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Simultaneous Pell equations

by

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1. Introduction. In this paper, we investigate positive integer solutions (x, y, z) of some special forms of the simultaneous Diophantine equations

$$\begin{cases} ax^2 - bz^2 = \delta_1, \\ cy^2 - dz^2 = \delta_2, \end{cases} \tag{1}$$

where a, b, c and d are positive integers, δ_1 and δ_2 are integers such that $\gcd(ab, \delta_1) = \gcd(cd, \delta_2) = 1$. By work of Thue [10] and Siegel [8], (1) has at most finitely many solutions if $(b, \delta_1) \neq k(d, \delta_2)$, where k is an integer. Considering (1) as an elliptic equation $ac(xy)^2 = (bz^2 + \delta_1)(dz^2 + \delta_2)$, one may apply the theory of linear forms in logarithms to effectively bound all solutions (x, y, z) of (1) and we can study the solutions of (1) via the arithmetic elliptic curves. The usual way to solve (1) completely is to combine lower bounds for the linear forms in logarithms of algebraic numbers with techniques from computational Diophantine approximations. Anglin [2] devotes Section 4.6 of his textbook to the description of an algorithm for solving some special forms of (1). For elementary arguments in certain cases of (1), see Walsh [11, 12], Bennett and Walsh [5] and the author's [14].

For the special Diophantine equations

$$x^2 - az^2 = y^2 - bz^2 = 1 \tag{2}$$

where a and b are distinct positive integers, Anglin [1] showed that (2) has at most one positive solution (x, y, z) whenever $\max(a, b) \leq 200$. Bennett [4], sharpening work of Masser and Rickert [6], proved that (2) has at most 3 positive solutions. The author [13], by using a different gap principle (to ensure that solutions do not lie too close together), showed that (2) has at most 2 positive solutions (x, y, z) provided that $\max(a, b) > 1.4 \cdot 10^{57}$.

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In the present paper, we will study the more general simultaneous Pell equations (1). In Section 2, some general information on (1) with $\delta_i \in \{\pm 1, \pm 2, \pm 4\}$, $i = 1, 2$, is obtained. On the other hand, in Theorems 2.1 and 2.2 we precisely say when (1) has infinitely many integer solutions (x, y, z). Section 3 is devoted to the simultaneous Pell equations

$$ax^2 - bz^2 = cy^2 - dz^2 = 1 \tag{3}$$

where a, b, c and d are positive integers with $b \neq d$. In Section 4 we obtain some similar results to those in Section 3, but on the simultaneous Pell equations

$$x^2 - ay^2 = y^2 - bz^2 = 1 \tag{4}$$

where a and b are positive integers.

In Section 5, by using the same idea as in [13], we apply a result of Baker and Wüstholz [3], namely a lower bound for linear forms in logarithms of three algebraic numbers, to prove the following main theorems of this paper. Denote by $N(a, b, c, d)$ and $N(a, b)$ the number of positive integer solutions of (3) and (4), respectively. We have:

THEOREM 1.1. *Let a, b, c and d be positive integers with $b \neq d$ and $\max(a, b, c, d) \geq 1.16 \cdot 10^{59}$. Then $N(a, b, c, d) \leq 2$.*

Theorem 1.1 is a generalization of Theorem 1.4 of [13]. For equations (4), we have similar results which slightly improve Theorem 7.1 of [4]:

THEOREM 1.2. *Let a and b be positive integers with $a > 3.31 \cdot 10^{35}$. Then $N(a, b) \leq 2$.*

THEOREM 1.3. *Let a and b be positive integers and let $x_1 + y_1\sqrt{a}$ be the fundamental solution of $x^2 - ay^2 = 1$ (i.e. x_1 and y_1 are the smallest positive integers satisfying $x_1^2 - ay_1^2 = 1$). If $y_1 > a^{0.005}$ and $a > 6.4 \cdot 10^{2326}$, then $N(a, b) \leq 1$.*

Of course, we can obtain similar results for other forms of equation (1). Since the method is essentially similar, we omit them here.

