h,k)$ is

$$\ll \bigg|\sum_{K<k,h\le 8^\varepsilon K} b(k)\overline{b(h)}\Big(\frac{h}{k}\Big)^{1/2}R(T;h,k)\bigg|;$$

here

$$
\begin{aligned}
(4.16)\qquad &\sum_{K<k,h\le 8^\varepsilon K} b(k)\overline{b(h)}\left(\frac{h}{k}\right)^{1/2} R(T;h,k)\\
&= \sum_{K<k\le 8^\varepsilon K}\ \sum_{K<h\le 8^\varepsilon K} \frac{b(k)}{k^{1/2}}\cdot \overline{b(h)}h^{1/2}\cdot k^* \sum_{l=1}^{\infty}\frac{d(l)}{l}\\
&\quad\times \frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} U(s,T)\left(\frac{l}{h^*k^*}\right)^s\left(e\left(l\frac{\overline{h^*}}{k^*}\right)-\cos(\pi s)e\left(-l\frac{\overline{h^*}}{k^*}\right)\right)ds\\
&= \sum_{l=1}^{\infty}\frac{d(l)}{l}\cdot\frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} U(s,T)l^s\\
&\quad\times\Bigg(\sum_{K<k\le 8^\varepsilon K}\ \sum_{K<h\le 8^\varepsilon K}\frac{b(k)}{k^{1/2}}\cdot\overline{b(h)}h^{1/2}\frac{k^*}{(h^*k^*)^s}e\left(l\frac{\overline{h^*}}{k^*}\right)\\
&\quad -\cos(\pi s)\sum_{K<k\le 8^\varepsilon K}\ \sum_{K<h\le 8^\varepsilon K}\frac{b(k)}{k^{1/2}}\cdot\overline{b(h)}h^{1/2}\frac{k^*}{(h^*k^*)^s}e\left(-l\frac{\overline{h^*}}{k^*}\right)\Bigg)ds\\
&= \sum_{l=1}^{\infty}\frac{d(l)}{l}\cdot\frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} U(s,T)l^s(Q(l,s)-\cos(\pi s)Q(-l,s))\,ds,
\end{aligned}
$$

where

$$
\begin{aligned}
(4.17)\qquad Q(l,s) &= \sum_{K<k\le 8^\varepsilon K}\ \sum_{K<h\le 8^\varepsilon K}\frac{b(k)}{k^{1/2}}\cdot\overline{b(h)}h^{1/2}\cdot\frac{k^*}{(h^*k^*)^s}e\left(l\frac{\overline{h^*}}{k^*}\right)\\
&= \sum_{K<k\le 8^\varepsilon K}\ \sum_{K<h\le 8^\varepsilon K} b(k)\overline{b(h)}\cdot\frac{1}{(k^*h^*)^{s-1/2}}e\left(l\frac{\overline{h^*}}{k^*}\right).
\end{aligned}
$$

For $s=-\varepsilon+it$, by the discussion in Section 5 of [4],

$$(4.18)\qquad U(s,T)l^s \ll \left(\frac{T}{l}\right)^{\varepsilon}(|t|+1)^{1/2+\varepsilon}\exp\left(\left(\frac{1}{2T}-\frac{3}{2}\pi\right)|t|\right),$$

$$(4.19)\qquad U(s,T)l^s\cos(\pi s) \ll \left(\frac{T}{l}\right)^{\varepsilon}(|t|+1)^{1/2+\varepsilon}\exp\left(\left(\frac{1}{2T}-\frac{\pi}{2}\right)|t|\right).$$

In the following we shall estimate $Q(l,s)$ for $s=-\varepsilon+it$:

