On a decomposition of integer vectors, II

by

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1. Introduction. Let \( n = (n_1, \ldots, n_{k+1}) \) be an integer vector in \( \mathbb{Z}^{k+1}, k > 1 \) and let

\[ h(n) = \max_{i \in \{1, \ldots, k+1\} |n_i|. \]

A. Schinzel [7] has proved the following

**Theorem A.** For every non-zero vector \( n \in \mathbb{Z}^{k+1} \) there exist linearly independent vectors \( p, q \in \mathbb{Z}^{k+1} \) such that \( n = up + vq \) with \( u, v \in \mathbb{Z} \) and

\[ h(p)h(q) \leq 2(h(n))^{1-1/k}. \]

The exponent \( 1 - 1/k \) on the right hand side of (1) is the best possible (see [6, Remark 1]). Later S. Chaladus and A. Schinzel [3] have showed that for every non-zero vector \( n \in \mathbb{Z}^3 \) the inequality (1) can be replaced by

\[ h(p)h(q) < \frac{2}{\sqrt{3}}(h(n))^{1/2}, \]

where the constant on the right hand side is the best possible. The author [1] has proved a more precise result, which gives estimates that depend on the initial vector \( n \in \mathbb{Z}^3 \). The aim of this paper is to prove the following

**Theorem 1.** For every integer vector \( n \in \mathbb{Z}^{k+1}, k > 1 \) with \( h(n) > 1 \) there exist linearly independent vectors \( p, q \in \mathbb{Z}^{k+1} \) such that \( n = up + vq \), \( u, v \in \mathbb{Z} \) and

\[ h(p)h(q) < \frac{2}{(k+1)^{1/k}}(h(n))^{1-1/k}. \]

For the cases of \( k = 1 \) or \( h(n) = 1 \), \( p \) and \( q \) can be trivially found with \( h(p)h(q) = 1 \).

We use the following notation. By \( C^n_{r}(a_1, \ldots, a_n) \) we denote the \( n \)-dimensional hypercube given by

\[ C^n_{r}(a_1, \ldots, a_n) = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i - a_i| \leq r/2, i \in \{1, \ldots, n\} \}, \]

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where \( a_i \in \mathbb{R}, \ i \in \{1, \ldots, n\} \). We write
\[
H(x_i = a) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i = a\}
\]
for \( a \in \mathbb{R}, \ i \in \{1, \ldots, n\} \).

By \( V_n(K) \) and \( \Delta(K) \) we denote the \( n \)-dimensional volume and the critical determinant of a set \( K \) respectively. The convex hull of sets \( A_1, \ldots, A_n \) is denoted by \( \text{conv}(A_1, \ldots, A_n) \).

2. On the critical determinant of polygons. Let \( \mathbf{n} = (n_1, \ldots, n_{k+1}) \) be an integer vector with \( k > 1 \), \( h(\mathbf{n}) > 1 \). We may assume without loss of generality that \( 0 \leq n_1 \leq \ldots \leq n_{k+1} \) and \( \gcd(n_1, \ldots, n_{k+1}) = 1 \). Let \( n_i = n_{i+1} \) for some \( i \in \{1, \ldots, n\} \). Consider the vector
\[
\mathbf{n}' = (n_1, \ldots, n_i, n_{i+2}, \ldots, n_{k+1}) \in \mathbb{Z}^k.
\]
If the theorem is true for \( \mathbf{n}' \) then there exist linearly independent vectors \( \mathbf{p}', \mathbf{q}' \in \mathbb{Z}^k \) such that \( \mathbf{n}' = u\mathbf{p}' + v\mathbf{q}' \), \( u, v \in \mathbb{Z} \) and
\[
h(\mathbf{p}')h(\mathbf{q}') < \frac{2}{k^{1/(k-1)}}(h(\mathbf{n}'))^{1-1/(k-1)}.
\]
Suppose \( \mathbf{p}' = (p'_1, \ldots, p'_k), \mathbf{q}' = (q'_1, \ldots, q'_k) \). Put
\[
\mathbf{p} = (p'_1, \ldots, p'_i, p'_{i+1}, \ldots, p'_k), \quad \mathbf{q} = (q'_1, \ldots, q'_i, q'_{i+1}, \ldots, q'_k).
\]
Then \( \mathbf{p}, \mathbf{q} \) are linearly independent vectors in \( \mathbb{Z}^{k+1} \) such that \( \mathbf{n} = u\mathbf{p} + v\mathbf{q} \) and
\[
h(\mathbf{p})h(\mathbf{q}) < \frac{2}{k^{1/(k-1)}}(h(\mathbf{n}))^{1-1/(k-1)}.
\]
Moreover,
\[
\frac{2}{k^{1/(k-1)}}(h(\mathbf{n}))^{1-1/(k-1)} < \frac{2}{(k+1)^{1/k}}(h(\mathbf{n}))^{1-1/k}
\]
and thus, we may assume without loss of generality that \( 0 < n_1 < \ldots < n_{k+1} \).

Let \( \mathbf{m} = (m_1, \ldots, m_{k+1}) \) be an integer vector such that \( \mathbf{m} \) and \( \mathbf{n} \) are linearly independent. Consider the polygon
\[
\mathcal{P} : \quad |m_ix - n_ix| \leq 1, \quad i \in \{1, \ldots, k+1\}
\]
and sets
\[
\mathcal{H}_{pqr} : \quad |m_ix - n_ix| \leq 1, \quad i \in \{p, q, r\},
\]
where \( p, q, r \in \{1, \ldots, k+1\}, \ p < q < r \) and vectors \( (m_p, m_q, m_r), (n_p, n_q, n_r) \) are linearly independent. Define
\[
\alpha_i = \frac{n_i}{n_{k+1}}, \quad \delta_i = m_i - m_{k+1}\alpha_i, \quad i \in \{1, \ldots, k\}.
\]
For fixed \( p, q, r \in \{1, \ldots, k+1\} \) with \( p < q < r \), consider the integer vectors \( (m_p, m_q, m_r) \) and \( (n_p, n_q, n_r) \). Suppose these vectors are linearly indepen-
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Set
\[ \alpha'_p = \frac{n_p}{n_r}, \quad \alpha'_q = \frac{n_q}{n_r}, \quad \delta'_p = m_p - m_{k+1}\alpha'_p, \quad \delta'_q = m_q - m_{k+1}\alpha'_q. \]

Let \( \xi > 0 \). In [1] we have found a set \( B_\xi = B_\xi(\alpha'_p, \alpha'_q) \subset \mathbb{R}^2 \) such that \( (\delta'_p, \delta'_q) \in B_\xi(\alpha'_p, \alpha'_q) \) if and only if
\[
V_2(H_{pqr}) \geq \frac{4}{n_r \xi}.
\]

(4)

By Lemma 1 of [1], \( B_\xi(\alpha'_p, \alpha'_q) \) is a centrally symmetric, convex set. We shall find a set \( G_{pqr} \subset \mathbb{R}^k \) such that \( (\delta_1, \ldots, \delta_k) \in G_{pqr} \) if and only if
\[
V_2(H_{pqr}) \geq \frac{4}{n_{k+1} \xi}.
\]

(5)

In the case \( r = k + 1 \), obviously,
\[ G_{pqr} = \{(x_1, \ldots, x_k) \in \mathbb{R}^k : (x_p, x_q) \in B_\xi(\alpha_p, \alpha_q)\}. \]

Let \( r < k + 1 \). Then
\[ \alpha'_r = \frac{\alpha_p}{\alpha_r}, \quad \alpha'_q = \frac{\alpha_q}{\alpha_r}, \quad \delta'_p = \delta_p - \frac{\alpha_p}{\alpha_r} \delta_r, \quad \delta'_q = \delta_q - \frac{\alpha_q}{\alpha_r} \delta_r. \]

Moreover, (5) can be rewritten as
\[
V_2(H_{pqr}) \geq \frac{4}{n_r \gamma}, \quad \gamma = \frac{n_{k+1}}{n_r} \xi.
\]

Therefore
\[ G_{pqr} = \left\{(x_1, \ldots, x_k) \in \mathbb{R}^k : \left(x_p - \frac{\alpha_p}{\alpha_r} x_r, x_q - \frac{\alpha_q}{\alpha_r} x_r \right) \in B_\gamma(\frac{\alpha_p}{\alpha_r}, \frac{\alpha_q}{\alpha_r}) \right\}. \]

The above definition, as well as the definition for \( r = k + 1 \) applies only if \( (m_p, m_q, m_r) \) and \( (n_p, n_q, n_r) \) are linearly independent.

