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A remark on Chen's theorem

by

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1. Introduction. Let p, p' denote primes and P_2 denote an almost prime with at most two prime factors. For sufficiently large x it is conjectured by Hardy and Littlewood [8] that

$$\sum_{\substack{p \le x \\ p+2=p'}} 1 = (1+o(1))\frac{Cx}{\log^2 x},$$

where

$$C = 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right).$$

This conjecture still remains open. The best result in this respect is due to Chen Jingrun [1] who showed in 1973 that

$$\pi_{1,2}(x) > \frac{0.335Cx}{\log^2 x},$$

where

$$\pi_{1,2}(x) = \sum_{\substack{p \le x \\ p+2=P_2}} 1.$$

The constant 0.335 was improved successively to $0.3445, 0.3772, 0.405, 0.71, 1.015, 1.05$ by Halberstam [7], Chen Jingrun [2, 3], Fouvry and Grupp [4], H. Q. Liu [10] and J. Wu [12] respectively.

In this paper we obtain the following result.

THEOREM.
$$\pi_{1,2}(x) > \frac{1.0974Cx}{\log^2 x}.$$

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2. Some lemmas. We denote by $\tau_k(n)$ the usual divisor function for $k \geq 2$ and $\tau_2(n) = \tau(n)$. For the definition of well-factorable functions we refer the reader to [4]. In the following we denote by $\lambda(q)$ a well-factorable function of level Q and of order k.

LEMMA 1 [4]. *For an arithmetical function λ' of level Q' and of order k', $Q' \leq Q$, $\lambda * \lambda'$ is a well-factorable function of level QQ' and of order $k + k'$.*

Let $\mathcal{A}$ denote a finite set of integers, $\mathcal{P}$ an infinite set of primes and $\overline{\mathcal{P}}$ the set of primes that do not belong to $\mathcal{P}$. Let $z \geq 2$. Put

$$P(z) = \prod_{p<z,\, p\in\mathcal{P}} p, \qquad \mathcal{P}(q) = \{p \mid p \in \mathcal{P},\ (p,q) = 1\},$$

$$S(\mathcal{A}, z) = S(\mathcal{A}; \mathcal{P}, z) = \sum_{a\in\mathcal{A},\, (a,P(z))=1} 1, \qquad \mathcal{A}_d = \{a \mid a \in \mathcal{A},\ a \equiv 0 \pmod d\}.$$

LEMMA 2 [9]. *Let*

$$|\mathcal{A}_d| = \frac{\omega(d)}{d}X + r_d, \qquad \mu(d) \neq 0, \qquad (d, \overline{\mathcal{P}}) = 1,$$

$$\frac{V(z_1)}{V(z_2)} \leq \frac{\log z_2}{\log z_1}\left(1 + \frac{K_1}{\log z_1}\right), \qquad K_1 > 1,\ z_2 > z_1 \geq 2,$$

$$\sum_{\substack{z_1 \leq p < z_2 \\ p\in\overline{P}}} \sum_{\alpha\geq 2} \frac{\omega(p^\alpha)}{p^\alpha} \leq \frac{K_2}{\log 3z_1}, \qquad K_2 > 1,$$

where $\omega(d)$ is a multiplicative function, $0 \leq \omega(p) < p$, $X > 1$ is independent of d, and

$$V(z) = \prod_{p|P(z)} \left(1 - \frac{\omega(p)}{p}\right).$$

Then for $0 < \varepsilon < 10^{-5}, 2 \leq z \leq Q^{1/2}$, we have

$$S(\mathcal{A}, \mathcal{P}, z) \geq XV(z)(f(s) - E) - \sum_{l<L} \sum_{q} \lambda_l^-(q) r(q),$$

$$S(\mathcal{A}, \mathcal{P}, z) \leq XV(z)(F(s) + E) + \sum_{l<L} \sum_{q} \lambda_l^+(q) r(q),$$

where $\lambda_l^\pm$ are well-factorable functions of level Q and

$$L = \exp(8\varepsilon^{-3}), \qquad E \ll \varepsilon + \varepsilon^{-8} \exp(K_1 + K_2) \log^{-1/3} Q,$$

$$s = \frac{\log Q}{\log z}, \quad |\lambda_l^{\pm}(q)| \le 1, \quad \lambda_l^{\pm}(q) = 0 \quad \text{for } (q, P(z)) = 1.$$

$f(s)$ and $F(s)$ are determined by the following differential-difference equation:

$$\begin{cases} F(s) = 2e^{\gamma}/s, & f(s) = 0, \quad 0 < s \le 2, \\ (sF(s))' = f(s-1), & (sf(s))' = F(s-1), \quad s \ge 2. \end{cases}$$

Here and below γ is Euler's constant.