$$
\begin{aligned}
(4.20)\quad Q(l,s) &= \sum_{K<k\le 8^\varepsilon K}\ \sum_{K<h\le 8^\varepsilon K} b(k)\overline{b(h)}\cdot\frac{1}{(k^*h^*)^{s-1/2}}e\left(l\frac{\overline{h^*}}{k^*}\right)\\
&= \sum_{K<k\le 8^\varepsilon K}\ \sum_{K<h\le 8^\varepsilon K} b(k)\overline{b(h)}\cdot\frac{(k,h)^{2s-1}}{(kh)^{s-1/2}}e\left(l\frac{\overline{h^*}}{k^*}\right)\\
&= \sum_{d\le 8^\varepsilon K} d^{2s-1} \sum_{K<k\le 8^\varepsilon K}\ \sum_{\substack{K<h\le 8^\varepsilon K\\ (k,h)=d}} \frac{b(k)\overline{b(h)}}{(kh)^{s-1/2}}e\left(l\frac{\overline{(h/d)}}{k/d}\right)\\
&= \sum_{d\le 8^\varepsilon K} d^{2s-1} \sum_{N<n_1\le 2^\varepsilon N}\ \sum_{N<n_2\le 2^\varepsilon N}\ \sum_{M<m_1\le 2^\varepsilon M}\\
&\qquad\times \sum_{\substack{M<m_2\le 2^\varepsilon M\\ (m_1n_1^2,m_2n_2^2)=d}} \frac{a(n_2)}{m_2^{1/2}}\frac{\overline{a(n_1)}}{m_1^{1/2}}\cdot\frac{1}{(m_1m_2n_1^2n_2^2)^{s-1/2}}e\left(l\frac{\overline{(m_1n_1^2/d)}}{m_2n_2^2/d}\right)\\
&= \sum_{d\le 8^\varepsilon K} d^{2s-1} \sum_{N<n_1\le 2^\varepsilon N}\ \sum_{N<n_2\le 2^\varepsilon N} \frac{a(n_2)\overline{a(n_1)}}{(n_1n_2)^{2s-1}}\\
&\qquad\qquad\times \sum_{M<m_1\le 2^\varepsilon M}\ \sum_{\substack{M<m_2\le 2^\varepsilon M\\ (m_1n_1^2,m_2n_2^2)=d}} \frac{1}{(m_1m_2)^s}e\left(l\frac{\overline{(m_1n_1^2/d)}}{m_2n_2^2/d}\right)\\
&= \sum_{d\le 8^\varepsilon K} d^{2s-1} \sum_{N<n_1\le 2^\varepsilon N}\ \sum_{N<n_2\le 2^\varepsilon N} \frac{a(n_2)\overline{a(n_1)}}{(n_1n_2)^{2s-1}}\cdot B(l,s,n_1,n_2,d),
\end{aligned}
$$

where

$$
\begin{aligned}
(4.21)\quad & B(l,s,n_1,n_2,d)\\
&\qquad = \sum_{\substack{M<m_2\le 2^\varepsilon M\\ d\mid m_2n_2^2}}\ \sum_{\substack{M<m_1\le 2^\varepsilon M\\ (m_1n_1^2,m_2n_2^2)=d}} \frac{1}{(m_1m_2)^s}e\left(l\frac{\overline{(m_1n_1^2/d)}}{m_2n_2^2/d}\right).
\end{aligned}
$$

We shall estimate

$$
\sum_{\substack{M<m_1\le M_1\\ (m_1n_1^2,m_2n_2^2)=d}} e\left(l\frac{\overline{(m_1n_1^2/d)}}{m_2n_2^2/d}\right)
$$

for $M<M_1\le 2^\varepsilon M$. Let $(m_1,d)=d_1$. Write $d=d_1d_2$. We see $(d_2,m_1/d_1)=1$.

Hence $d \mid m_1 n_1^2$, so $d_2 \mid n_1^2$, therefore $d_2 \le 4^\varepsilon N^2$. By Lemma 17,

$$\sum_{\substack{M<m_1\le M_1\\(m_1n_1^2,m_2n_2^2)=d}} e\bigg(l\frac{\overline{(m_1n_1^2/d)}}{m_2n_2^2/d}\bigg)$$

$$= \sum_{d_1|d}\bigg(\sum_{\substack{M<m_1\le M_1\\(m_1,d)=d_1\\(\frac{m_1}{d_1}\cdot\frac{n_1^2}{d_2},\frac{m_2n_2^2}{d})=1}} e\bigg(l\frac{\overline{(m_1n_1^2/d)}}{m_2n_2^2/d}\bigg)\bigg)$$

$$= \sum_{d_1|d}\bigg(\sum_{\substack{M/d_1<m_1'\le M_1/d_1\\(m_1',d_2)=1\\(m_1',m_2n_2^2/d)=1\\(n_1^2/d_2,m_2n_2^2/d)=1}} e\bigg(l\frac{\overline{(n_1^2/d_2)}\cdot\overline{m_1'}}{m_2n_2^2/d}\bigg)\bigg)$$

$$= \sum_{\substack{d_1|d\\(n_1^2/d_2,m_2n_2^2/d)=1}}\bigg(\sum_{\substack{M/d_1<m_1'\le M_1/d_1\\(m_1',m_2n_2^2/d)=1\\(m_1',d_2)=1}} e\bigg(l\overline{\Big(\frac{n_1^2}{d_2}\Big)}\cdot\frac{\overline{m_1'}}{m_2n_2^2/d}\bigg)\bigg)$$

$$\ll \sum_{\substack{d_1|d\\(n_1^2/d_2,m_2n_2^2/d)=1}}\bigg(l\overline{\Big(\frac{n_1^2}{d_2}\Big)},\frac{m_2n_2^2}{d}\bigg)^{1/2}\Big(\frac{m_2n_2^2}{d}\Big)^{1/2+\varepsilon}d_2^\varepsilon$$

$$\ll \Big(l,\frac{m_2n_2^2}{d}\Big)^{1/2}\sum_{d_1|d}\Big(\frac{m_2n_2^2}{d}\Big)^{1/2+\varepsilon}d_2^\varepsilon$$

$$\ll \Big(\sum_{d_1|d}1\Big)\Big(l,\frac{m_2n_2^2}{d}\Big)^{1/2}\Big(\frac{MN^2}{d}\Big)^{1/2+\varepsilon}d^\varepsilon$$

$$\ll \Big(l,\frac{m_2n_2^2}{d}\Big)^{1/2}\Big(\frac{MN^2}{d}\Big)^{1/2+\varepsilon}d^{2\varepsilon},$$

where we note $d_2 \le 4^\varepsilon N^2$, hence $M_1/d_1 < 2m_2n_2^2/d$.