Let us find a set \( M_\xi \subset \mathbb{R}^k \) such that \( (\delta_1, \ldots, \delta_k) \in M_\xi \) if and only if
\[
\Delta(P) \geq \frac{1}{n_{k+1} \xi}.
\]

By the theorem of K. Mahler (Theorem 5 in [5])
\[
\Delta(P) = \frac{1}{4} \min V_2(H_{pqr}),
\]

where the minimum on the right is over all \( p, q, r \in \{1, \ldots, k + 1\} \) such that \( p < q < r \) and vectors \( (m_p, m_q, m_r) \) and \( (n_p, n_q, n_r) \) are linearly independent.

Therefore
\[
M_\xi = \bigcap G_{pqr}.
\]

(6)

As is easily seen, all sets \( G_{pqr} \) on the right hand side of (6) are centrally symmetric and convex. Thus, \( M_\xi \) is centrally symmetric, convex set as well.
3. Definitions of $E$ and $F$. Define two sequences of sets \( \{E^i\}_{i=1}^k \) and \( \{F^i\}_{i=1}^k \) as follows. For \( i = 1 \), let

\[
E^1 = F^1 = [-1, 1] \subset \mathbb{R}.
\]

For \( i \in \{2, \ldots, k\} \) set

\[
E^i_0 = \{ (x_1, \ldots, x_{i-1}, 0) \in \mathbb{R}^i : (x_1, \ldots, x_{i-1}) \in E^{i-1} \},
\]

\[
E^i_1 = \{ (x_1, \ldots, x_{i-1}, 1) \in \mathbb{R}^i : (x_1, \ldots, x_{i-1}) \in C_1^{i-1}(1/2, \ldots, 1/2) \},
\]

\[
F^i_0 = \{ (x_1, \ldots, x_{i-1}, 0) \in \mathbb{R}^i : (x_1, \ldots, x_{i-1}) \in F^{i-1} \},
\]

\[
F^i_1 = \{ (x_1, \ldots, x_{i-1}, 1) \in \mathbb{R}^i : (x_1, \ldots, x_{i-1}) \in C_1^{i-1}(\alpha_1/2, \ldots, \alpha_{i-1}/2) \},
\]

\[
E^i = \operatorname{conv}(E^i_0, E^i_1, -E^i_1) \subset \mathbb{R}^i,
\]

\[
F^i = \operatorname{conv}(F^i_0, F^i_1, -F^i_1) \subset \mathbb{R}^i.
\]

We write

\[
E^i_\lambda = E^i \cap H(x_i = \lambda), \quad F^i_\lambda = F^i \cap H(x_i = \lambda), \quad \lambda \in (0, 1).
\]

Define \( E = E^k \) and \( F = F^k \). Then \( E, F \) are \( k \)-dimensional polytopes with volumes

\[
V_k(E) = 2 \int_0^1 V_{k-1}(E^{k-1}_\lambda) \, d\lambda, \quad V_k(F) = 2 \int_0^1 V_{k-1}(F^{k-1}_\lambda) \, d\lambda.
\]

4. The definition and properties of \( \widehat{E} \). Consider the set

\[
\widehat{E} = \{ (x_1, \ldots, x_k) \in \mathbb{R}^k : |x_i| \leq 1, \ |x_i - x_j| \leq 1, \ i, j \in \{1, \ldots, k\}, \ i \neq j \}.
\]

It will be proved in Lemma 7 that \( \widehat{E} = E \).

**Lemma 1.**

\[
V_k(\widehat{E}) = k + 1, \quad \Delta(\widehat{E}) = (k + 1)2^{-k}.
\]

**Proof.** H. G. ApSimon [2] (see also [4, Ch. 5, §32]) proved that the body \( \mathcal{A} \subset \mathbb{R}^k \) defined by the inequalities

\[
|x_i| \leq 1 + c_i, \quad |x_i - x_j| \leq c_i + c_j - 2, \quad i, j \in \{1, \ldots, k\}, \ i \neq j,
\]

where \( c_i > 1 \) for \( i \in \{1, \ldots, k\} \), has a unique critical lattice \( \Lambda \) with the basis

\[
(c_1 + 1, 2, \ldots, 2), \ (2, c_2 + 1, \ldots, 2), \ldots, (2, 2, \ldots, c_k + 1)
\]

and

\[
d(\Lambda) = \Delta(\mathcal{A}) = 2^{-k}V_k(\mathcal{A}) = \left( \prod_{i=1}^k (c_i - 1) \right) \left( 1 + 2 \sum_{i=1}^k (c_i - 1)^{-1} \right).
\]

Put \( c_i = 3 \) for \( i \in \{1, \ldots, k\} \); then \( \mathcal{A} = 4\widehat{E} \) and therefore

\[
V_k(\widehat{E}) = 4^{-k}V_k(\mathcal{A}) = k + 1, \quad \Delta(\widehat{E}) = 4^{-k}\Delta(\mathcal{A}) = (k + 1)2^{-k}.
\]

\[\blacksquare\]
5. Connection between $E$ and $\hat{E}$. Let us give another, equivalent definition of $\hat{E}$. We define a sequence $\{\hat{E}^i\}_{i=1}^k$. For $i = 1$ put $\hat{E}^1 = [-1, 1] \subset \mathbb{R}$. For $i \in \{2, \ldots, k\}$ set

$$\hat{A}^i = \{(x_1, \ldots, x_i) \in \mathbb{R}^i : (x_1, \ldots, x_{i-1}) \in \hat{E}^{i-1}\},$$
$$\hat{B}^i = \{(x_1, \ldots, x_i) \in \mathbb{R}^i : |x_i| \leq 1, |x_i - x_j| \leq 1, j \in \{1, \ldots, i - 1\}\},$$
$$\hat{E}^i = \hat{A}^i \cap \hat{B}^i.$$  

Then, obviously, $\hat{E} = \hat{E}^k$.

**Lemma 2.** $E \subset \hat{E}$ and therefore $V_k(E) \leq k + 1$.

**Proof.** It suffices to show that $E^i \subset \hat{E}^i$ for $i \in \{1, \ldots, k\}$. We use induction on $i$. The case $i = 1$ is trivial. Furthermore,

$$\hat{E}^i \cap H(x_i = 1) = E^{i-1}_1, \quad \hat{E}^i \cap H(x_i = -1) = -E^{i-1}_1$$

and by the inductive hypothesis $E^{i-1}_0 \subset \hat{E}^i \cap H(x_i = 0)$. Since $\hat{E}^i$ is convex, we obtain $E^i \subset \hat{E}^i$. ■

6. Geometric tools. Let $K_0, K_1$ be bounded convex sets in $\mathbb{R}^{l-1}$, $l > 1$. Define

$$K_0^{l-1} = \{(x_1, \ldots, x_{l-1}, 0) \in \mathbb{R}^l : (x_1, \ldots, x_{l-1}) \in K_0\},$$
$$K_1^{l-1} = \{(x_1, \ldots, x_{l-1}, 1) \in \mathbb{R}^l : (x_1, \ldots, x_{l-1}) \in K_1\},$$
$$K_1^{l-1} = \text{conv}(K_0^{l-1}, K_1^{l-1}) \cap H(x_l = \lambda), \quad \lambda \in [0, 1].$$