LEMMA 3 [7]. *We have*

$$F(s) = \begin{cases} \dfrac{2e^{\gamma}}{s}, & 0 < s \le 3, \\ \dfrac{2e^{\gamma}}{s}\left(1 + \displaystyle\int_2^{s-1} \frac{\log(t-1)}{t}\, dt\right), & 3 \le s \le 5; \end{cases}$$

$$f(s) = \begin{cases} \dfrac{2e^{\gamma}\log(s-1)}{s}, & 2 \le s \le 4, \\ \dfrac{2e^{\gamma}}{s}\left(\log(s-1) + \displaystyle\int_3^{s-1} \frac{dt}{t} \int_2^{t-1} \frac{\log(u-1)}{u}\, du\right), & 4 \le s \le 6. \end{cases}$$

LEMMA 4 [4]. *Let $Q = x^{4/7-\varepsilon}$ and $\varepsilon > 0$. For any given $A > 0$ and $|a| \le \log^A x$,*

$$\sum_{(q,a)=1} \lambda(q)\left(\pi(x;q,a) - \frac{\operatorname{Li} x}{\varphi(q)}\right) = O_{A,\varepsilon,a}\left(\frac{x}{\log^A x}\right).$$

LEMMA 5 [4]. *Let (α_m) and (β_n) be two sequences satisfying the following conditions*:

- $M \ge x^{\varepsilon}$, $\alpha_m = 0$ *for* $m \notin [M, 2M]$, $|\alpha_m| \le \tau_k(m)$;
- $N \ge x^{\varepsilon}$, $\beta_n = 0$ *for* $n \notin [N, 2N]$, $|\beta_n| \le \tau_k(n)$;
- *for any given* $e \ge 1$, $d \ge 1$, $(d,l) = 1$, $A > 0$,

$$\sum_{\substack{n \equiv l\,(d) \\ (n,e)=1}} \beta_n = \frac{1}{\varphi(d)} \sum_{(n,de)=1} \beta_n + O\left(\frac{N\tau(e)^B}{\log^A N}\right);$$

- *if* $p \mid n \to p < \exp(\log^{1/2} x)$ *then* $\beta_n = 0$,

where k and B are constants. Let $MN \le x, v = \log N/\log x$, $Q = x^{\theta(v)-2\varepsilon}$ where $\theta(v)$ is defined by

$$\theta(v)=\begin{cases}(1+v)/2, & 0\le v\le 1/10,\\ (13+2v)/24, & 1/10\le v\le 1/6,\\ (3+2v)/6, & 1/6\le v\le 1/4,\\ (2-v)/3, & 1/4\le v\le 2/7,\\ (2+v)/4, & 2/7\le v\le 2/5,\\ 1-v, & 2/5\le v\le 1/2,\\ 1/2, & 1/2\le v\le 1.\end{cases}$$

Then for any $A>0$ and $|a|\le \log^A x$,

$$\sum_{(q,a)=1}\lambda(q)\biggl(\sum_{mn\equiv a\,(q)}\alpha_m\beta_n-\frac{1}{\varphi(q)}\sum_{(mn,q)=1}\alpha_m\beta_n\biggr)=O_{A,\varepsilon,k,B}\biggl(\frac{x}{\log^A x}\biggr).$$

Lemma 6 [5]. *Let $\xi(\cdot)$ denote an arithmetical function such that*

$$|\xi(q)|\le\log x,\qquad \xi(q)=0\qquad \textit{for } q>Q_1.$$

Then

$$\sum_{(qq_1,a)=1}\lambda(q)\xi(q_1)\biggl(\pi(x;qq_1,a)-\frac{\operatorname{Li}x}{\varphi(qq_1)}\biggr)=O_{A,\varepsilon,a}\biggl(\frac{x}{\log^A x}\biggr)$$

if either

$$\begin{aligned}&Q_1\le Q,\qquad Q_1Q\le x^{4/7-\varepsilon},\qquad \textit{or}\\ &Q_1\ge Q,\qquad Q_1^6Q\le x^{2-\varepsilon},\qquad \textit{or}\\ &\xi(q)=\Lambda(q),\qquad Q_1Q\le x^{11/20-\varepsilon},\qquad Q_1\le x^{1/3-\varepsilon}.\end{aligned}$$

Lemma 7 [5]. *Let $\eta>0$ and define*

$$g(t)=\begin{cases}4/7, & 0\le t\le 2/7-\eta,\\ 11/20, & 2/7-\eta\le t\le 1/3-\eta,\\ 1/2, & 1/3-\eta\le t\le 1/2-\eta.\end{cases}$$

Then for any $A>0,\varepsilon>0$ and $|a|\le\log^A x$,

$$\sum_{x^t\le p<2x^t}\sum_{(q,a)=1}\lambda(q)\biggl(\pi(x;pq,a)-\frac{\operatorname{Li}x}{\varphi(pq)}\biggr)=O_{A,k,a}\biggl(\frac{x}{\log^A x}\biggr),$$

where $Q=x^{g(t)-t-\varepsilon}$.