For positive integers $l > 1$, $m > 1$ and $a > 1$, let $n(l, m)$ and $c(l, a)$ be integers with

$$n(l, m) = \frac{(m + \sqrt{m^2 - 1})^{2l} - (m - \sqrt{m^2 - 1})^{2l}}{4\sqrt{m^2 - 1}}$$

and

$$4c(l, a) - 1 = \frac{(\sqrt{a} + \sqrt{a - 1})^l - (\sqrt{a} - \sqrt{a - 1})^l}{2\sqrt{a - 1}}, \quad l \equiv 3 \pmod 4.$$

Then the simultaneous Pell equations

$$x^2 - (m^2 - 1)z^2 = y^2 - (n(l, m)^2 - 1)z^2 = 1, \quad x, y, z \in \mathbb{Z}$$

and

$$ax^2 - (a-1)z^2 = c(l,a)y^2 - (c(l,a)-1)z^2 = 1, \qquad x, y, z \in \mathbb{Z}$$

have two positive solutions

$$(x,y,z) = (m, n(l,m), 1),$$
$$(x,y,z) = \left(\frac{(m+\sqrt{m^2-1})^{2l} + (m-\sqrt{m^2-1})^{2l}}{2}, 2n(l,m)^2 - 1, 2n(l,m)\right)$$

and

$$(x,y,z) = (1,1,1),$$
$$(x,y,z) = \left(\frac{(\sqrt{a}+\sqrt{a-1})^{l} + (\sqrt{a}-\sqrt{a-1})^{l}}{2\sqrt{a}}, 4c(l,a) - 3, 4c(l,a) - 1\right),$$

respectively. We call (a_1, b_1, c_1, d_1) an *equivalent form* of (a,b,c,d) if $(a_1, b_1, c_1, d_1) = (a/a_0^2, b/b_0^2, c/c_0^2, d/d_0^2)$, where a_0, b_0, c_0, d_0 are positive integers. We think a more general result is true.

CONJECTURE 1.1. *Apart from*

$$(a,b,c,d) = (1, m^2-1, 1, n^2(l,m) - 1), \ (a, a-1, c(l,a), c(l,a) - 1)$$

and their equivalent forms, $N(a,b,c,d) \le 1$.

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2. General results. Let $\delta_i \in \{\pm 1, \pm 2, \pm 4\}$, $i = 1, 2$, a, b, c, and d be positive integers such that neither ab nor cd is a perfect square and $\gcd(ab, \delta_1) = \gcd(cd, \delta_2) = 1$. In this section, we investigate the following more general simultaneous Pell equations

$$\begin{cases} ax^2 - bz^2 = \delta_1, \\ cy^2 - dz^2 = \delta_2. \end{cases} \tag{5}$$

To discuss solutions (x,y,z) of (5), without loss of generality, we may assume that both $ax^2 - bz^2 = \delta_1$ and $cy^2 - dz^2 = \delta_2$ are solvable in positive integers.

DEFINITION 2.1. If $ax^2 - by^2 = \delta$, $\delta \in \{\pm 1, \pm 2, \pm 4\}$, is solvable in positive integers, let $x_0\sqrt{a} + y_0\sqrt{b}$ be the smallest value of $x\sqrt{a} + y\sqrt{b}$ such that (x,y) is a positive solution of $ax^2 - by^2 = \delta$. Then $x_0\sqrt{a} + y_0\sqrt{b}$ is said to be the *smallest solution* of this equation.

Let $x_0\sqrt{a}+z_0\sqrt{b}$ and $y_0\sqrt{c}+z_0^\star\sqrt{d}$ be the smallest solutions of $ax^2-bz^2=\delta_1$ and $cy^2-dz^2=\delta_2$, respectively. Put

$$\alpha=\frac{x_0\sqrt{a}+z_0\sqrt{b}}{\sqrt{|\delta_1|}},\qquad \beta=\frac{y_0\sqrt{c}+z_0^\star\sqrt{d}}{\sqrt{|\delta_2|}},$$
$$\overline{\alpha}=\frac{x_0\sqrt{a}-z_0\sqrt{b}}{\sqrt{|\delta_1|}},\qquad \overline{\beta}=\frac{y_0\sqrt{c}-z_0^\star\sqrt{d}}{\sqrt{|\delta_2|}}.$$

Define

$$U_l=\begin{cases}\dfrac{\alpha^l-\overline{\alpha}^l}{\sqrt{4b/|\delta_1|}} & \text{if } 2\nmid l \text{ or } (b,\delta_1)\neq(1,-1),(1,-4),\\[2ex] \dfrac{\alpha^l+\overline{\alpha}^l}{\sqrt{4/|\delta_1|}} & \text{if } 2\,|\,l \text{ and } (b,\delta_1)=(1,-1) \text{ or } (1,-4).\end{cases}$$