**Lemma 3.** For all $\lambda, \eta \in [0, 1]$,

$$\text{conv}(K_1^{l-1}, K_1^{l-1}) \cap H(x_l = \mu) = K_\mu^{l-1}, \quad \mu = \eta + \lambda(1 - \eta).$$

**Proof.** We have

$$\text{conv}(K_1^{l-1}, K_1^{l-1}) = \bigcup_{a \in [0, 1]} \{(1 - a)K_1^{l-1} + aK_0^{l-1}\}.$$  

From the equation $x_l = (1 - a)\eta + a = \mu$, we obtain $a = \lambda$ and hence

$$\text{conv}(K_1^{l-1}, K_1^{l-1}) \cap H(x_l = \mu) = (1 - \lambda)K_1^{l-1} + \lambda K_0^{l-1}.$$  

Furthermore, $K_1^{l-1} = (1 - \eta)K_0^{l-1} + \eta K_1^{l-1}$ and thus, by the convexity of $K_1^{l-1}$, we obtain

$$\text{conv}(K_1^{l-1}, K_1^{l-1}) \cap H(x_l = \mu) = (1 - \lambda)((1 - \eta)K_0^{l-1} + \eta K_1^{l-1}) + \lambda K_1^{l-1}$$

$$= (1 - \lambda)(1 - \eta)K_0^{l-1} + \lambda + \eta(1 - \lambda))K_1^{l-1} = (1 - \mu)K_0^{l-1} + \mu K_1^{l-1}. \quad \blacksquare$$
Define
\[ K_{00}^{l-1} = \{ (x_1, \ldots, x_{l-1}, 0, 0) \in \mathbb{R}^{l+1} : (x_1, \ldots, x_{l-1}) \in K_0 \}, \]
\[ K_{10}^{l-1} = \{ (x_1, \ldots, x_{l-1}, 1, 0) \in \mathbb{R}^{l+1} : (x_1, \ldots, x_{l-1}) \in K_1 \}, \]
\[ K_{01}^{l-1} = \{ (x_1, \ldots, x_{l-1}, 0, 1) \in \mathbb{R}^{l+1} : (x_1, \ldots, x_{l-1}) \in K_1 \}. \]

**Lemma 4.** For every \( \lambda \in [0, 1] \),
\[ V_{l-1}(\text{conv}(K_{00}^{l-1}, K_{10}^{l-1}) \cap H(x_l = \lambda)) = V_{l-1}(\text{conv}(K_{00}^{l-1}, K_{01}^{l-1}) \cap H(x_{l+1} = \lambda)). \]

**Proof.** We can obtain \( \text{conv}(K_{00}^{l-1}, K_{01}^{l-1}) \) from \( \text{conv}(K_{00}^{l-1}, K_{10}^{l-1}) \) by exchanging the variables \( x_l, x_{l+1} \).

For \( a \in \mathbb{R} \) define
\[ K_{a1}^{l-1} = \{ (x_1, \ldots, x_{l-1}, a, 1) \in \mathbb{R}^{l+1} : (x_1, \ldots, x_{l-1}) \in K_1 \}. \]

**Lemma 5.** For all \( \lambda, \eta \in [0, 1] \), \( a \in \mathbb{R} \),
\[ V_{l-1}(\text{conv}(\text{conv}(K_{00}^{l-1}, K_{10}^{l-1}) \cap H(x_l = \eta), K_{a1}^{l-1}) \cap H(x_{l+1} = \lambda)) = V_{l-1}(\text{conv}(\text{conv}(K_{00}^{l-1}, K_{10}^{l-1}) \cap H(x_l = \eta), K_{10}^{l-1}) \cap H(x_l = \mu)), \]
where \( \mu = \eta + \lambda(1 - \eta) \).

**Proof.** Define
\[ L_{\eta} = \text{conv}(K_{00}^{l-1}, K_{10}^{l-1}) \cap H(x_l = \eta), \]
\[ M_{\lambda} = \text{conv}(L_{\eta}, K_{a1}^{l-1}) \cap H(x_{l+1} = \lambda), \]
\[ N_{\mu} = \text{conv}(L_{\eta}, K_{10}^{l-1}) \cap H(x_l = \mu). \]

We show that \( V_{l-1}(M_{\lambda}) = V_{l-1}(N_{\mu}) \). Consider the translation \( T \):
\[ x_1' = x_1, \ldots, x_{l-1}' = x_{l-1}, \ x_l' = x_l - \eta, \ x_{l+1}' = x_{l+1}. \]

All volumes are invariant under \( T \). Let
\[ L_{\eta} \xrightarrow{T} U, \quad K_{10}^{l-1} \xrightarrow{T} V, \quad K_{a1}^{l-1} \xrightarrow{T} W. \]

Therefore
\[ M_{\lambda} \xrightarrow{T} Y = \text{conv}(U, W) \cap H(x_{l+1}' = \lambda), \]
\[ N_{\mu} \xrightarrow{T} X = \text{conv}(U, V) \cap H(x_l' = \mu - \eta). \]

Consider the sets \( U, V, X \), and the affine transformation \( \phi \):
\[ x_1'' = x_1', \ldots, x_{l-1}'' = x_{l-1}', \ x_l'' = (1 - \eta)^{-1}x_l', \ x_{l+1}'' = x_{l+1}'. \]

Let
\[ U \xrightarrow{\phi} U_{\phi}, \quad V \xrightarrow{\phi} V_{\phi}, \quad X \xrightarrow{\phi} X_{\phi}. \]
Since $U \subset H(x'_l=0)$, we have $U = U_\phi$. Further, $V_\phi \subset H(x''_l = 1) \cap H(x''_{l+1} = 0)$ and
\[
X_\phi = \text{conv}(U_\phi, V_\phi) \cap H(x''_l = (1-\eta)^{-1}(\mu - \eta)) = \text{conv}(U_\phi, V_\phi) \cap H(x''_l = (1-\eta)^{-1}(\eta + \lambda(1-\eta) - \eta)) = \text{conv}(U_\phi, V_\phi) \cap H(x''_l = \lambda).
\]
Obviously, $V_{l-1}(X_\phi) = V_{l-1}(X)$. Consider the sets $U$, $W$, $Y$, and the affine transformation $\psi$:
\[
x'_1 = x'_1, \ldots, x'_{l-1} = x'_{l-1}, x''_l = x'_l + (\eta - a)x'_{l+1}, x''_{l+1} = x'_{l+1}.
\]
Let
\[
U \xrightarrow{\psi} U_\psi, \quad W \xrightarrow{\psi} W_\psi, \quad Y \xrightarrow{\psi} Y_\psi.
\]
Since $U \subset H(x'_l = 0) \cap H(x'_{l+1} = 0)$, we have $U = U_\psi$. Further, we have the inclusion $W_\psi \subset H(x''_l = 0) \cap H(x''_{l+1} = 1)$ and $Y_\psi = \text{conv}(U_\psi, W_\psi) \cap H(x''_{l+1} = \lambda)$. Obviously, $V_{l-1}(Y_\psi) = V_{l-1}(Y)$. Lemma 4 with $K^{l-1}_{00} = U = U_\phi = U_\psi$, $K^{l-1}_{10} = V_\psi$, $K^{l-1}_{01} = W_\psi$ implies $V_{l-1}(X_\phi) = V_{l-1}(X_\psi)$ and thus $V_{l-1}(M_\lambda) = V_{l-1}(N_\mu)$.

**Lemma 6.** For all $\lambda, \eta \in [0, 1]$, $a \in \mathbb{R}$,
\[
V_{l-1}(\text{conv}(\text{conv}(K^{l-1}_{00}, K^{l-1}_{10}) \cap H(x_l = \eta), K^{l-1}_{a1}) \cap H(x_{l+1} = \lambda)) = V_{l-1}(\text{conv}(K^{l-1}_{00}, K^{l-1}_{10}) \cap H(x_l = \mu)),
\]
where $\mu = \eta + \lambda(1-\eta)$.