Lemma 8 [11]. *Let*

$$x>1,\qquad z=x^{1/u},\qquad Q(z)=\prod_{p<z}p.$$

Then for $u \geq u_0 > 1$, *we have*

$$\sum_{\substack{n \leq x \\ (n,Q(z))=1}} 1 = w(u)\frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right),$$

where $w(u)$ *is determined by the following differential-difference equation*:

$$\begin{cases} w(u) = 1/u, & 1 \leq u \leq 2, \\ (uw(u))' = w(u-1), & u \geq 2. \end{cases}$$

Moreover,

$$w(u) < \frac{1}{1.763} \quad \text{for } u \geq 2.$$

3. Weighted sieve method. Let x be a sufficiently large real number and put

$$\mathcal{A} = \{a \mid a = p+2,\ p \leq x\}, \tag{3.1}$$

$$\mathcal{P} = \{p \mid (p,2) = 1\}. \tag{3.2}$$

LEMMA 9 [3]. *Let* $0 < \alpha < \beta < 1/3$ *and* $\alpha + 3\beta > 1$. *Then*

$$\begin{aligned} \pi_{1,2}(x) &\geq \sum_{\substack{a\in\mathcal{A} \\ (a,2P(x^\alpha))=1}} \left(1 - \frac{1}{2}\varrho_1(a) - \frac{1}{2}\varrho_2(a) - \varrho_3(a) + \frac{1}{2}\varrho_4(a)\right) + O(x^{1-\alpha}) \\ &\geq S(\mathcal{A}, x^\alpha) - \frac{1}{2}\sum_{x^\alpha \leq p < x^\beta} S(\mathcal{A}_p, x^\alpha) \\ &\quad - \frac{1}{2}\sum_{x^\alpha \leq p_1 < x^\beta \leq p_2 < (x/p_1)^{1/2}} S(\mathcal{A}_{p_1p_2}, p_2) - \sum_{x^\beta \leq p_1 < p_2 < (x/p_1)^{1/2}} S(\mathcal{A}_{p_1p_2}, p_2) \\ &\quad + \frac{1}{2}\sum_{x^\alpha \leq p_1 < p_2 < p_3 < x^\beta} S(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(p_1), p_2) + O(x^{1-\alpha}), \end{aligned}$$

where

$$\varrho_1(a) = \sum_{\substack{p \mid a \\ x^\alpha \leq p < x^\beta}} 1;$$

$$\varrho_2(a) = \begin{cases} 1, & a = p_1p_2p_3,\ x^\alpha \leq p_1 < x^\beta \leq p_2 < p_3, \\ 0, & \text{otherwise}; \end{cases}$$

$$\varrho_3(a) = \begin{cases} 1, & a = p_1p_2p_3,\ x^\beta \leq p_1 < p_2 < p_3, \\ 0, & \text{otherwise}; \end{cases}$$

$$\varrho_4(a) = \begin{cases} 1, & a = p_1p_2p_3n,\ x^\alpha \leq p_1 < p_2 < p_3 < x^\beta,\ (a, 2p_1^{-1}P(p_2)) = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. Since the second inequality can be deduced from the first one easily, it suffices to prove the first inequality. Let

$$v_2(a) = \sum_{p^m | a} m, \qquad \lambda(a) = \begin{cases} 1, & v_2(a) \le 2, \\ 0, & v_2(a) > 2. \end{cases}$$

Then

$$\pi_{1,2}(x) \ge \sum_{\substack{a \in \mathcal{A} \\ (a, P(x^\alpha)) = 1}} \lambda(a) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(x^\alpha)) = 1}} \mu^2(a)\lambda(a) + O(x^{1-\alpha}).$$

On the other hand,

$$\sum_{\substack{a \in \mathcal{A} \\ (a, P(x^\alpha)) = 1}} \left(1 - \frac{1}{2}\varrho_1(a) - \frac{1}{2}\varrho_2(a) - \varrho_3(a) + \frac{1}{2}\varrho_4(a)\right)$$
$$= \sum_{\substack{a \in \mathcal{A} \\ (a, P(x^\alpha)) = 1}} \mu^2(a)\left(1 - \frac{1}{2}\varrho_1(a) - \frac{1}{2}\varrho_2(a) - \varrho_3(a) + \frac{1}{2}\varrho_4(a)\right) + O(x^{1-\alpha}).$$

For

$$\mu^2(a) = 1, \qquad (a, P(x^\alpha)) = 1,$$

we have three cases:

1) $v_2(a) \le 2$. Then $\varrho_4(a) = 0$, and

$$\lambda(a) = 1 \ge 1 - \frac{1}{2}\varrho_1(a) - \frac{1}{2}\varrho_2(a) - \varrho_3(a) + \frac{1}{2}\varrho_4(a).$$