Similarly, define U_k' to be $(\beta^k-\overline{\beta}^k)/\sqrt{4d/|\delta_2|}$ or $(\beta^k+\overline{\beta}^k)/\sqrt{4/|\delta_2|}$. First we have:

Lemma 2.1 ([9]). *Let $x_1\sqrt{a}+y_1\sqrt{b}$ be the smallest solution of $ax^2-by^2=\delta$, $\delta\in\{1,2,4\}$. Then every positive solution (x,y) of this equation can be given by*

$$\frac{x\sqrt{a}+y\sqrt{b}}{\sqrt{|\delta|}}=\left(\frac{x_1\sqrt{a}+y_1\sqrt{b}}{\sqrt{|\delta|}}\right)^n,\qquad n>0,$$

with $2\nmid n$ if $\min(a,b)>1$ or $(a,\delta)\neq(1,1),(1,4)$.

Recall that if $(b,\delta_1)\neq k(d,\delta_2)$, then (1) has only finitely many solutions. If $(b,\delta_1)=k(d,\delta_2)$ and (1) has infinitely many solutions, then $ax^2=kcy^2$. Therefore, without loss of generality we may assume that $k=1$. We have:

Theorem 2.1. *Let a, b and c be positive integers, $\delta\in\{\pm1,\pm2,\pm4\}$ such that neither ab nor bc is a perfect square and $\gcd(abc,\delta)=1$. Then the simultaneous Pell equations*

$$\begin{cases}ax^2-bz^2=\delta,\\ cy^2-bz^2=\delta\end{cases}\tag{6}$$

have a positive integer solution (x,y,z) if and only if each equation in (6) is solvable in positive integers and ac is a perfect square. Moreover, if the system has a positive integer solution, then it has infinitely many integer solutions (x,y,z).

Proof. It suffices to prove that if each equation in (6) is solvable and ac is a perfect square, then (6) have infinitely many integer solutions. Let $a=a_0a_1^2$ with a_0 square-free. Since ac is a perfect square, a_0 is moreover the square-free part of c, say, $c=a_0c_1^2$. Let $x_0'\sqrt{a_0}+z_0'\sqrt{b}$ be the smallest solution of $a_0x^2 - bz^2 = \delta$. From Lemma 2.1 we have

$$\frac{x_0\sqrt{a}+z_0\sqrt{b}}{\sqrt{|\delta|}} = \left(\frac{x_0'\sqrt{a_0}+z_0'\sqrt{b}}{\sqrt{|\delta|}}\right)^m, \qquad \frac{y_0\sqrt{c}+z_0^\star\sqrt{b}}{\sqrt{|\delta|}} = \left(\frac{x_0'\sqrt{a_0}+z_0'\sqrt{b}}{\sqrt{|\delta|}}\right)^n$$

for some positive integers m and n. Put

$$\frac{x_t\sqrt{a_0}+z_t\sqrt{b}}{\sqrt{|\delta|}} = \left(\frac{x_0'\sqrt{a_0}+z_0'\sqrt{b}}{\sqrt{|\delta|}}\right)^{mnt}, \qquad t = 1, 2, \ldots.$$

Then $a_1 \mid x_t$ and $c_1 \mid x_t$. Hence $(x_t/a_1, x_t/c_1, z_t)$ is a positive integer solution of (6) for every $t = 1, 2, \ldots$. ■

More generally we have:

THEOREM 2.2. *Let a, b and c be positive integers, δ a nonzero integer such that neither ab nor bc is a perfect square and* $\gcd(abc, \delta) = 1$. *Then the simultaneous Pell equations*

$$(7) \qquad \begin{cases} ax^2 - bz^2 = \delta, \\ cy^2 - bz^2 = \delta \end{cases}$$

have a positive integer solution (x, y, z) only if each equation in (7) is solvable in positive integers and ac is a perfect square. Moreover, if they have a positive integer solution, then they have infinitely many integer solutions (x, y, z).