**Proof.** By Lemma 5 it is enough to prove that
\[
V_{l-1}(\text{conv}(\text{conv}(K^{l-1}_{00}, K^{l-1}_{10}) \cap H(x_l = \eta), K^{l-1}_{10}) \cap H(x_l = \mu)) = V_{l-1}(\text{conv}(K^{l-1}_{00}, K^{l-1}_{10}) \cap H(x_l = \mu)).
\]
Since $K^{l-1}_{00}, K^{l-1}_{10} \subset H(x_{l+1} = 0)$, we have
\[
V_{l-1}(\text{conv}(\text{conv}(K^{l-1}_{00}, K^{l-1}_{10}) \cap H(x_l = \eta), K^{l-1}_{10}) \cap H(x_l = \mu)) = V_{l-1}(\text{conv}(K^{l-1}_{00}, K^{l-1}_{10}) \cap H(x_l = \mu)) = V_{l-1}(\text{conv}(K^{l-1}_{00}, K^{l-1}_{10}) \cap H(x_l = \mu)),
\]
where the last equality is valid in view of Lemma 3.

**7. A property of $E$**

**Lemma 7.** For every $\lambda \in [0, 1]$,
(i) $E^a_\lambda = \text{conv}(E^n_0, E^n_1) \cap H(x_{n+1} = \lambda)$,
(ii) $V_n(E^a_\lambda) = -n\lambda + n + 1$, $n \in \{1, \ldots, k-1\}$.

Moreover, $V_k(E) = k + 1$ and thus $E = \hat{E}$.
Proof. We use induction on $n$. The case $n = 1$ is trivial. The affine transformation
\[ x'_1 = x_1, \ldots, x'_{n-1} = x_{n-1}, \ x'_n = x_n - \frac{1}{2} x_{n+1}, \ x'_{n+1} = x_{n+1} \]
does not change the volume of a body and the volume of its intersection with any hyperplane $H(x_{n+1} = a)$. Thus, we may assume without loss of generality that
\[ E^n_1 = \{(x_1, \ldots, x_n, 1) \in \mathbb{R}^{n+1} : (x_1, \ldots, x_n) \in C^n_1(1/2, \ldots, 1/2, 0)\}. \]
Therefore
\[
V_n(\text{conv}(E^n_0, E^n_1) \cap H(x_{n+1} = \lambda))
= 2 \int_0^{1-\lambda/2} V_{n-1}(\text{conv}(E^n_0, E^n_1) \cap H(x_{n+1} = \lambda) \cap H(x_n = \xi)) \, d\xi.
\]
Let us show that $\text{conv}(E^n_0, E^n_1) \cap H(x_{n+1} = \lambda) \cap H(x_n = \xi)$, where $\lambda \in (0, 1)$, contains the set
\[ E_1^1(\xi) = \text{conv}(E^n_0 \cap H(x_n = 0), E^n_1 \cap H(x_n = \xi/\lambda)) \cap H(x_{n+1} = \lambda) \]
for $0 \leq \xi \leq \lambda/2$, and
\[ E_2(\xi) = \text{conv}(E^n_0 \cap H(x_n = \eta), E^n_1 \cap H(x_n = 1/2)) \cap H(x_{n+1} = \lambda) \]
for $\lambda/2 < \xi \leq 1 - \lambda/2$, $\eta = (\xi - \lambda/2)/(1 - \lambda)$. Let $0 \leq \xi \leq \lambda/2$ and $x = (x_1, \ldots, x_{n+1}) \in E_1^1(\xi)$. Then there exist $x' = (x'_1, \ldots, x'_{n-1}, 0, 0) \in E^n_0 \cap H(x_n = 0)$ and $x'' = (x''_1, \ldots, x''_{n-1}, \xi/\lambda, 1) \in E^n_1 \cap H(x_n = \xi/\lambda)$ such that
\[ x = (1 - a)x' + ax'', \quad a \in (0, 1). \]
Since $x_{n+1} = (1 - a)x'_{n+1} + ax''_{n+1} = \lambda$, we obtain $a = \lambda$. Furthermore,
\[ x_n = (1 - \lambda)x'_n + \lambda x''_n = \xi. \]
Since $E^n_0 \cap H(x_n = 0) \subset E^n_0$ and $E^n_0 \cap H(x_n = \xi/\lambda) \subset E^n_1$, we have $x' \in E^n_0$, $x'' \in E^n_1$, and thus
\[ x \in \text{conv}(E^n_0, E^n_1) \cap H(x_{n+1} = \lambda) \cap H(x_n = \xi). \]
By similar arguments, the case $\lambda/2 < \xi \leq 1 - \lambda/2$ can be settled as well. Therefore
\[ V_n(\text{conv}(E^n_0, E^n_1) \cap H(x_{n+1} = \lambda)) \geq 2I_1 + 2I_2, \]
where
\[ I_1 = \int_0^{\lambda/2} V_{n-1}(E_1^1(\xi)) \, d\xi, \quad I_2 = \int_{\lambda/2}^{1-\lambda/2} V_{n-1}(E_2(\xi)) \, d\xi. \]
Put \( l = n \), \( K_0 = E_0^{n-1} \), \( K_1 = C_1^{n-1} (1/2, \ldots, 1/2) \). Then in view of part (i) of the inductive hypothesis we have, in the notation of Section 6,

\[
K_0^{n-1} = E_0^{n-1}, \quad K_1^{n-1} = E_1^{n-1}, \quad K_\lambda^{n-1} = E_\lambda^{n-1}, \quad \lambda \in [0, 1]
\]

and

\[
K_{00}^{n-1} = E_0^n \cap H(x_n = 0), \quad K_{10}^{n-1} = E_0^n \cap H(x_n = 1).
\]

Assume \( \lambda \in (0, 1) \), and consider the integral \( I_1 \). Put \( \eta = 0 \), \( a = \xi / \lambda \). Then

\[
\text{conv}(K_{00}^{n-1}, K_{10}^{n-1}) \cap H(x_n = \eta) = K_{00}^{n-1}, \quad K_{a1}^{n-1} = E_a^n \cap H(x_n = \xi / \lambda)
\]

and by Lemma 6 we obtain

\[
V_{n-1}(E_1(\xi)) = V_{n-1}(\text{conv}(E_0^{n-1}, E_1^{n-1}) \cap H(x_n = \lambda)) = V_{n-1}(E_\lambda^{n-1})
\]

\[
= (1 - n) \lambda + n,
\]

where the last two equalities are valid by the inductive hypothesis. Therefore

\[
I_1 = ((1 - n) \lambda + n) \int_0^{\lambda/2} d\xi = \frac{\lambda}{2} ((1 - n) \lambda + n).
\]

Consider the integral \( I_2 \). Put \( \eta = (\xi - \lambda/2)/(1 - \lambda) \), \( a = 1/2 \). Then

\[
\text{conv}(K_{00}^{n-1}, K_{10}^{n-1}) \cap H(x_n = \eta) = E_0^n \cap H(x_n = \eta), \quad K_{a1}^{n-1} = E_1^n \cap H(x_n = 1/2)
\]

and by Lemma 6 we obtain

\[
V_{n-1}(E_2(\xi)) = V_{n-1}(\text{conv}(E_0^{n-1}, E_1^{n-1}) \cap H(x_n = \mu)) = V_{n-1}(E_\mu^{n-1}),
\]

where \( \mu = \eta + \lambda(1 - \eta) = \xi + \lambda/2 \). By the inductive hypothesis

\[
V_{n-1}(E_\mu^{n-1}) = (1 - n) \mu + n = (1 - n)(\xi + \lambda/2) + n.
\]