2) $v_2(a) = 3$. If $\varrho_1(a) = 0$, then $\varrho_3(a) = 1, \varrho_2(a) = \varrho_4(a) = 0$, and

$$\lambda(a) = 0 = 1 - \frac{1}{2}\varrho_1(a) - \frac{1}{2}\varrho_2(a) - \varrho_3(a) + \frac{1}{2}\varrho_4(a).$$

If $\varrho_1(a) = 1$, then $\varrho_3(a) = \varrho_4(a) = 0$, and $\varrho_2(a) = 1$, hence

$$\lambda(a) = 0 = 1 - \frac{1}{2}\varrho_1(a) - \frac{1}{2}\varrho_2(a) - \varrho_3(a) + \frac{1}{2}\varrho_4(a).$$

If $\varrho_1(a) = 2$, then $\varrho_2(a) = \varrho_3(a) = \varrho_4(a) = 0$, and

$$\lambda(a) = 0 = 1 - \frac{1}{2}\varrho_1(a) - \frac{1}{2}\varrho_2(a) - \varrho_3(a) + \frac{1}{2}\varrho_4(a).$$

If $\varrho_1(a) = 3$, then $\varrho_2(a) = \varrho_3(a) = 0, \varrho_4(a) = 1$, and

$$\lambda(a) = 0 = 1 - \frac{1}{2}\varrho_1(a) - \frac{1}{2}\varrho_2(a) - \varrho_3(a) + \frac{1}{2}\varrho_4(a).$$

3) $v_2(a) \ge 4$. Then $\varrho_1(a) \ge 2$. If $\varrho_1(a) = 2$, then $\varrho_2(a) = \varrho_3(a) = \varrho_4(a) = 0$, and

$$\lambda(a) = 0 = 1 - \frac{1}{2}\varrho_1(a) - \frac{1}{2}\varrho_2(a) - \varrho_3(a) + \frac{1}{2}\varrho_4(a).$$

If $\varrho_1(a) \ge 3$, then $\varrho_2(a) = \varrho_3(a) = 0, \varrho_4(a) = 1$, and

$$\lambda(a) = 0 \ge 1 - \frac{1}{2}\varrho_1(a) - \frac{1}{2}\varrho_2(a) - \varrho_3(a) + \frac{1}{2}\varrho_4(a).$$

Combining the above arguments we complete the proof of Lemma 9.

LEMMA 10. *We have*

$$\begin{aligned}
2\pi_{1,2}(x) \ge 2S(\mathcal{A}, x^{1/10.5}) &- \frac{1}{2}\sum_{x^{1/10.5} \le p < x^{1/3.0015}} S(\mathcal{A}_p, x^{1/10.5}) \\
&- \frac{1}{2}\sum_{x^{1/10.5} \le p_1 < x^{1/3.0015} \le p_2 < (x/p_1)^{1/2}} S(\mathcal{A}_{p_1p_2}, p_2) \\
&- \sum_{x^{1/3.0015} \le p_1 < p_2 < (x/p_1)^{1/2}} S(\mathcal{A}_{p_1p_2}, p_2) \\
&- \frac{1}{2}\sum_{x^{1/10.5} \le p < x^{1/7.68}} S(\mathcal{A}_p, p) - \frac{1}{2}\sum_{x^{1/10.5} \le p < x^{1/3.449}} S(\mathcal{A}_p, x^{1/10.5}) \\
&+ \frac{1}{2}\sum_{x^{1/10.5} \le p_1 < p_2 < x^{1/7.68}} S(\mathcal{A}_{p_1p_2}, x^{1/10.5}) \\
&+ \frac{1}{2}\sum_{x^{1/10.5} \le p_1 < x^{1/7.68} \le p_2 < x^{8/21}p_1^{-1}} S(\mathcal{A}_{p_1p_2}, x^{1/10.5}) \\
&- \frac{1}{2}\sum_{x^{1/7.68} \le p_1 < x^{1/3.449} \le p_2 < (x/p_1)^{1/2}} S(\mathcal{A}_{p_1p_2}, p_2) \\
&- \sum_{x^{1/3.449} \le p_1 < p_2 < (x/p_1)^{1/2}} S(\mathcal{A}_{p_1p_2}, p_2) \\
&- \frac{1}{2}\sum_{x^{1/10.5} \le p_1 < p_2 < p_3 < p_4 < x^{1/7.68}} S(\mathcal{A}_{p_1p_2p_3p_4}; \mathcal{P}(p_1), p_2) \\
&- \frac{1}{2}\sum_{x^{1/10.5} \le p_1 < p_2 < p_3 < x^{1/7.68} \le p_4 < x^{8/21}p_3^{-1}} S(\mathcal{A}_{p_1p_2p_3p_4}; \mathcal{P}(p_1), p_2) \\
&+ O(x^{9.5/10.5}) \\
= 2\Sigma - \frac{1}{2}\Sigma_1 - \frac{1}{2}\Sigma_2 - \Sigma_3 &- \frac{1}{2}\Sigma_4 - \frac{1}{2}\Sigma_5 + \frac{1}{2}\Sigma_6 + \frac{1}{2}\Sigma_7 \\
&- \frac{1}{2}\Sigma_8 - \Sigma_9 - \frac{1}{2}\Sigma_{10} - \frac{1}{2}\Sigma_{11} + O(x^{9.5/10.5}).
\end{aligned}$$