Proof. It suffices to prove the last assertion. Let $a = a_0a_1^2$, $c = a_0c_1^2$ with a_0 square-free, and let (x, y, z) be a positive solution of (7). Then $x\sqrt{a}+z\sqrt{b} = y\sqrt{c}+z\sqrt{b}$. Let $x_0 + y_0\sqrt{a_0ba_1^2c_1^2}$ be the fundamental solution of $x^2 - a_0ba_1^2c_1^2y^2 = 1$. Put

$$x_t\sqrt{a_0} + z_t\sqrt{b} = (x\sqrt{a} + z\sqrt{b})(x_0 + y_0\sqrt{a_0ba_1^2c_1^2})^t, \quad t = 1, 2, \ldots.$$

It is easy to see that $a_1 \mid x_t$, $c_1 \mid x_t$ and $(x_t/a_1, x_t/c_1, z_t)$ is a positive integer solution of (7) for every $t = 1, 2, \ldots$. ■

By Theorems 2.1 and 2.2, we may assume that $(b, \delta_1) \neq (d, \delta_2)$ throughout the paper. We have the following lemmas:

LEMMA 2.2 ([7]). *If $(b, \delta_1) \neq (1, -1), (1, -4)$ and $m \mid n$, or n/m is an odd integer, then $U_m \mid U_n$.*

LEMMA 2.3. *Let k_0, k_1, k_2 and q be positive integers with $k_2 = 2qk_1 \pm k_0$, $0 \le k_0 \le k_1$. Then $U_{k_2} \equiv \pm U_{k_0} \pmod{U_{k_1}}$.*

Proof. Note that k_0 and k_2 have the same parities. We divide the proof into two cases.

Case I: $2\nmid k_2$ or $(b,\delta_1)\neq(1,-1)$ or $(1,-4)$. We have

$$\frac{\alpha^{2qk_1+k_0}-\overline{\alpha}^{2qk_1+k_0}}{\sqrt{4b/|\delta_1|}}-(\alpha\overline{\alpha})^{qk_1}\frac{\alpha^{k_0}-\overline{\alpha}^{k_0}}{\sqrt{4b/|\delta_1|}}=\frac{(\alpha^{qk_1+k_0}+\overline{\alpha}^{qk_1+k_0})(\alpha^{qk_1}-\overline{\alpha}^{qk_1})}{\sqrt{4b/|\delta_1|}}$$

and

$$\frac{\alpha^{2qk_1-k_0}-\overline{\alpha}^{2qk_1-k_0}}{\sqrt{4b/|\delta_1|}}+(\alpha\overline{\alpha})^{qk_1-k_0}\frac{\alpha^{k_0}-\overline{\alpha}^{k_0}}{\sqrt{4b/|\delta_1|}}=\frac{(\alpha^{qk_1-k_0}+\overline{\alpha}^{qk_1-k_0})(\alpha^{qk_1}-\overline{\alpha}^{qk_1})}{\sqrt{4b/|\delta_1|}}.$$

Thus $U_{k_2}\equiv\pm U_{k_0}\ (\operatorname{mod} U_{k_1})$.

Case II: $2\mid k_2$ and $(b,\delta_1)=(1,-1)$ or $(1,-4)$. If $2\mid q$ or $2\nmid k_1$, then

$$\frac{\alpha^{2qk_1\pm k_0}+\overline{\alpha}^{2qk_1\pm k_0}}{\sqrt{4/|\delta_1|}}-(\alpha\overline{\alpha})^{qk_1}\frac{\alpha^{k_0}+\overline{\alpha}^{k_0}}{\sqrt{4/|\delta_1|}}=\frac{(\alpha^{qk_1\pm k_0}-\overline{\alpha}^{qk_1\pm k_0})(\alpha^{qk_1}-\overline{\alpha}^{qk_1})}{\sqrt{4/|\delta_1|}}.$$

If $2\nmid q$ and $2\mid k_1$, then

$$\frac{\alpha^{2qk_1\pm k_0}+\overline{\alpha}^{2qk_1\pm k_0}}{\sqrt{4/|\delta_1|}}+(\alpha\overline{\alpha})^{qk_1}\frac{\alpha^{k_0}+\overline{\alpha}^{k_0}}{\sqrt{4b/|\delta_1|}}=\frac{(\alpha^{qk_1\pm k_0}+\overline{\alpha}^{qk_1\pm k_0})(\alpha^{qk_1}+\overline{\alpha}^{qk_1})}{\sqrt{4/|\delta_1|}}.$$

Thus again $U_{k_2}\equiv\pm U_{k_0}\ (\operatorname{mod} U_{k_1})$. ■

Lemma 2.4. *Let the notations be as above and $(b,\delta_1)\neq(d,\delta_2)$. Let z_1 be the smallest positive integer z of the solutions (x,y,z) of* (5). *Then $z_1\mid z$ for any solution (x,y,z) of* (5).