Therefore

\[
I_2 = \int_{\lambda/2}^{1-\lambda/2} ((1 - n)(\xi + \lambda/2) + n) d\xi
\]

\[
= (1 - n) \int_{\lambda/2}^{1-\lambda/2} \xi d\xi + ((1 - n) \lambda/2 + n) \int_{\lambda/2}^{1-\lambda/2} d\xi
\]

\[
= (1 - n)(1 - \lambda)/2 + (1 - n)(1 - \lambda)\lambda/2 + n(1 - \lambda).
\]

Thus

\[
2I_1 + 2I_2
\]

\[
= \lambda((1 - n) \lambda + n) + (1 - n)(1 - \lambda) + \lambda(1 - n)(1 - \lambda) + 2n(1 - \lambda)
\]

\[
= \lambda^2 - n\lambda^2 + n\lambda + 1 - \lambda - n + n\lambda + \lambda - \lambda^2 - n\lambda + n\lambda^2 + 2n - 2n\lambda
\]

\[
= - n\lambda + n + 1.
\]
Therefore
\[ V_n(\text{conv}(E_0^n, E_1^n) \cap H(x_{n+1} = \lambda)) \geq -n\lambda + n + 1, \quad \lambda \in (0, 1). \]

Consider the case \( \lambda = 0 \). By the inductive hypothesis
\[ V_n(\text{conv}(E_0^n, E_1^n) \cap H(x_{n+1} = 0)) = V_n(E_0^n) = 2 \int_0^1 V_{n-1}(E_{\lambda}^{n-1}) d\lambda \]
\[ = 2 \int_0^1 (1 - n)\lambda + n) d\lambda = n + 1. \]

For the case of \( \lambda = 1 \) we have
\[ V_n(\text{conv}(E_0^n, E_1^n) \cap H(x_{n+1} = 1)) = V_n(E_1^n) = 1. \]

Moreover, in view of the inclusion
\[ \text{conv}(E_0^n, E_1^n) \cap H(x_{n+1} = \lambda) \subset E_{\lambda}^n, \quad \lambda \in [0, 1], \]
we have
\[ V_n(E_{\lambda}^n) \geq -n\lambda + n + 1, \quad \lambda \in (0, 1), \]
and obviously (ii) implies (i). Suppose there exists \( \lambda \in (0, 1) \), such that \( V_n(E_{\lambda}^n) > -n\lambda + n + 1 \). Then, by the continuity of the function \( f(\lambda) = V_n(E_{\lambda}^n) \) on \( (0, 1) \), we obtain
\[ V_{n+1}(E_{\lambda}^{n+1}) = 2 \int_0^1 V_n(E_{\lambda}^n) d\lambda > 2 \int_0^1 (-n\lambda + n + 1) d\lambda = n + 2. \]

But Lemma 2 gives \( V_{n+1}(E_{\lambda}^{n+1}) \leq n + 2 \), a contradiction. Therefore, \( V_n(E_{\lambda}^n) = -n\lambda + n + 1 \) for every \( \lambda \in [0, 1] \). Finally,
\[ V_k(E) = 2 \int_0^1 V_{k-1}(E_{\lambda}^{k-1}) d\lambda = k + 1 \]
and therefore \( E = \hat{E} \).

Remark. The equality \( E = \hat{E} \) has been proved by A. Schinzel, by a different argument.

8. A property of \( F \)

Lemma 8. For every \( \lambda \in [0, 1] \),
\[ V_n(\text{conv}(F_0^n, F_1^n) \cap H(x_{n+1} = \lambda)) \geq V_n(E_{\lambda}^n), \quad n \in \{1, \ldots, k - 1\}. \]
Moreover, \( V_k(F) \geq V_k(E) = k + 1 \).

Proof. We use induction on \( n \). The case \( n = 1 \) is trivial. The affine transformation
\[ x_1' = x_1, \ldots, x_{n-1}' = x_{n-1}, \quad x_n' = x_n - \frac{\alpha_n}{2} x_{n+1}, \quad x_{n+1}' = x_{n+1} \]
does not change the volume of a body, and the volume of its intersection with any hyperplane $H(x_{n+1} = a)$. Thus, we may assume without loss of generality that

$$F^n_1 = \{(x_1, \ldots, x_n, 1) \in \mathbb{R}^{n+1} : (x_1, \ldots, x_n) \in C^n_1(\alpha_1/2, \ldots, \alpha_{n-1}/2, 0)\}.$$ 

Therefore

$$V_n(\text{conv}(F^n_0, F^n_1) \cap H(x_{n+1} = \lambda)) \quad 1-\lambda/2$$

$$= 2 \int_0^{\lambda/2} V_{n-1}(\text{conv}(F^n_0, F^n_1) \cap H(x_{n+1} = \lambda) \cap H(x_n = \xi)) \, d\xi.$$ 

Define

$$F^{n-1}_{00} = \{(x_1, \ldots, x_n, 0, 0) \in \mathbb{R}^{n+1} : (x_1, \ldots, x_n) \in F^{n-1}_0\}.$$ 

Then, in particular,

$$F^{n-1}_{00} = \{(x_1, \ldots, x_n, 0) \in \mathbb{R}^{n+1} : (x_1, \ldots, x_n) \in F^{n-1}_0\}.$$ 

Arguments similar to ones of Section 7 show that $\text{conv}(F^n_0, F^n_1) \cap H(x_{n+1} = \lambda) \cap H(x_n = \xi)$, where $\lambda \in (0, 1)$, contains the set

$$F_1(\xi) = \text{conv}(F^{n-1}_{00}, F^n_1 \cap H(x_n = \xi/\lambda)) \cap H(x_{n+1} = \lambda)$$

for $0 \leq \xi \leq \lambda/2$, and

$$F_2(\xi) = \text{conv}(\text{conv}(F^{n-1}_{00}, F^n_0 \cap H(x_n = 1)) \cap H(x_n = \eta), F^n_1 \cap H(x_n = 1/2)) \cap H(x_{n+1} = \lambda)$$

for $\lambda/2 < \xi \leq 1 - \lambda/2, \eta = (\xi - \lambda/2)/(1 - \lambda)$. Therefore

$$V_n(\text{conv}(F^n_0, F^n_1) \cap H(x_{n+1} = \lambda)) \geq 2J_1 + 2J_2,$$

where

$$J_1 = \int_0^{\lambda/2} V_{n-1}(F_1(\xi)) \, d\xi, \quad J_2 = \int_{\lambda/2}^{1-\lambda/2} V_{n-1}(F_2(\xi)) \, d\xi.$$ 

Put $l = n$, $K_0 = F^{n-1}_0$, $K_1 = C^{n-1}_1(\alpha_1/2, \ldots, \alpha_{n-1}/2)$. Then, in the notation of Section 6, we have

$$K^{n-1}_{00} = F^{n-1}_0, \quad K^{n-1}_1 = F^{n-1}_1,$$

$$K^{n-1}_\lambda = \text{conv}(F^{n-1}_0, F^{n-1}_1) \cap H(x_n = \lambda), \quad \lambda \in [0, 1],$$

$$K^{n-1}_{00} = F^{n-1}_0, \quad K^{n-1}_{10} = F^n_0 \cap H(x_n = 1).$$

Assume $\lambda \in (0, 1)$, and consider the integral $J_1$. Put $\eta = 0, a = \xi/\lambda$. Then

$$\text{conv}(K^{n-1}_{00}, K^{n-1}_{10}) \cap H(x_n = \eta) = F^{n-1}_{00}, \quad K^{n-1}_{a1} = F^n_1 \cap H(x_n = \xi/\lambda)$$

and by Lemma 6 we obtain

$$V_{n-1}(F_1(\xi)) = V_{n-1}(\text{conv}(F^{n-1}_0, F^{n-1}_1) \cap H(x_n = \lambda)) \geq V_{n-1}(E^{n-1}_\lambda),$$