Proof. By Buchstab's identity

$$S(\mathcal{A}, z_2) = S(\mathcal{A}, z_1) - \sum_{z_1 \le p < z_2} S(\mathcal{A}_p, p), \qquad 2 \le z_1 \le z_2,$$

we have

$$S(\mathcal{A},x^{1/7.68})=S(\mathcal{A},x^{1/10.5})-\frac{1}{2}\sum_{x^{1/10.5}\le p<x^{1/7.68}}S(\mathcal{A}_p,p)-\frac{1}{2}\sum_{x^{1/10.5}\le p<x^{1/7.68}}S(\mathcal{A}_p,x^{1/10.5})+\frac{1}{2}\sum_{x^{1/10.5}\le p_1<p_2<x^{1/7.68}}S(\mathcal{A}_{p_1p_2},x^{1/10.5})-\frac{1}{2}\sum_{x^{1/10.5}\le p_1<p_2<p_3<x^{1/7.68}}S(\mathcal{A}_{p_1p_2p_3},p_1),\tag{3.3}$$

$$\begin{aligned}&\sum_{x^{1/7.68}\le p<x^{1/3.449}}S(\mathcal{A}_p,x^{1/7.68})\\
&\le\sum_{x^{1/7.68}\le p<x^{2/7}}S(\mathcal{A}_p,x^{1/10.5})\\
&\quad-\sum_{x^{1/10.5}\le p_1<x^{1/7.68}\le p_2<x^{2/7}}S(\mathcal{A}_{p_1p_2},p_1)+\sum_{x^{2/7}\le p<x^{1/3.449}}S(\mathcal{A}_p,x^{1/10.5})\\
&\le\sum_{x^{1/7.68}\le p<x^{1/3.449}}S(\mathcal{A}_p,x^{1/10.5})-\sum_{x^{1/10.5}\le p_1<x^{1/7.68}\le p_2<x^{8/21}p_1^{-1}}S(\mathcal{A}_{p_1p_2},p_1)\\
&=\sum_{x^{1/7.68}\le p<x^{1/3.449}}S(\mathcal{A}_p,x^{1/10.5})\\
&\quad-\sum_{x^{1/10.5}\le p_1<x^{1/7.68}\le p_2<x^{8/21}p_1^{-1}}S(\mathcal{A}_{p_1p_2},x^{1/10.5})\\
&\quad+\sum_{x^{1/10.5}\le p_1<p_2<x^{1/7.68}\le p_3<x^{8/21}p_2^{-1}}S(\mathcal{A}_{p_1p_2p_3},p_1),\end{aligned}\tag{3.4}$$

$$\begin{aligned}&\sum_{x^{1/10.5}\le p_1<p_2<p_3<x^{1/3.001}}S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(p_1),p_2)\\
&\qquad-\sum_{x^{1/10.5}\le p_1<p_2<p_3<x^{1/7.68}}S(\mathcal{A}_{p_1p_2p_3},p_1)\\
&\qquad-\sum_{x^{1/10.5}\le p_1<p_2<x^{1/7.68}\le p_3<x^{8/21}p_2^{-1}}S(\mathcal{A}_{p_1p_2p_3},p_1)\\
&\ge\sum_{x^{1/10.5}\le p_1<p_2<p_3<x^{1/7.68}}\left(S(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(p_1),p_2)-S(\mathcal{A}_{p_1p_2p_3},p_1)\right)\end{aligned}\tag{3.5}$$

$$+ \sum_{x^{1/10.5} \le p_1 < p_2 < x^{1/7.68} \le p_3 < x^{8/21} p_2^{-1}} (S(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(p_1), p_2) - S(\mathcal{A}_{p_1p_2p_3}, p_1))$$

$$\ge - \sum_{x^{1/10.5} \le p_1 < p_2 < p_3 < p_4 < x^{1/7.68}} S(\mathcal{A}_{p_1p_2p_3p_4}; \mathcal{P}(p_1), p_2)$$

$$- \sum_{x^{1/10.5} \le p_1 < p_2 < p_3 < x^{1/7.68} \le p_4 < x^{8/21} p_3^{-1}} S(\mathcal{A}_{p_1p_2p_3p_4}; \mathcal{P}(p_1), p_2).$$

By Lemma 9 with $(\alpha,\beta) = (1/10.5, 1/3.0015)$ and $(\alpha,\beta) = (1/7.68, 1/3.449)$ and (3.3)–(3.5), we complete the proof of Lemma 10.