By the above estimate and partial summation, for $s = -\varepsilon + it$, we have

$$\sum_{\substack{M<m_1\le 2^\varepsilon M\\(m_1n_1^2,m_2n_2^2)=d}} \frac{1}{m_1^s} e\bigg(l\frac{\overline{(m_1n_1^2/d)}}{m_2n_2^2/d}\bigg)$$

$$\ll (|t|+1)M^\varepsilon\Big(l,\frac{m_2n_2^2}{d}\Big)^{1/2}\Big(\frac{MN^2}{d}\Big)^{1/2+\varepsilon}d^{2\varepsilon}.$$

By Lemma 16 we get

$$\begin{aligned}
&B(l,s,n_1,n_2,d)\\
&\qquad \ll (|t|+1)\left(\frac{MN^2}{d}\right)^{1/2+\varepsilon} M^{2\varepsilon}d^{2\varepsilon}\sum_{\substack{M<m_2\le 2^\varepsilon M\\ d\mid m_2n_2^2}}\left(l,\frac{m_2n_2^2}{d}\right)^{1/2}\\
&\qquad \le (|t|+1)\left(\frac{MN^2}{d}\right)^{1/2+\varepsilon} M^{2\varepsilon}d^{2\varepsilon}\sum_{M<m_2\le 2^\varepsilon M}(l,m_2)^{1/2}(l,n_2^2)^{1/2}\\
&\qquad \ll (|t|+1)\left(\frac{MN^2}{d}\right)^{1/2+\varepsilon} M^{1+2\varepsilon}d^{2\varepsilon}(l,n_2^2)^{1/2}l^{\varepsilon/4}.
\end{aligned}$$

By Lemma 16 again,

$$\begin{aligned}
Q(l,s) &\ll (|t|+1)(MN^2)^{1/2+\varepsilon}M^{1+2\varepsilon}\sum_{d\le 8^\varepsilon K}\frac{1}{d^{3/2+\varepsilon}}\\
&\qquad \times \sum_{N<n_1\le 2^\varepsilon N}\sum_{N<n_2\le 2^\varepsilon N}|a(n_1)a(n_2)|N^{2(1+2\varepsilon)}(l,n_2^2)^{1/2}l^{\varepsilon/4}\\
&\ll (|t|+1)M^{3/2+3\varepsilon}N^{2+8\varepsilon}\sum_{N<n_2\le 2^\varepsilon N}(n_2,l)l^{\varepsilon/4}\\
&\ll (|t|+1)M^{3/2+3\varepsilon}N^{3+8\varepsilon}l^{\varepsilon/2}.
\end{aligned}$$

Consequently, the contribution of $R(T;h,k)$ is

$$\begin{aligned}
&\ll \sum_{l=1}^{\infty}\frac{d(l)}{l}\left(\frac{T}{l}\right)^{\varepsilon}\int_{-\infty}^{\infty}(|t|+1)^{3/2+\varepsilon}\exp\left(\left(\frac{1}{2T}-\frac{\pi}{2}\right)|t|\right)dt\cdot M^{3/2+3\varepsilon}N^{3+8\varepsilon}l^{\varepsilon/2}\\
&\ll T^\varepsilon M^{3/2+3\varepsilon}N^{3+8\varepsilon}\sum_{l=1}^{\infty}\frac{d(l)}{l^{1+\varepsilon/2}}\ll T^\varepsilon M^{3/2+3\varepsilon}N^{3+8\varepsilon}\ll\frac{T}{N^{1-8\varepsilon}}.
\end{aligned}$$

3. *The contribution of* $R_0(T;h,k)$. Using Lemma 14, (4.12) and the estimates in item 2, we see that the contribution of $R_0(T;h,k)$ is

$$\ll\left|\sum_{K<k,h\le 8^\varepsilon K} b(k)\overline{b(h)}\left(\frac{h}{k}\right)^{1/2}R_0(T;h,k)\right|,$$