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where the last inequality is valid by the inductive hypothesis. Therefore

\[ J_1 \geq V_{n-1}(E_{\lambda}^{n-1}) \int_0^{\lambda/2} d\xi. \]

Consider the integral \( J_2 \). Put \( \eta = (\xi - \lambda/2)/(1 - \lambda), a = 1/2 \). Then

\[ \text{conv}(K_{a_0}^{n-1}, K_{10}^{n-1}) \cap H(x_n = \eta) = \text{conv}(F_{00}^{n-1}, F_0^{n-1} \cap H(x_n = 1)) \cap H(x_n = \eta), \]

\[ K_{a_1}^{n-1} = F_1^n \cap H(x_n = 1/2) \]

and by Lemma 6 we obtain

\[ V_{n-1}(F_2(\xi)) = V_{n-1}(\text{conv}(F_{0}^{n-1}, F_1^{n-1}) \cap H(x_n = \mu)) \geq V_{n-1}(E_{\mu}^{n-1}), \]

where \( \mu = \xi + \lambda/2 \) and the last inequality is valid by the inductive hypothesis. Therefore

\[ J_2 \geq - \int_{\lambda/2}^{1-\lambda/2} V_{n-1}(E_{\xi+\lambda/2}^{n-1}) d\xi \]

and

\[ V_n(\text{conv}(F_0^n, F_1^n) \cap H(x_{n+1} = \lambda)) \]

\[ \geq 2J_1 + 2J_2 \geq 2V_{n-1}(E_{\lambda}^{n-1}) \int_0^{\lambda/2} d\xi + 2 \int_{\lambda/2}^{1-\lambda/2} V_{n-1}(E_{\xi+\lambda/2}^{n-1}) d\xi \]

\[ = - n\lambda + n + 1 = V_n(E_{\lambda}^{n}), \quad \lambda \in (0, 1). \]

Consider the case \( \lambda = 0 \). Note that

\[ \text{conv}(F_0^{n-1}, F_1^{n-1}) \cap H(x_n = \lambda) \subset F_\lambda^{n-1}, \quad \lambda \in [0, 1], \]

thus, by the inductive hypothesis,

\[ V_n(\text{conv}(F_0^n, F_1^n) \cap H(x_{n+1} = 0)) = V_n(F_0^n) = 2 \int_0^{1} V_{n-1}(F_{\lambda}^{n-1}) d\lambda \]

\[ \geq 2 \int_0^{1} V_{n-1}(\text{conv}(F_0^{n-1}, F_1^{n-1}) \cap H(x_n = \lambda)) d\lambda \geq 2 \int_0^{1} V_{n-1}(E_{\lambda}^{n-1}) d\lambda \]

\[ = 2 \int_0^{1} ((1 - n)\lambda + n) d\lambda = n + 1. \]

In the case \( \lambda = 1 \) we have

\[ V_n(\text{conv}(F_0^n, F_1^n) \cap H(x_{n+1} = 1)) = V_n(F_1^n) = 1. \]

Furthermore,

\[ \text{conv}(F_0^n, F_1^n) \cap H(x_{n+1} = \lambda) \subset F_\lambda^n, \quad \lambda \in [0, 1], \]
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thus, finally,

\[
V_k(F) = 2 \int_0^1 V_{k-1}(F_{1}^{k-1}) \, d\lambda
\]

\[
\geq 2 \int_0^1 (\text{conv}(F_{0}^{k-1}, F_{1}^{k-1}) \cap H(x_k = \lambda)) \, d\lambda
\]

\[
\geq 2 \int_0^1 V_{k-1}(E_{\lambda}^{k-1}) \, d\lambda = V_k(E) = k + 1. \]

9. Properties of \( G_{pqr} \)

**Lemma 9.** For all \( p, q \in \{1, \ldots, k\} \), \( p < q \) and \( r = k + 1 \) the set \( G_{pqr} \), if defined (see p. 375), contains points \((x_1, \ldots, x_k) \in \mathbb{R}^k\) for which one of the following conditions is valid:

(i) \((x_p, x_q) = (\pm(\xi, 0))\),

(ii) \((x_p, x_q) = (\pm((\alpha_p/2 \pm 1/2)\xi, 0))\),

(iii) \((x_p, x_q) = (\pm((\alpha_p/2 \pm 1/2)\xi, \xi))\),

(iv) \((x_p, x_q) = (\pm((\alpha_p/2 \pm 1/2)\xi, (\alpha_q/2 \pm 1/2)\xi))\),

for any combination of \( \pm \).

**Proof.** By the definition of \( G_{pqr} \) we have to check the inclusion

\[(x_p, x_q) \in \mathcal{B}_\xi(\alpha_p, \alpha_q)\]

for all cases (i)–(iv). It is easy to verify by using formulae (8)–(13) of [1] that \( \mathcal{B}_\xi(\alpha_p, \alpha_q) \) contains the centrally symmetric, convex octagon \( \mathcal{O}_\xi(\alpha_p, \alpha_q) \) with vertices

\[
\pm P^1 = (\pm(\xi, (1 - \alpha_q)/(1 - \alpha_p)), \pm P^2 = (\pm(\xi(1 + \alpha_p)/(1 + \alpha_q), \xi), \pm P^3 = (\pm(-\xi(1 - \alpha_p)/(1 + \alpha_q), \xi), \pm P^4 = (\pm(-\xi, (1 - \alpha_q)/(1 + \alpha_p)).
\]

In cases (i)–(iii) the inclusion \((x_p, x_q) \in \mathcal{O}_\xi(\alpha_p, \alpha_q)\) is obviously valid and therefore (7) is true. Consider case (iv). Since case (iii) is proved and \((\alpha_q/2 + 1/2)\xi \leq \xi\), we have

\[
\pm((\alpha_p/2 \pm 1/2)\xi, (\alpha_q/2 + 1/2)\xi) \in \mathcal{O}_\xi(\alpha_p, \alpha_q).
\]

Since \(|(\alpha_q/2 - 1/2)\xi| \leq \min(|P^1_y|, |P^4_y|)\) the points \(\pm((\alpha_p/2 \pm 1/2)\xi, (\alpha_q/2 - 1/2)\xi)\) lie in the rectangle with vertices \(P^1, P^4, -P^1, -P^4\) and therefore

\[
\pm((\alpha_p/2 + 1/2)\xi, (\alpha_q/2 - 1/2)\xi) \in \mathcal{O}_\xi(\alpha_p, \alpha_q). \]

**Lemma 10.** For all \( p, q, r \in \{1, \ldots, k\} \), \( p < q < r \), the set \( G_{pqr} \), if defined, contains points \((x_1, \ldots, x_k) \in \mathbb{R}^k\) for which one of the following conditions is valid:
for any combination of \( \pm \).