4. Proof of the Theorem. In this section, the sets $\mathcal{A}$ and $\mathcal{P}$ are defined by (3.1) and (3.2) respectively. Let

$$X = \operatorname{Li} x \sim \frac{x}{\log x}.$$

For $(d,2) = 1$,

$$r_d = \pi(x; d, -2) - \frac{\operatorname{Li} x}{\varphi(d)}, \qquad \omega(d) = \frac{d}{\varphi(d)}, \qquad \mu(d) \ne 0.$$

1) *Evaluation of* $\Sigma, \Sigma_4, \Sigma_6, \Sigma_7$. Let $Q = x^{4/7-\varepsilon}$. By Mertens' theorem we have

$$V(z) = \frac{e^{-\gamma} C}{\log z}\left(1 + O\left(\frac{1}{\log z}\right)\right). \tag{4.1}$$

By Lemmas 2–4 and some routine arguments we get

$$\begin{aligned}\Sigma &\ge 3.5(1 + O(\varepsilon))\frac{Cx}{\log^2 x}\left(\log 5 + \int_2^4 \frac{\log(s-1)}{s} \log\frac{5}{s+1}\, ds\right) \\ &\ge 5.8946937 \frac{Cx}{\log^2 x}.\end{aligned} \tag{4.2}$$

Let λ' denote the characteristic function of the primes in the interval $[L, L')$, where $x^{1/10.5} \le L < L' \le 2L < x^{1/7.68}$ and λ denote a well-factorable function of level QL^{-1}. Then $L' < QL^{-1}$, by Lemma 1, $\lambda * \lambda'$ is a well-factorable function of level Q. By Lemmas 2–4 and some routine arguments we get

$$\begin{aligned}\Sigma_4 &\le 3.5(1 + O(\varepsilon))\frac{Cx}{\log^2 x}\left(\log\frac{35}{23.72}\left(1 + \int_2^{16.72/7} \frac{\log(s-1)}{s}\, ds\right)\right. \\ &\qquad \left. + \int_{16.72/7}^4 \frac{\log(s-1)}{s}\log\frac{5}{s+1}\, ds\right) \\ &\le 1.61893 \frac{Cx}{\log^2 x}.\end{aligned} \tag{4.3}$$

Let λ_1', λ_2' denote the characteristic functions of the primes in the intervals $[L_1, L_1')$ and $[L_2, L_2')$ respectively, where $x^{1/10.5+4\varepsilon} \le L_1 < L_1' \le 2L_1 < x^{1/7.68}$, $x^{1/7.68} \le L_2 < L_2' \le 2L_2 < x^{8/21}(2L_1)^{-1}$, and λ denote a well-factorable function of level $Q(L_1L_2)^{-1}$. Then $L_1' < Q(L_1L_2)^{-1}, L_2' < QL_2^{-1}$, by Lemma 1, $\lambda * \lambda_1'$ is a well-factorable function of level QL_2^{-1}, $(\lambda * \lambda_1') * \lambda_2'$ is a well-factorable function of level Q. By Lemmas 2–4 and some routine arguments we get

$$\Sigma_7 \ge 3.5(1+O(\varepsilon)) \times \frac{Cx}{\log^2 x} \int_{1/10.5}^{1/7.68} \int_{1/7.68}^{8/21-t_1} \frac{\log(5-10.5(t_1+t_2))}{t_1t_2(1-1.75(t_1+t_2))}\, dt_1\, dt_2. \tag{4.4}$$

Similarly,

$$\Sigma_6 \ge 3.5(1+O(\varepsilon))\frac{Cx}{\log^2 x} \int_{1/10.5}^{1/7.68} \int_{t_1}^{1/7.68} \frac{\log(5-10.5(t_1+t_2))}{t_1t_2(1-1.75(t_1+t_2))}\, dt_1\, dt_2,$$

$$\Sigma_6 + \Sigma_7 \ge 1.188865\frac{Cx}{\log^2 x}. \tag{4.5}$$

2) *Evaluation of* Σ_1, Σ_5. We have

$$\begin{aligned}\Sigma_1 &= \Bigg(\sum_{x^{1/10.5}\le p<x^{2/7-\varepsilon}} + \sum_{x^{2/7-\varepsilon}\le p<x^{0.29}} + \sum_{x^{0.29}\le p<x^{1/3.0015}}\Bigg) S(\mathcal{A}_p, x^{1/10.5}) \\ &= S_1 + S_2 + S_3.\end{aligned} \tag{4.6}$$