Proof. Let (x_1,y_1,z_1) be the positive solution of (5) with smallest positive integer z, and (x,y,z) be any solution of (5). Then from the definitions of U_l and U'_k, we have

$$z_1=U_{l_1}=U'_{k_1},\qquad z=U_l=U'_k$$

for some positive integers l, k, l_1 and k_1. If $l_1\mid l$ and l/l_1 is odd, then $z_1\mid z$, whence $k_1\mid k$ and k/k_1 is odd. If $k_1\mid k$ and k/k_1 is odd, then $z_1\mid z$, whence $l_1\mid l$ and l/l_1 is odd.

Suppose that $z_1\nmid z$. By the above discussion there are positive integers q_1, q, l_0 and l such that

$$l=2q_1l_1\pm l_0,\quad 0\le l_0<l_1,\qquad k=2qk_1\pm k_0,\quad 0\le k_0<k_1.$$

By Lemma 2.3 we have $z\equiv U_l\equiv\pm U_{l_0}\ (\operatorname{mod} U_{l_1})$ and $z\equiv\pm U'_{k_0}\ (\operatorname{mod} U'_{k_1})$. Hence

$$U_{l_0}\equiv\pm U'_{k_0}\ (\operatorname{mod} z_1). \tag{8}$$

If $2\mid ll_1$, then $(b,\pm\delta_1)=(1,-1)$ or $(1,-4)$, and $\alpha=(u+v\sqrt{a})/2\ge(3+\sqrt{5})/2$, where u and v are positive integers with $u^2-v^2a=4$. If $2\nmid ll_1$, then k_0 is odd, and $k_0\le k_1-2$, and $\alpha=(u\sqrt{a}+v\sqrt{b})/|\delta_1|\ge(\sqrt{5}+1)/2$, where u and v are positive integers with $u^2a-v^2b=\delta_1$. Hence in both cases we have $U_{l_0} < \frac{1}{2}U_{l_1}$. Similarly, $U'_{k_0} < \frac{1}{2}U'_{k_1}$. Hence $\max(U_{l_0}, U'_{k_0}) < \frac{1}{2}\max(U_{l_1}, U'_{k_1}) = \frac{1}{2}z_1$, and so (8) holds if and only if $U_{l_0} = U'_{k_0}$. Therefore equations (5) have a solution (x_0, y_0, z_0) with $z_0 = U_{l_0} = U'_{k_0} < z_1$. This contradicts our assumption that z_1 is the smallest such solution, so $z_1 \mid z$, which proves our lemma. ∎

Thanks to Lemma 2.4, when considering the number of solutions (x, y, z) of (5), without loss of generality, we may assume that $(a, c, b, d, \delta_1, \delta_2) = (1, 1, m^2 - \delta_1, n^2 - \delta_2, \delta_1, \delta_2)$ or $(m^2 + \delta_1, n^2 + \delta_2, 1, 1, -\delta_1, -\delta_2)$, $\delta_i \in \{1, 4\}$, $i = 1, 2$, or $(a, c, a - \delta_1, c - \delta_2, \delta_1, \delta_2)$, $\delta_i \in \{\pm 1, \pm 2, \pm 4\}$, $i = 1, 2$. We will keep this assumption hereafter, whereby (5) has a trivial solution.

3. Lemmas for $ax^2 - bz^2 = cy^2 - dz^2 = 1$. Throughout this section we assume that $c > a > 1$, $\alpha = \sqrt{a} + \sqrt{a-1}$ and $\beta = \sqrt{c} + \sqrt{c-1}$. Suppose that (x, y, z) is a positive integer solution of (3); then

$$z = \frac{\alpha^l - \alpha^{-l}}{2\sqrt{a-1}} = \frac{\beta^k - \beta^{-k}}{2\sqrt{c-1}} \tag{9}$$

for some positive odd integers l and k. Since $c > a$, from (9) it is clear that

$$\sqrt{\frac{c-1}{a-1}}\,\alpha^l > \beta^k > \alpha^l, \qquad \left(\frac{\beta}{\alpha}\right)^2 > \sqrt{\frac{c-1}{a-1}},$$

so if $k > 1$ and $l > 1$, then $l > k$.