**Proof.** By the definition of \( \mathcal{G}_{pqr} \) we have to check the inclusion
\[
(x_p - (\alpha_p/\alpha_r)x_r, x_q - (\alpha_q/\alpha_r)x_r) \in \mathcal{B}_{\xi/\alpha_r}(\alpha_p/\alpha_r, \alpha_q/\alpha_r)
\]
for all cases (i)–(vi). Note that \( \mathcal{B}_{\xi/\alpha_r}(\alpha_p/\alpha_r, \alpha_q/\alpha_r) \) contains the octagon \( \mathcal{O}_{\xi/\alpha_r}(\alpha_p/\alpha_r, \alpha_q/\alpha_r) \) with vertices
\[
\pm Q^1 = \pm \left( \frac{\xi}{\alpha_r}, \frac{\xi(\alpha_r - \alpha_q)}{\alpha_r(\alpha_r - \alpha_p)} \right), 
\pm Q^2 = \pm \left( \frac{\xi(\alpha_r + \alpha_p)}{\alpha_r(\alpha_r + \alpha_q)}, \frac{\xi}{\alpha_r} \right),
\pm Q^3 = \pm \left( -\frac{\xi(\alpha_r - \alpha_p)}{\alpha_r(\alpha_r + \alpha_q)}, \frac{\xi}{\alpha_r} \right), 
\pm Q^4 = \pm \left( -\frac{\xi(\alpha_r - \alpha_q)}{\alpha_r(\alpha_r + \alpha_p)}, \frac{\xi}{\alpha_r} \right).
\]
Let us find the abscissae of points of the intersection of \( \mathcal{O}_{\xi/\alpha_r}(\alpha_p/\alpha_r, \alpha_q/\alpha_r) \) with the line \( y = \xi \). The equation of the line through \( Q^1, Q^2 \) is
\[
y + \frac{\alpha_r + \alpha_q}{\alpha_r - \alpha_p} x - \frac{2\xi}{\alpha_r - \alpha_p} = 0.
\]
By putting \( y = \xi \) we obtain
\[
R_x = \xi \frac{2 - \alpha_r + \alpha_p}{\alpha_r + \alpha_q}.
\]
The equation of the line through \( Q^3, Q^4 \) is
\[
y - \frac{\alpha_r + \alpha_q}{\alpha_r + \alpha_p} x - \frac{2\xi}{\alpha_r + \alpha_p} = 0.
\]
By putting \( y = \xi \) we obtain
\[
L_x = -\xi \frac{2 - \alpha_r - \alpha_p}{\alpha_r + \alpha_q}.
\]
(i), (ii). Here \( x_r = 0 \) and we have the obvious inclusions
\[
\pm(\xi, 0), \pm((\alpha_p/2 \pm 1/2)\xi, 0) \in \mathcal{O}_{\xi/\alpha_r}(\alpha_p/\alpha_r, \alpha_q/\alpha_r).
\]
(iii) Here \( x_r = 0 \) and thus, by symmetry, it is enough to prove
\[
((\alpha_p/2 \pm 1/2)\xi, \xi) \in \mathcal{O}_{\xi/\alpha_r}(\alpha_p/\alpha_r, \alpha_q/\alpha_r).
\]
Let us show that \( ((\alpha_p/2\pm1/2)\xi, \xi) \) lies on the segment \( (L_x, \xi)(R_x, \xi) \). Indeed,
\[
(\alpha_p/2 - 1/2)\xi - L_x = \frac{(3 - \alpha_p)(1 - \alpha_r) + (1 - \alpha_q)(1 - \alpha_p)}{2(\alpha_r + \alpha_q)} \xi \geq 0
\]
and

\[ R_x - (\alpha_p/2 + 1/2)\xi = \frac{3(1 - \alpha_r) + 1 - \alpha_q + \alpha_p(1 - \alpha_r) + \alpha_p(1 - \alpha_q)}{2(\alpha_r + \alpha_q)} \xi \geq 0. \]

Since \((L_x, \xi)(R_x, \xi) \subset \mathcal{O}_{\xi/\alpha_r}(\alpha_p/\alpha_r, \alpha_q/\alpha_r)\), case (iii) is settled.

(iv) Here \(x_r = 0\), and thus, by symmetry, it is enough to prove

\((\alpha_p/2 \pm 1/2)\xi, (\alpha_q/2 \pm 1/2)\xi) \in \mathcal{O}_{\xi/\alpha_r}(\alpha_p/\alpha_r, \alpha_q/\alpha_r). \]

Since \((\alpha_q/2 + 1/2)\xi \leq \xi\), case (iii) implies

\((\alpha_p/2 \pm 1/2)\xi, (\alpha_q/2 + 1/2)\xi) \in \mathcal{O}_{\xi/\alpha_r}(\alpha_p/\alpha_r, \alpha_q/\alpha_r). \]

Note that

\[(8) \quad |R_x| - |L_x| = \frac{2\alpha_p}{\alpha_r + \alpha_q} \geq 0 \]

and therefore the inequalities \((\alpha_p/2 - 1/2)\xi \geq L_x\) and \((\alpha_q/2 - 1/2)\xi > -\xi\) imply

\((\alpha_p/2 - 1/2)\xi, (\alpha_q/2 - 1/2)\xi) \in \mathcal{O}_{\xi/\alpha_r}(\alpha_p/\alpha_r, \alpha_q/\alpha_r). \]

The equation of the line through \((\alpha_p/2 + 1/2)\xi, (\alpha_q/2 - 1/2)\xi)\) and \((\xi, 0)\) is

\[ y - \frac{1 - \alpha_q}{1 - \alpha_p} x + \frac{1 - \alpha_q}{1 - \alpha_p} \xi = 0. \]

It intersects \(x = 0\) at \((0, -\xi(1 - \alpha_q)/(1 - \alpha_p))\). Since \(-\xi(1 - \alpha_q)/(1 - \alpha_p) \geq -\xi,\) we obtain

\((0, -\xi(1 - \alpha_q)/(1 - \alpha_p)) \in \mathcal{O}_{\xi/\alpha_r}(\alpha_p/\alpha_r, \alpha_q/\alpha_r). \]

Moreover, \((\xi, 0) \in \mathcal{O}_{\xi/\alpha_r}(\alpha_p/\alpha_r, \alpha_q/\alpha_r)\) and by convexity,

\((\alpha_p/2 + 1/2)\xi, (\alpha_q/2 - 1/2)\xi) \in \mathcal{O}_{\xi/\alpha_r}(\alpha_p/\alpha_r, \alpha_q/\alpha_r). \]

(v) By symmetry it is enough to consider the points

\((\alpha_p/2 \pm 1/2)\xi, (\alpha_q/2 \pm 1/2)\xi, \xi)\).

The square with vertices \((\alpha_p/2 \pm 1/2)\xi, (\alpha_q/2 \pm 1/2)\xi)\) lies in the square with vertices \((\xi(\alpha_p \pm 1)/(1 + \alpha_r), \xi(\alpha_q \pm 1)/(1 + \alpha_r)).\) Therefore, by the convexity of \(\mathcal{G}_{pqr}\), it is enough to consider the points

\((\xi(\alpha_p \pm 1)/(1 + \alpha_r), \xi(\alpha_q \pm 1)/(1 + \alpha_r), \xi). \]

Since

\[ \frac{\alpha_p \pm 1}{1 + \alpha_r} \xi - \frac{\alpha_p}{\alpha_r} \xi = \frac{\pm \alpha_r - \alpha_p}{\alpha_r(1 + \alpha_r)} \xi, \quad \frac{\alpha_q \pm 1}{1 + \alpha_r} \xi - \frac{\alpha_q}{\alpha_r} \xi = \frac{\pm \alpha_r - \alpha_q}{\alpha_r(1 + \alpha_r)} \xi, \]

we have to show that

\[ \left( \frac{\pm \alpha_r - \alpha_p}{\alpha_r(1 + \alpha_r)} \xi, \frac{\pm \alpha_r - \alpha_q}{\alpha_r(1 + \alpha_r)} \xi \right) \in \mathcal{O}_{\xi/\alpha_r} \left( \frac{\alpha_p}{\alpha_r}, \frac{\alpha_q}{\alpha_r} \right). \]