By Lemma 4 and the arguments used in [12], we get

$$\begin{aligned}S_1 &\le 3.5(1+O(\varepsilon))\frac{Cx}{\log^2 x}\Bigg(\log 5 + \int_2^4 \frac{\log(s-1)}{s}\log\frac{5(5-s)}{s+1}\, ds\Bigg) \\ &\le 6.679727\frac{Cx}{\log^2 x}.\end{aligned} \tag{4.7}$$

By Lemmas 6 and 7 and the arguments used in [12], we have

$$S_2 \le (1+O(\varepsilon))\frac{Cx}{\log^2 x}\log\frac{29}{26}, \tag{4.8}$$

$$S_3 \le \frac{40Cx}{11\log^2 x}\log\frac{52}{37.74785}. \tag{4.9}$$

By (4.6)–(4.9) we get

$$\Sigma_1 \le 3.5(1+O(\varepsilon))\frac{Cx}{\log^2 x}\Bigg(\log 5 + \int_2^4 \frac{\log(s-1)}{s}\log\frac{5(5-s)}{s+1}\, ds\Bigg) \tag{4.10}$$

$$+\frac{Cx}{\log^2 x}\log\frac{29}{26}+\frac{40Cx}{11\log^2 x}\log\frac{52}{37.7485}$$
$$\le 7.95371\frac{Cx}{\log^2 x}.$$

Similarly,

$$\Sigma_5 \le 3.5(1+O(\varepsilon))\frac{Cx}{\log^2 x}\bigg(\log 5+\int_2^4\frac{\log(s-1)}{s}\log\frac{5(5-s)}{s+1}\,ds\bigg) + \frac{Cx}{\log^2 x}\log\frac{1}{0.898} \le 6.78732\frac{Cx}{\log^2 x}. \tag{4.11}$$

3) *Evaluation of* Σ_{10},Σ_{11}. We have

$$\begin{aligned}\Sigma_{10} &= \sum_{x^{1/10.5}\le p_1<p_2<p_3<p_4<x^{1/7.68}}\ \sum_{\substack{a\in\mathcal{A},\,p_1p_2p_3p_4\mid a\\(a,p_1^{-1}P(p_2))=1}} 1+O(x^{9.5/10.5})\\ &= \sum_{x^{1/10.5}\le p_1<p_2<p_3<p_4<x^{1/7.68}}\ \sum_{\substack{p=p_1p_2p_3p_4n-2\\1\le n\le x/p_1p_2p_3p_4,\,(n,p_1^{-1}P(p_2))=1}} 1\\ &\quad + O(x^{9.5/10.5})\\ &= S(\mathcal{B},x^{1/2})+O(x^{9.5/10.5}),\end{aligned} \tag{4.12}$$

where

$$\mathcal{B}=\{mp_1p_2p_3p_4-2\mid x^{1/10.5}\le p_1<p_2<p_3<p_4<x^{1/7.68},\ mp_1p_2p_3p_4\le x+2,(m,p_1^{-1}P(p_2))=1\}.$$

Applying the splitting technique used in [4] to remove the dependence of mp_4 on p_1,p_2,p_3, by Lemma 5 with $p_1p_2p_3$ for n, we get

$$\sum_{(q,2)=1}\lambda(q) \times\bigg(\sum_{mp_1p_2p_3p_4\equiv 2\,(q)}\alpha_{mp_4}\beta_{p_1p_2p_3}-\frac{1}{\varphi(q)}\sum_{(mp_1p_2p_3p_4,q)=1}\alpha_{mp_4}\beta_{p_1p_2p_3}\bigg) = O_\varepsilon\bigg(\frac{x}{\log^{10}x}\bigg) \tag{4.13}$$

with $Q = x^{4/7-\varepsilon}$. Lemmas 2 and 3, when combined with (4.13), give

$$
\begin{aligned}
(4.14)\quad & S(\mathcal{B}, x^{1/2}) \\
& \le 3.5(1+O(\varepsilon))\frac{C|\mathcal{B}|}{\log x} \\
& = 3.5(1+O(\varepsilon))\frac{C}{\log x}\sum_{x^{1/10.5}\le p_1<p_2<p_3<p_4<x^{1/7.68}}\ \sum_{\substack{1\le m\le x/(p_1p_2p_3p_4)\\ (m,p_1^{-1}P(p_2))=1}} 1 \\
& \quad + O(x^{9.5/10.5}) \\
& < \frac{3.5C}{1.763\log x}(1+O(\varepsilon))\sum_{x^{1/10.5}\le p_1<p_2<p_3<p_4<x^{1/7.68}} \frac{x}{p_1p_2p_3p_4\log p_2} \\
& \quad + O(x^{9.5/10.5}) \\
& = \frac{3.5Cx}{1.763\log^2 x}(1+O(\varepsilon))\int_{1/10.5}^{1/7.68}\frac{dt_1}{t_1}\int_{t_1}^{1/7.68}\frac{1}{t_2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)\log\frac{1}{7.68t_2}\,dt_2.
\end{aligned}
$$