Let

$$\Lambda = \frac{1}{2}\log\frac{c-1}{a-1} + l\log\alpha - k\log\beta. \tag{10}$$

Then (9) implies that

$$0 < \Lambda = \log(1 - \beta^{-2k}) - \log(1 - \alpha^{-2l}) < -\log(1 - \alpha^{-2l}) < \frac{\alpha^2}{\alpha^2 - 1}\,\alpha^{-2l}.$$

It follows that

$$\log\Lambda < -2l\log\alpha + \log\frac{\alpha^2}{\alpha^2 - 1}. \tag{11}$$

Suppose that $N(a, c) \ge 3$. Let (x_i, y_i, z_i) $(i = 1, 2, 3)$ be the first three positive solutions of (3), say,

$$z_i = \frac{\alpha^{l_i} - \alpha^{-l_i}}{2\sqrt{a-1}} = \frac{\beta^{k_i} - \beta^{-k_i}}{2\sqrt{c-1}}$$

for some positive integers l_i and k_i $(i = 1, 2, 3)$ with $1 = k_1 < k_2 < k_3$ and $1 = l_1 < l_2 < l_3$. By the same discussions as in the proof of Lemma 2.4, we have:

LEMMA 3.1. *With the above notations, either $l_2 \mid l_3$ and $k_2 \mid k_3$, or $l_3 = 2ql_2 \pm 1$ and $k_3 = 2q_1k_2 \pm 1$ for some positive integers q and q_1.*

Proof. If $l_2 \mid l_3$, then $z_2 \mid z_3$, whence $k_2 \mid k_3$. Conversely, if $k_2 \mid k_3$, then $l_2 \mid l_3$. Now if $l_2 \nmid l_3$, then $k_2 \nmid k_3$, and let

$$l_3 = 2ql_2 \pm l_0, \quad 0 < l_0 < l_2, \qquad k_3 = 2q_1k_2 \pm k_0, \quad 0 < k_0 < k_2,$$

for some positive integers q, q_1, k_0 and l_0. By the same argument as in the proof of Lemma 2.4, we have $l_0 = k_0 = 1$, and the same plus or minus sign occurs by Lemma 2.3 (Case I). ∎

LEMMA 3.2. *If $k_2 \neq 3$, then $l_3 > 3.5 \cdot l_2\beta$.*

Proof. We assume first that $l_2 \mid l_3$. By Lemma 3.1 we have $l_3 = ql_2$, $k_3 = q_1k_2$ for odd positive integers q and q_1. Therefore

$$\frac{z_3}{z_2} = \frac{U_{ql_2}}{U_{l_2}} = \frac{U'_{q_1k_2}}{U'_{k_2}},$$

which implies that $q > q_1$. Considering these equations modulo z_2^2, we have

$$q(ax_2^2)^{(q-1)/2} \equiv q_1(cy_2^2)^{(q_1-1)/2} \pmod{z_2^2}. \tag{12}$$

Since $ax_2^2 \equiv cy_2^2 \equiv 1 \pmod{z_2^2}$, we get $q \equiv q_1 \pmod{z_2^2}$. Hence $q > z_2^2 > \beta^8$ and

$$l_3 > l_2\beta^8.$$

Next assume that $l_2 \nmid l_3$. By Lemma 3.1 we have $l_3 = 2ql_2 \pm 1$ and $k_3 = 2q_1k_2 \pm 1$ for some positive integers q and q_1. From $z_3 = U'_{k_3} = U_{l_3}$ we have $q > q_1$. Note that $\beta^{2k_2} = 2z_2^2(c-1) + 1 + 2y_2z_2\sqrt{c(c-1)}$, so

$$z_3 = U'_{k_3} \equiv 2cq_1y_2z_2 \pm 1 \pmod{2z_2^2(c-1)}. \tag{13}$$

Similarly,

$$z_3 = U_{l_3} \equiv 2aqx_2z_2 \pm 1 \pmod{2z_2^2(a-1)}. \tag{14}$$

From (13) and (14) we get

$$aqx_2 \equiv cq_1y_2 \pmod{z_2}. \tag{15}$$

Since $ax_2^2 \equiv cy_2^2 \equiv 1 \pmod{z_2^2}$, we have

$$cq_1^2 \equiv aq^2 \pmod{z_2}.$$

If $aq^2 \neq cq_1^2$, then $cq^2 > \max(cq_1^2, aq^2) > z_2 > \beta^4 + \beta^2 + 1$ and $l_3 = 2ql_2 \pm 1 \geq 3.5 \cdot l_2\beta$.