where

$$\begin{aligned}
&\sum_{K<k,h\le 8^\varepsilon K} b(k)\overline{b(h)}\left(\frac{h}{k}\right)^{1/2}R_0(T;h,k)\\
&\quad=\sum_{K<k\le 8^\varepsilon K}\sum_{K<h\le 8^\varepsilon K} b(k)\overline{b(h)}\left(\frac{h}{k}\right)^{1/2}\left(\frac14-\frac{1}{\pi i}\sum_{a=1}^{k^*}\beta(a,k^*)\right.\\
&\qquad\qquad\qquad\left.\times\sum_{0<b<k^*/2}\eta(b,k^*)e\left(ab\frac{\overline{h^*}}{k^*}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{d\le 8^\varepsilon K} \sum_{K<k\le 8^\varepsilon K} \sum_{\substack{K<h\le 8^\varepsilon K\\(k,h)=d}} b(k)\overline{b(h)}\Big(\frac{h}{k}\Big)^{1/2}\Bigg(\frac14-\frac{1}{\pi i}\sum_{a=1}^{k/d}\beta\Big(a,\frac{k}{d}\Big)\\
&\qquad\qquad \times \sum_{0<b<k/(2d)} \eta\Big(b,\frac{k}{d}\Big)e\Big(ab\frac{\overline{(h/d)}}{k/d}\Big)\Bigg)\\
&= \sum_{d\le 8^\varepsilon K} \sum_{N<n_1\le 2^\varepsilon N} \sum_{N<n_2\le 2^\varepsilon N} \sum_{M<m_2\le 2^\varepsilon M} \sum_{\substack{M<m_1\le 2^\varepsilon M\\(m_1n_1^2,m_2n_2^2)=d}} \frac{a(n_2)}{m_2^{1/2}}\cdot\frac{\overline{a(n_1)}}{m_1^{1/2}}\\
&\qquad \times \Big(\frac{m_1n_1^2}{m_2n_2^2}\Big)^{1/2}\Bigg(\frac14-\frac{1}{\pi i}\sum_{a=1}^{m_2n_2^2/d}\beta\Big(a,\frac{m_2n_2^2}{d}\Big)\\
&\qquad\qquad \times \sum_{0<b<m_2n_2^2/(2d)} \eta\Big(b,\frac{m_2n_2^2}{d}\Big)e\Big(ab\frac{\overline{(m_1n_1^2/d)}}{m_2n_2^2/d}\Big)\Bigg)\\
&= \sum_{d\le 8^\varepsilon K} \sum_{N<n_1\le 2^\varepsilon N} \sum_{N<n_2\le 2^\varepsilon N} a(n_2)\overline{a(n_1)}\Big(\frac{n_1}{n_2}\Big) \sum_{M<m_2\le 2^\varepsilon M}\frac{1}{m_2}\\
&\qquad \times \sum_{\substack{M<m_1\le 2^\varepsilon M\\(m_1n_1^2,m_2n_2^2)=d}} \Bigg(\frac14-\frac{1}{\pi i}\sum_{a=1}^{m_2n_2^2/d}\beta\Big(a,\frac{m_2n_2^2}{d}\Big)\\
&\qquad\qquad \times \sum_{0<b<m_2n_2^2/(2d)} \eta\Big(b,\frac{m_2n_2^2}{d}\Big)e\Big(ab\frac{\overline{(m_1n_1^2/d)}}{m_2n_2^2/d}\Big)\Bigg)\\
&\ll \sum_{d\le 8^\varepsilon K} \sum_{N<n_1\le 2^\varepsilon N} \sum_{N<n_2\le 2^\varepsilon N} |a(n_1)a(n_2)| \sum_{M<m_2\le 2^\varepsilon M}\frac{1}{m_2} \sum_{\substack{M<m_1\le 2^\varepsilon M\\(m_1n_1^2,m_2n_2^2)=d}} 1\\
&\quad + \sum_{d\le 8^\varepsilon K} \sum_{N<n_1\le 2^\varepsilon N} \sum_{N<n_2\le 2^\varepsilon N} |a(n_1)a(n_2)| \sum_{\substack{M<m_2\le 2^\varepsilon M\\ d\mid m_2n_2^2}}\frac{1}{m_2}\\
&\cdot \sum_{a=1}^{m_2n_2^2/d}\Big|\beta\Big(a,\frac{m_2n_2^2}{d}\Big)\Big| \sum_{0<b<m_2n_2^2/(2d)} \eta\Big(b,\frac{m_2n_2^2}{d}\Big)\Big| \sum_{\substack{M<m_1\le 2^\varepsilon M\\(m_1n_1^2,m_2n_2^2)=d}} e\Big(ab\frac{\overline{(m_1n_1^2/d)}}{m_2n_2^2/d}\Big)\Big|\\
&\ll \sum_{N<n_1\le 2^\varepsilon N} \sum_{N<n_2\le 2^\varepsilon N} \sum_{M<m_1\le 2^\varepsilon M} \sum_{M<m_2\le 2^\varepsilon M} \frac{|a(n_1)a(n_2)|}{m_2}\\
&\quad + \sum_{d\le 8^\varepsilon K} \sum_{N<n_1\le 2^\varepsilon N} \sum_{N<n_2\le 2^\varepsilon N} |a(n_1)a(n_2)| \sum_{\substack{M<m_2\le 2^\varepsilon M\\ d\mid m_2n_2^2}}\frac{1}{m_2}
\end{aligned}
$$