By (4.12) and (4.14) we get

$$
\begin{aligned}
(4.15)\qquad \Sigma_{10} \le \frac{3.5}{1.763}(1+O(\varepsilon))\frac{Cx}{\log^2 x} \\
\times \int_{1/10.5}^{1/7.68}\frac{dt_1}{t_1}\int_{t_1}^{1/7.68}\frac{1}{t_2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)\log\frac{1}{7.68t_2}\,dt_2.
\end{aligned}
$$

By a similar method we get

$$
\begin{aligned}
(4.16)\quad \Sigma_{11} \le \frac{3.5}{1.763}(1+O(\varepsilon))\frac{Cx}{\log^2 x} \\
\times \int_{1/10.5}^{1/7.68}\frac{dt_1}{t_1}\int_{t_1}^{1/7.68}\frac{1}{t_2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)\log\left(7.68\left(\frac{8}{21}-t_2\right)\right)dt_2.
\end{aligned}
$$

By (4.15) and (4.16) we obtain

$$
\begin{aligned}
(4.17)\quad & \Sigma_{10}+\Sigma_{11} \\
& \le \frac{3.5}{1.763}(1+O(\varepsilon))\frac{Cx}{\log^2 x}\int_{1/10.5}^{1/7.68}\frac{dt_1}{t_1}\int_{t_1}^{1/7.68}\frac{1}{t_2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)\log\left(\frac{8}{21t_2}-1\right)dt_2 \\
& \le 0.070549\frac{Cx}{\log^2 x}.
\end{aligned}
$$

4) *Evaluation of* $\Sigma_2, \Sigma_3, \Sigma_8, \Sigma_9$. By the arguments used in [10] we get

$$
\begin{aligned}
(4.18)\qquad \Sigma_2 &\le 2(1+O(\varepsilon))\frac{Cx}{\log^2 x}\biggl(4\int_{1/3.0015}^{2.0015/6.003} \frac{\log\frac{9.5-10.5t}{2.0015-3.0015t}}{t(2+t)(1-t)}\,dt \\
&\quad + 4\int_{2.0015/6.003}^{0.4} \frac{\log\frac{(9.5-10.5t)(1-2t)}{t}}{t(2+t)(1-t)}\,dt \\
&\quad + \int_{0.4}^{9.5/21} \frac{\log\frac{(9.5-10.5t)(1-2t)}{t}}{t(1-t)^2}\,dt\biggr) \\
&\le 1.7711\frac{Cx}{\log^2 x},
\end{aligned}
$$

$$
\begin{aligned}
(4.19)\qquad \Sigma_3 &\le 8(1+O(\varepsilon))\frac{Cx}{\log^2 x}\int_{1/3.0015}^{1/3} \frac{\log(1/t-2)}{t(2+t)(1-t)}\,dt \\
&\le 0.000003\frac{Cx}{\log^2 x},
\end{aligned}
$$

$$
\begin{aligned}
(4.20)\qquad \Sigma_8 &\le 2(1+O(\varepsilon))\frac{Cx}{\log^2 x}\biggl(4\int_{1/3.449}^{2.449/6.898} \frac{\log\frac{6.68-7.68t}{2.449-3.449t}}{t(2+t)(1-t)}\,dt \\
&\quad + 4\int_{2.449/6.898}^{0.4} \frac{\log\frac{(6.68-7.68t)(1-2t)}{t}}{t(2+t)(1-t)}\,dt \\
&\quad + \int_{0.4}^{6.68/15.36} \frac{\log\frac{(6.68-7.68t)(1-2t)}{t}}{t(1-t)^2}\,dt\biggr) \\
&\le 1.90815\frac{Cx}{\log^2 x},
\end{aligned}
$$

$$
(4.21)\qquad \Sigma_9 \le 8(1+O(\varepsilon))\frac{Cx}{\log^2 x}\int_{1/3.449}^{1/3} \frac{\log(1/t-2)}{t(2+t)(1-t)}\,dt \le 0.134048\frac{Cx}{\log^2 x}.
$$

Proof of the Theorem. By (4.2), (4.3), (4.5), (4.10), (4.11), (4.17)–(4.21), $2\pi_{1,2}(x)$ is bounded below by

$$
\begin{aligned}
&\biggl(2\cdot 5.89469 - \frac{7.95372}{2} - \frac{1.7711}{2} - 0.000003 - \frac{1.61893}{2} - \frac{6.78732}{2} \\
&\quad + \frac{1.18886}{2} - \frac{1.90815}{2} - 0.134048 - \frac{0.070549}{2}\biggr)\frac{Cx}{\log^2 x} > \frac{2.1948Cx}{\log^2 x},
\end{aligned}
$$

and so

$$\pi_{1,2}(x) > \frac{1.0974Cx}{\log^2 x}.$$

The Theorem is proved.

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