If $aq^2 = cq_1^2$, then $a = a_1^2u$, $c = c_1^2u$, $q = c_1t$, $q_1 = a_1t$, where a_1, c_1, u, t are positive integers with $\gcd(a_1, c_1) = 1$ and $a_1 < c_1$. Since $u(a_1^2x_2^2 - c_1^2y_2^2) = ax_2^2 - cy_2^2 = (a-c)z_2^2 = u(a_1^2 - c_1^2)z_2^2$ and $\gcd(z_2, a_1c_1ux_2y_2) = 1$, we see that

$$a_1x_2 + c_1y_2 = r\xi^2, \qquad c_1y_2 - a_1x_2 = s\eta^2, \qquad \gcd(\xi, s\eta) \mid 2, \tag{16}$$

where r, s, ξ, η are positive integers such that $z_2 = \xi\eta$ or $2\xi\eta$ and $rs = c_1^2 - a_1^2$ or $4(c_1^2 - a_1^2)$. Now, by (15), $a_1c_1u(a_1x_2 - c_1y_2)t \equiv 0 \pmod{z_2}$, and so $(c_1y_2 - a_1x_2)t \equiv 0 \pmod{z_2}$, hence $2t \equiv 0 \pmod{\xi}$. Therefore

$$cq^2 = c_1^4ut^2 \ge \frac{c_1^4u}{4}\,\xi^2 \ge \frac{c_1^4u}{16(c_1^2 - a_1^2)}\,(a_1x_2 + c_1y_2) > z_2$$

and $l_3 \ge 3.5 \cdot l_2\beta$. The lemma is proved. ∎

If $k_2 = 3$, then $z_2 = U_3' = 4c - 1$. However:

LEMMA 3.3. *If $k_2 = 3$ and $\beta > 1000$, then $l_3 > 1.8 \cdot l_2\beta^{2/3}$.*

Proof. The proof is similar to that of Lemma 3.2. If $3 \mid k_2$, let $k_3 = 2q_1$ and $l_3 = l_2q$ for some positive integers q and q_1, so that

$$q(ax_2^2)^{(q-1)/2} \equiv q_1(cy_2^2)^{(q_1-1)/2} \pmod{z_2^2}.$$

Note that $z_2 = 4c - 1$, and $ax_2^2 \equiv cy_2^2 \equiv 1 \pmod{z_2^2}$, so $q > z_2^2 = (4c-1)^2 > \beta^4$, and $l_3 = ql_2 > l_2\beta^4$. If $3 \nmid k_3$, let $k_3 = 6q_1 \pm 1$ and $l_3 = 2ql_2 \pm 1$ for some positive integers q and q_1. We have

$$z_3 \equiv 2cq_1y_2z_2 \pm 1 \equiv 2aqx_2z_2 \pm 1 \pmod{z_2^2},$$

and so

$$cq_1y_2 \equiv aqx_2 \pmod{z_2}. \tag{17}$$

Since $z_2 = 4c - 1$, we get $l \ge 7$ and $ax_2^2 \equiv cy_2^2 \equiv 1 \pmod{z_2^2}$. We thus have

$$q_1^2 \equiv 4cq_1^2 \equiv 4aq^2 \pmod{z_2},$$

whereby it follows that $q^2 \ge c/a > 0.9\beta^{2/3}$ ($\beta > 1000$) and $l_3 = 2l_2q \pm 1 > 1.8 \cdot l_2\beta^{2/3}$. The lemma is proved. ∎

4. Some lemmas for $x^2 - ay^2 = y^2 - bz^2 = 1$. In this section we give some lemmas related to the simultaneous equations

$$\begin{cases} x^2 - ay^2 = 1, \\ y^2 - bz^2 = 1. \end{cases} \tag{18}$$

By Lemma 2.4 we may assume that $\alpha = x_1 + y_1\sqrt{a}$ and $\beta = y_1 + \sqrt{y_1^2 - 1}$, where $b = y_1^2 - 1$. Suppose that (x, y, z) is a positive integer solution of (4). Then

$$y = \frac{\alpha^l - \alpha^{-l}}{2\sqrt{a}} = \frac{\beta^k + \beta^{-k}}{2} \tag{19}$$

for some positive odd integers l and k. From (19) we have

$$\sqrt{\frac{1}{a}}\,\alpha^l > \beta^k. \tag{20}$$

Hence if $k > 1$ and $l > 1$, then $k > l$.