This is easily seen from the expressions for the coordinates of the vertices of \(\mathcal{O}_{\xi/\alpha_r}(\alpha_p/\alpha_r, \alpha_q/\alpha_r). \)
(vi) By symmetry it is enough to consider the points
\[ ((\alpha_p/2 \pm 1/2)\xi, (\alpha_q/2 \pm 1/2)\xi, (\alpha_r/2 \pm 1/2)\xi). \]
Since \((\alpha_r/2 + 1/2)\xi \leq \xi\) and case (v) is settled, it is enough to consider the points
\[ ((\alpha_p/2 \pm 1/2)\xi, (\alpha_q/2 \pm 1/2)\xi, (\alpha_r/2 - 1/2)\xi). \]
We have
\[ \left( \frac{\alpha_p}{2} \pm \frac{1}{2} \right) \xi - \frac{\alpha_p}{\alpha_r} \left( \frac{\alpha_r}{2} - \frac{1}{2} \right) \xi = \frac{\alpha_p \pm \alpha_r}{2\alpha_r}, \]
\[ \left( \frac{\alpha_q}{2} \pm \frac{1}{2} \right) \xi - \frac{\alpha_q}{\alpha_r} \left( \frac{\alpha_r}{2} - \frac{1}{2} \right) \xi = \frac{\alpha_q \pm \alpha_r}{2\alpha_r}. \]
It is easy to verify that
\[ \left( \frac{\alpha_p \pm \alpha_r}{2\alpha_r}, \frac{\alpha_q \pm \alpha_r}{2\alpha_r} \right) \in \mathcal{O}_{\xi/\alpha_r} \left( \frac{\alpha_p}{\alpha_r}, \frac{\alpha_q}{\alpha_r} \right). \]

10. Properties of \( \mathcal{M}_\xi \)

**Lemma 11.** \( \xi F \subset \mathcal{M}_\xi \).

**Proof.** By the definition of \( \mathcal{M}_\xi \) we have to show that \( \xi F \subset \mathcal{G}_{pqr} \) for all \( p, q, r \in \{1, \ldots, k+1\} \) such that \( p < q < r \) and the set \( \mathcal{G}_{pqr} \) is defined. Let \( \mathbf{x} = (x_1, \ldots, x_k) \) be a vertex of the polytope \( \xi F \). By the definition of \( F \) either
\[ \mathbf{x} = \pm (\xi, 0, \ldots, 0) \quad \text{or} \quad \mathbf{x} = \pm (\xi v_1, \ldots, \xi v_{i-1}, \xi, 0, \ldots, 0), \]
where \((v_1, \ldots, v_{i-1})\) is a vertex of \( \mathcal{C}_{i-1}^i(\alpha_1/2, \ldots, \alpha_{i-1}/2) \). Therefore in the case \( p, q, r \in \{1, \ldots, k+1\}, p < q < r \) and \( r = k+1 \), \( \mathbf{x} \) is as in Lemma 9. Thus \( \mathbf{x} \in \mathcal{G}_{pqr} \). For \( p, q, r \in \{1, \ldots, k\} \) with \( p < q < r \), \( \mathbf{x} \) is as in Lemma 10. Thus also \( \mathbf{x} \in \mathcal{G}_{pqr} \) and by the convexity of \( \mathcal{G}_{pqr} \) we have \( \xi F \subset \mathcal{G}_{pqr} \). ■

**Lemma 12.** \( \xi F \neq \mathcal{M}_\xi \) and therefore \( V_k(\mathcal{M}_\xi) > (k+1)\xi^k \).

**Proof.** We shall find a plane \( P \), such that \( \xi F \cap P \neq \mathcal{M}_\xi \cap P \). Let
\[ P = \{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_3 = x_4 = \ldots = x_k = 0\}. \]
Then
\[ \xi F \cap P = \{(x_1, \ldots, x_k) \in \mathbb{R}^k : (x_1, x_2) \in \mathcal{X}\}, \]
where \( \mathcal{X} \) is a polygon. Let us find the set \( \mathcal{M}_\xi \cap P \). By definition, \( \mathcal{M}_\xi = \bigcap \mathcal{G}_{pqr} \), thus
\[ \mathcal{M}_\xi \cap P = \left( \bigcap \mathcal{G}_{pqr} \right) \cap P = \bigcap (\mathcal{G}_{pqr} \cap P). \]
If \( r = k+1 \), then
Therefore, while if $r \leq k$ then

$$G_{pq} \cap P = \begin{cases} \{(x_1, \ldots, x_k) \in \mathbb{R}^k : (x_1, x_2) \in \mathcal{B}_\xi(\alpha_1, \alpha_2)\} & \text{for } p = 1, q = 2, \\ \{(x_1, \ldots, x_k) \in \mathbb{R}^k : |x_1| \leq \xi\} & \text{for } p = 1, q > 2, \\ \{(x_1, \ldots, x_k) \in \mathbb{R}^k : |x_2| \leq \xi\} & \text{for } p = 2, q > 2, \\ P & \text{for } p, q > 2. \end{cases}$$

Thus, if $\mathcal{Y}$ is a polygon then $\mathcal{Y} = S_1$. But by Lemma 2 of [1], $\mathcal{B}_\xi(\alpha_1, \alpha_2) \neq S_1$, hence $\mathcal{Y} \neq S_1$ and thus $\mathcal{X} \neq \mathcal{Y}$. Therefore $\xi F \cap P \neq \mathcal{M}_\xi \cap P$ and, finally, $\xi F \neq M_{\xi}$.

11. Proof of the main theorem. Let

$$\xi = \frac{2}{((k+1)n_{k+1})^{1/k}}.$$ 

Then by Lemma 12 there exists $\varepsilon > 0$ such that

$$V_k(M_{\xi-\varepsilon}) > 2^k/n_{k+1},$$ 

where $M_{\xi-\varepsilon}$ is defined on replacing $\xi$ by $\xi - \varepsilon$ in the definition of $M_{\xi}$.

Consider the $(k+1)$-dimensional cylinder $D$ with base $M_{\xi-\varepsilon}$ and axis

$$\tau \alpha_1, \ldots, \tau \alpha_k, \tau, \quad \tau \in (-n_{k+1}, n_{k+1}).$$
Then $D$ is a centrally symmetric, convex set. By (10),
\[ V_{k+1}(D) = 2n_{k+1}V_k(M_{\xi - \varepsilon}) > 2^{k+1}. \]
In virtue of Minkowski’s first theorem, the set $D$ contains a non-zero integer vector $m = (m_1, \ldots, m_{k+1})$. By (11) we find $|m_{k+1}| < n_{k+1}$. Therefore in view of condition $\gcd(n_1, \ldots, n_{k+1}) = 1$, the vectors $m$ and $n$ are linearly independent.

Consider the hyperplane $H(x_{k+1} = m_{k+1})$ with induced coordinates $x_1, \ldots, x_k$. On this hyperplane the point $(m_1, \ldots, m_k)$ lies in the translation of $M_{\xi - \varepsilon}$ with center at $(m_{k+1}\alpha_1, \ldots, m_{k+1}\alpha_k)$ and therefore the point $(\delta_1, \ldots, \delta_k)$, where $\delta_1, \ldots, \delta_k$ are defined by (3), lies in $M_{\xi - \varepsilon}$. Consider the polygon $P$ defined by (2). By the definition of $M_{\xi - \varepsilon}$,
\[ \Delta(P) \geq \frac{1}{n_{k+1}(\xi - \varepsilon)} > \frac{1}{n_{k+1}\xi}. \]
$P$ has anomaly 1 (see [4, Ch. 3, §18, Theorem 5]). In virtue of Minkowski’s second theorem there exist two linearly independent integer vectors $(x_1, y_1)$ and $(x_2, y_2)$ such that
\[ |m_{i}y_{j} - n_{i}x_{j}| \leq \lambda_{j}, \quad i \in \{1, \ldots, k + 1\}, \quad j \in \{1, 2\} \]
and
\[ \lambda_1\lambda_2\Delta(P) \leq 1. \]
Put
\[ p = y_1m - x_1n, \quad q = y_2m - x_2n. \]
Then $p$, $q$ are linearly independent, and by Theorem 2 of [7] we can assume without loss of generality that
\[ n = up + vq, \quad u, v \in \mathbb{Z}. \]
Finally, by (12)–(14) we obtain
\[ h(p)h(q) \leq \lambda_1\lambda_2 < n_{k+1}\xi = \frac{2}{(k+1)^{1/k}n_{k+1}^{1-1/k}} = \frac{2}{(k+1)^{1/k}(h(n))^{1-1/k}}. \]

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