$$\times \sum_{a=1}^{m_2n_2^2/d} \left|\beta\left(a, \frac{m_2n_2^2}{d}\right)\right| \sum_{0<b<m_2n_2^2/(2d)} \frac{1}{b}\left(ab, \frac{m_2n_2^2}{d}\right)^{1/2} \left(\frac{MN^2}{d}\right)^{1/2+\varepsilon} d^{2\varepsilon}$$

$$\ll N^{-2+2\varepsilon}M^{-1}(MN)^2 + N^{-2+2\varepsilon}M^{-1} \sum_{d\le 8^\varepsilon K} \sum_{N<n_1\le 2^\varepsilon N} \sum_{N<n_2\le 2^\varepsilon N}$$

$$\times \sum_{\substack{M<m_2\le 2^\varepsilon M\\ d|m_2n_2^2}} \left(\frac{MN^2}{d}\right)^{1/2+\varepsilon} d^{2\varepsilon} \sum_{a=1}^{m_2n_2^2/d} \left|\beta\left(a, \frac{m_2n_2^2}{d}\right)\right|\left(a, \frac{m_2n_2^2}{d}\right)^{1/2}$$

$$\times \sum_{0<b<m_2n_2^2/(2d)} \frac{1}{b}\left(b, \frac{m_2n_2^2}{d}\right)^{1/2}.$$

We have

$$\sum_{0<b<m_2n_2^2/(2d)} \frac{1}{b}\left(b, \frac{m_2n_2^2}{d}\right)^{1/2}$$

$$= \sum_{r|\frac{m_2n_2^2}{d}} r^{1/2} \sum_{\substack{0<b<m_2n_2^2/(2d)\\ (b,m_2n_2^2/d)=r}} \frac{1}{b} \le \sum_{r|\frac{m_2n_2^2}{d}} r^{1/2} \sum_{\substack{0<b\le m_2n_2^2/(2d)\\ r|b}} \frac{1}{b}$$

$$\ll \sum_{r|\frac{m_2n_2^2}{d}} \frac{1}{r^{1/2}} \log\left(\frac{2m_2n_2^2}{d}\right) \ll \left(\frac{m_2n_2^2}{d}\right)^{\varepsilon/4} \ll \left(\frac{MN^2}{d}\right)^{\varepsilon/4}$$

and

$$\sum_{a=1}^{m_2n_2^2/d} \left|\beta\left(a, \frac{m_2n_2^2}{d}\right)\right|\left(a, \frac{m_2n_2^2}{d}\right)^{1/2}$$

$$\ll \frac{m_2n_2^2}{d} + \sum_{1\le a\le m_2n_2^2/(2d)} \frac{m_2n_2^2/d}{a}\left(a, \frac{m_2n_2^2}{d}\right)^{1/2}$$

$$+ \sum_{m_2n_2^2/(2d)<a\le m_2n_2^2/d-1} \frac{m_2n_2^2/d}{m_2n_2^2/d-a}\left(\frac{m_2n_2^2}{d}-a, \frac{m_2n_2^2}{d}\right)^{1/2}$$

$$\ll \frac{m_2n_2^2}{d} + \frac{m_2n_2^2}{d} \sum_{1\le a\le m_2n_2^2/(2d)} \frac{1}{a}\left(a, \frac{m_2n_2^2}{d}\right)^{1/2} \ll \left(\frac{MN^2}{d}\right)^{1+\varepsilon/4}.$$

Therefore the contribution of $R_0(T;h,k)$ is

$$\ll MN^{2\varepsilon} + N^{-2+2\varepsilon}M^{-1}$$

$$\times \sum_{d\le 8^\varepsilon K} \sum_{N<n_1\le 2^\varepsilon N} \sum_{N<n_2\le 2^\varepsilon N} \sum_{M<m_2\le 2^\varepsilon M} \left(\frac{MN^2}{d}\right)^{1/2+\varepsilon} d^{2\varepsilon}\left(\frac{MN^2}{d}\right)^{1+\varepsilon/2}$$

$$\ll MN^{2\varepsilon} + N^{2\varepsilon}(MN^2)^{3/2+3\varepsilon/2} \sum_{d\le 8^\varepsilon K} \frac{1}{d^{3/2-\varepsilon/2}}$$
$$\ll MN^{2\varepsilon} + M^{3/2+3\varepsilon/2}N^{3+5\varepsilon} \ll \frac{T}{N^{1-8\varepsilon}}.$$

Combining all of the above, we get

$$\int_0^\infty e^{-t/T}|\zeta(1/2+it)|^2\Big|\sum_{K<k\le 8^\varepsilon K} \frac{b(k)}{k^{it}}\Big|^2 \ll \frac{T(\log T)^2}{N^{1-8\varepsilon}}.$$

Hence,

$$\log T \sum_{r\le \frac{1}{2\varepsilon\log 2}\log T+O(1)} \int_0^\infty e^{-t/T}|\zeta(1/2+it)|^2$$
$$\times \Big|\sum_{2^{\varepsilon r}<m\le 2^\varepsilon\cdot 2^{\varepsilon r}} \frac{1}{m^{1/2+it}}\Big|^2\Big|\sum_{N<n\le 2^\varepsilon N} \frac{a(n)}{n^{2it}}\Big|^2 dt \ll \frac{T(\log T)^4}{N^{1-8\varepsilon}}.$$

The proof of Proposition 3 is finished.

Proof of Proposition 2. We observe that the measure of the set of all t such that $T/2\le t\le T$ and $2t\in U_4$ is $\ll T^{11\varepsilon}(\log T)^{10}$.

We suppose firstly that $k=2m$ with a positive integer m. By Proposition 1 and Lemmas 8, 9 and 11,

$$\int_{T/2}^{T} |\zeta(1/2+it)|^4|\zeta(1+2it)|^{-2m}\,dt$$
$$= \int_{T/2\le t\le T,\,2t\notin U_4} |\zeta(1/2+it)|^4|\zeta(1+2it)|^{-2m}\,dt$$
$$+ \int_{T/2\le t\le T,\,2t\in U_4} |\zeta(1/2+it)|^4|\zeta(1+2it)|^{-2m}\,dt$$
$$\ll \int_{T/2\le t\le T,\,2t\notin U_4} |\zeta(1/2+it)|^4\bigg(\Big|\sum_{l\le X}\frac{\mu(l)}{l^{1+2it}}\cdot e^{-l/X}\Big|^{2m} + O(1)\bigg)\,dt$$
$$+ O(T(\log T)^4)$$
$$\ll \int_{T/2}^{T} |\zeta(1/2+it)|^4\Big|\sum_{l\le X}\frac{\mu(l)}{l^{1+2it}}\cdot e^{-l/X}\Big|^{2m} dt + O(T(\log T)^4)$$
$$= \int_{T/2}^{T} |\zeta(1/2+it)|^4\Big|\sum_{n\le X^m}\frac{a(n)}{n^{2it}}\Big|^2 dt + O(T(\log T)^4),$$

where

$$a(n) = \frac{1}{n} \sum_{l_1 \cdots l_m = n} \mu(l_1) \cdots \mu(l_m) \exp\left(-\frac{l_1 + \cdots + l_m}{X}\right),$$

U_4 is defined as in (3.2), and X is defined as in (3.8). We can see that

$$X^m = \exp\left(\frac{2m}{\varepsilon}(\log T)^{3/4}\right) \ll T^{1/16-2\varepsilon} \quad \text{and} \quad a(n) = O(n^{-1+\varepsilon}).$$

By Cauchy's inequality,

$$\begin{aligned} \left|\sum_{n \le X^m} \frac{a(n)}{n^{2it}}\right|^2 &= \left|\sum_{s \le \frac{m \log X}{\varepsilon \log 2} - 1} \frac{1}{2^{\varepsilon s/4}} \cdot 2^{\varepsilon s/4} \sum_{2^{\varepsilon s} < n \le 2^{\varepsilon} 2^{\varepsilon s}} \frac{a(n)}{n^{2it}}\right|^2 \\ &\le \sum_{s \le \frac{m \log X}{\varepsilon \log 2} - 1} \frac{1}{2^{\varepsilon s/2}} \sum_{s \le \frac{m \log X}{\varepsilon \log 2} - 1} 2^{\varepsilon s/2} \left|\sum_{2^{\varepsilon s} < n \le 2^{\varepsilon} 2^{\varepsilon s}} \frac{a(n)}{n^{2it}}\right|^2 \\ &\ll \sum_{s \le \frac{m \log X}{\varepsilon \log 2} - 1} 2^{\varepsilon s/2} \left|\sum_{2^{\varepsilon s} < n \le 2^{\varepsilon} 2^{\varepsilon s}} \frac{a(n)}{n^{2it}}\right|^2. \end{aligned}$$

Hence, Proposition 3 yields

$$\begin{aligned} &\int_{T/2}^{T} |\zeta(1/2 + it)|^4 \left|\sum_{n \le X^m} \frac{a(n)}{n^{2it}}\right|^2 dt \\ &\qquad \ll \sum_{s \le \frac{m \log X}{\varepsilon \log 2} - 1} 2^{\varepsilon s/2} \int_{T/2}^{T} |\zeta(1/2 + it)|^4 \left|\sum_{2^{\varepsilon s} < n \le 2^{\varepsilon} 2^{\varepsilon s}} \frac{a(n)}{n^{2it}}\right|^2 \\ &\qquad \ll \sum_{s \le \frac{m \log X}{\varepsilon \log 2} - 1} 2^{\varepsilon s/2} \cdot \frac{T(\log T)^4}{2^{\varepsilon s(1-8\varepsilon)}} \ll T(\log T)^4. \end{aligned}$$

Thus,

$$\int_{T/2}^{T} |\zeta(1/2 + it)|^4 |\zeta(1 + 2it)|^{-2m} \, dt \ll T(\log T)^4.$$

Therefore

$$\int_{1}^{T} |\zeta(1/2 + it)|^4 |\zeta(1 + 2it)|^{-2m} \, dt \ll T(\log T)^4.$$

For general $k > 0$, we have an even integer $2m$ such that $k < 2m$. By Hölder's inequality,

$$\begin{aligned}
&\int_1^T |\zeta(1/2+it)|^4 |\zeta(1+2it)|^{-k}\,dt \\
&\quad = \int_1^T |\zeta(1/2+it)|^{4\cdot\frac{2m-k}{2m}} \cdot |\zeta(1/2+it)|^{4\cdot\frac{k}{2m}} |\zeta(1+2it)|^{-k}\,dt \\
&\quad \le \Big(\int_1^T |\zeta(1/2+it)|^{4\cdot\frac{2m-k}{2m}\cdot\frac{2m}{2m-k}}\,dt\Big)^{\frac{2m-k}{2m}} \\
&\qquad \times \Big(\int_1^T |\zeta(1/2+it)|^{4\cdot\frac{k}{2m}\cdot\frac{2m}{k}} |\zeta(1+2it)|^{-k\cdot\frac{2m}{k}}\,dt\Big)^{\frac{k}{2m}} \\
&\quad = \Big(\int_1^T |\zeta(1/2+it)|^4\,dt\Big)^{\frac{2m-k}{2m}} \Big(\int_1^T |\zeta(1/2+it)|^4 |\zeta(1+2it)|^{-2m}\,dt\Big)^{\frac{k}{2m}} \\
&\quad \ll T(\log T)^4.
\end{aligned}$$

The proof of Proposition 2 is finished.

5. Proof of the Theorem. We shall apply Lemma 5. For $\mathrm{Re}(s) > 1$, let

$$f(s) = \sum_{n=1}^{\infty} \frac{d^2(n)}{n^s}.$$

By Lemma 6,

$$f(s) = \frac{\zeta^4(s)}{\zeta(2s)}.$$

It is easy to see that $\psi(n) = n^{\varepsilon}$, which is non-decreasing. As $\sigma \to 1^+$,

$$\sum_{n=1}^{\infty} \frac{d^2(n)}{n^{\sigma}} = \frac{\zeta^4(\sigma)}{\zeta(2\sigma)} = O\Big(\frac{1}{(\sigma-1)^4}\Big).$$

Let $c = 1+\varepsilon, Y = [x]+1/2$, $T = x^{3/4}$. Then

$$\begin{aligned}
\sum_{n\le x} d^2(n) &= \sum_{n<Y} d^2(n) + O_{\varepsilon}(x^{\varepsilon}) \\
&= \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \frac{\zeta^4(s)}{\zeta(2s)} \cdot \frac{Y^s}{s}\,ds + O_{\varepsilon}(x^{1/4+2\varepsilon}).
\end{aligned}$$

We move the line of integration to $\mathrm{Re}(s) = 1/2$. The residue of $\frac{\zeta^4(s)}{\zeta(2s)} \cdot \frac{Y^s}{s}$ at $s = 1$ is

$$YP(\log Y) = xP(\log x) + O(x^{\varepsilon}).$$

By Lemmas 8 and 10,

$$\frac{1}{2\pi i}\int\limits_{1/2+iT}^{1+\varepsilon+iT}\frac{\zeta^4(s)}{\zeta(2s)}\cdot\frac{Y^s}{s}\,ds \ll \max_{1/2\le\sigma\le 1+\varepsilon} T^{\frac{4}{3}(1-\sigma)+4\varepsilon}\log T\cdot\frac{x^\sigma}{T} \ll x^{1/4+4\varepsilon}.$$

In the same way,

$$\frac{1}{2\pi i}\int\limits_{1/2-iT}^{1+\varepsilon-iT}\frac{\zeta^4(s)}{\zeta(2s)}\cdot\frac{Y^s}{s}\,ds \ll x^{1/4+4\varepsilon}.$$

Hence,

$$E(x)=\frac{1}{2\pi i}\int\limits_{1/2-iT}^{1/2+iT}\frac{\zeta^4(s)}{\zeta(2s)}\cdot\frac{Y^s}{s}\,ds+O(x^{1/4+4\varepsilon}).$$

It follows from Proposition 2 that

$$\begin{aligned}E(x) &\ll x^{1/2}\sum_{k\le\frac{\log T}{\log 2}}\int\limits_{2^{k-1}}^{2^k}\frac{|\zeta(1/2+it)|^4}{|\zeta(1+2it)|}\,\frac{dt}{t}+O(x^{1/2})\\ &\ll x^{1/2}\sum_{k\le\frac{\log T}{\log 2}}\frac{1}{2^k}\int\limits_{1}^{2^k}\frac{|\zeta(1/2+it)|^4}{|\zeta(1+2it)|}\,dt+O(x^{1/2})\\ &\ll x^{1/2}\sum_{k\le\frac{\log T}{\log 2}}\frac{1}{2^k}\cdot 2^k k^4 \ll x^{1/2}(\log x)^5.\end{aligned}$$

Thus, the proof of the Theorem is complete.

6. Some remarks. By the method of this paper, we can prove that if k is any given positive number, and a is a given positive integer, then

$$\int\limits_1^T\frac{|\zeta(1/2+it)|^4}{|\zeta(1+ait)|^k}\,dt \ll_{k,a} T(\log T)^4.$$

We note that if $\mathrm{Re}(s)>1$, then

$$\sum_{n=1}^{\infty}\frac{d(n^3)}{n^s}=\frac{\zeta^4(s)}{\zeta^3(2s)}\cdot G_1(s), \tag{6.1}$$

where

$$G_1(s)=\prod_p\frac{1+2/p^s}{(1-1/p^s)(1+1/p^s)^3},$$

$G_1(s)$ is absolutely convergent for $\mathrm{Re}(s)>1/3$ (see [2, p. 95]). Using the method similar to that in this paper, we can prove the following proposition.

PROPOSITION 4. *We have*

$$\sum_{n\le x} d(n^3) = xP_1(\log x) + O(x^{1/2}(\log x)^5),$$

where $P_1(y)$ is a suitable cubic polynomial in y.

In 2006, M. Z. Garaev, F. Luca and W. G. Nowak [2] proved that as $x\to\infty$, if $y = y(x)$ satisfies

$$\frac{y}{x^{1/2}\log x\log\log x}\to\infty,$$

then

$$\sum_{x<n\le x+y} d^2(n) \sim \frac{6}{\pi^2} y(\log x)^3,$$

and that as $x\to\infty$, if $y = y(x)$ satisfies

$$\frac{y}{x^{1/2}\log x(\log\log x)^3}\to\infty,$$

then

$$\sum_{x<n\le x+y} d(n^3) \sim B_0 y(\log x)^3,$$

where B_0 is a positive constant.

Combining the method of this paper with that of [2], we can prove the following proposition.

PROPOSITION 5. *As $x\to\infty$, if $y = y(x)$ satisfies*

$$\frac{y}{x^{1/2}\log x}\to\infty,$$

then

$$\sum_{x<n\le x+y} d^2(n) \sim \frac{6}{\pi^2} y(\log x)^3 \quad \textit{and} \quad \sum_{x<n\le x+y} d(n^3) \sim B_0 y(\log x)^3.$$

Let $r(n)$ be the number of representations of n as the sum of two squares. In 2004, M. Kühleitner and W. G. Nowak [6] proved that

$$\sum_{n\le x} r^2(n) = 4x\log x + B_1 x + O(x^{1/2}(\log x)^3\log\log x), \tag{6.2}$$

where B_1 is a positive constant, and that

$$\sum_{n\le x} r(n^3) = A_2 x\log x + B_2 x + O(x^{1/2}(\log x)^3(\log\log x)^2), \tag{6.3}$$

where A_2, B_2 are positive constants.

Let $\mathbf{K} = \mathbb{Q}(i)$, and let $\zeta_{\mathbf{K}}(s)$ be the Dedekind ζ function in the field $\mathbf{K}$. If $\mathrm{Re}(s) > 1$, then

$$\sum_{n=1}^{\infty} \frac{r^2(n)}{n^s} = \frac{16\zeta_{\mathbf{K}}^2(s)}{(1+2^{-s})\zeta(2s)}, \qquad \sum_{n=1}^{\infty} \frac{r(n^3)}{n^s} = \frac{\zeta_{\mathbf{K}}^2(s)}{\zeta(2s)\zeta_{\mathbf{K}}(2s)} \cdot G_2(s),$$

where $G_2(s)$ is holomorphic and bounded for $\mathrm{Re}(s) > 1/3 + \varepsilon$ (see [6, (4.1) and (4.4)]).

If the result of Iwaniec [4] could be generalized to $\zeta_{\mathbf{K}}(s)$, then the error terms in (6.2) and (6.3) could be improved to $O(x^{1/2}(\log x)^3)$. Furthermore, the sums studied in [2]:

$$\sum_{x<n\le x+y} r^2(n), \qquad \sum_{x<n\le x+y} r(n^3), \qquad \sum_{x<n\le x+y} d(n)r(n)$$

could also be improved correspondingly.

Acknowledgements. The authors would like to thank the referee for his/her nice comments and suggestions. Chaohua Jia is supported by the National Natural Science Foundation of China (Grant No. 11371344) and the National Key Basic Research Program of China (Project No. 2013CB834202).

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Received on 18.11.2013
and in revised form on 23.3.2014 (7643)