Let

$$\Lambda = l\log\alpha - k\log\beta - \frac{1}{2}\log a. \tag{21}$$

Then (19) implies that

$$0 < \Lambda = \log(1+\beta^{-2k}) + \log(1-\alpha^{-2l}) < \overline{\beta}^{2k} + \frac{\alpha^2}{\alpha^2-1}\,\alpha^{-2l} < 1.5\cdot\overline{\beta}^{2k}.$$

Hence

$$\log\Lambda < -2k\log\beta + \log 1.5. \tag{22}$$

Suppose that $N_1(a,b) \ge 3$. Let (x_i, y_i, z_i) $(i = 1,2,3)$ be the first three positive solutions of (4), say,

$$z_i = \frac{\alpha^{l_i} - \alpha^{-l_i}}{2\sqrt{a}} = \frac{\beta^{k_i} + \beta^{-k_i}}{2}$$

for some positive integers l_i and k_i $(i = 1,2,3)$ with $1 = l_1 < l_2 < l_3$ and $1 = k_1 < k_2 < k_3$. By the same arguments as in Section 3, we have:

LEMMA 4.1. *With the above notations, either $l_2 \mid l_3$ and $k_2 \mid k_3$, or $l_3 = 2ql_2 \pm 1$ and $k_3 = 2q_1k_2 \pm 1$ for some positive integers q and q_1.*

LEMMA 4.2. *Suppose that (x,y,z) is a positive integer solution of* (18) *and $y = U_l = V_k'$. Then $l \equiv 1 \pmod 4$ and $k > 2y_1^2$.*

Proof. Note that from $\alpha^2 = 2ay_1^2 + 1 + 2x_1y_1\sqrt{a}$, we get $U_l/y_1 \equiv l \pmod{4y_1^2}$. Similarly, since $\beta^2 = 2y_1^2 - 1 + 2y_1\sqrt{y_1^2-1}$ we have $V_k'/y_1 \equiv (-1)^{(k-1)/2}k \pmod{4y_1^2}$. Therefore $l \equiv (-1)^{(k-1)/2}k \pmod{4y_1^2}$, whence $k > 2y_1^2$. The lemma is proved. ∎

LEMMA 4.3. *$k_3 > 5.5\cdot k_2\alpha$.*

Proof. We assume first that $l_2 \mid l_3$. By Lemma 4.1 we have $l_3 = ql_2$, $k_3 = q_1k_2$ for odd positive integers q and q_1. Therefore

$$\frac{z_3}{z_2} = \frac{U_{ql_2}}{U_{l_2}} = \frac{U'_{q_1k_2}}{U'_{k_2}}, \tag{23}$$

which implies that $q_1 > q$. By the same argument as in the proof of Lemma 4.2 we have

$$q \equiv q_1(-1)^{(q_1-1)/2} \pmod{4y_2^2}. \tag{24}$$

Hence $q > 2y_2^2 > 2\alpha^8$, so $k_3 > 2k_2\alpha^8$.

Next assume that $l_2 \nmid l_3$. By Lemma 4.1 we have $l_3 = 2ql_2 \pm 1$ and $k_3 = 2q_1k_2 \pm 1$ for some positive integers q and q_1. Similarly we have

$$y = V_{k_3} \equiv (-1)^{q_1}(y_1 \mp 2q_1y_2z_2(y_1^2-1)) \pmod{2y_2^2}, \tag{25}$$

$$z_3 = U_{l_3} \equiv \pm y_1 + 2x_1x_2y_2q \pmod{2y_2^2}. \tag{26}$$

Since $y_1 \le y_2/2$, we get

$$\mp(-1)^{q_1}q_1z_2(y_1^2-1) \equiv x_1x_2q \pmod{y_2}. \tag{27}$$

Note that $l \neq 3$ by Lemma 4.2. Since $x_2^2 \equiv 1 \pmod{y_2^2}, z_2^2(y_1^2-1) \equiv -1 \pmod{y_2^2}$, we have

$$q_1^2(y_1^2-1) \equiv x_1^2 q^2 \pmod{y_2}.$$

Since $y_1^2 - 1$ is never a square of an integer when $y_1 > 1$, it follows that $(q_1 y_1)^2$ or $(qx_1)^2 > y_2 > y_1(\alpha^4+\alpha^2+1)$. Hence $q_1 > q > 2.8 \cdot \alpha$, $k_3 = 2q_1 k_2 \pm 1 > 5.5 \cdot k_2 \alpha$. ■

5. Proofs of the main results. First we recall the following famous result of Baker and Wüstholz [3]. Let $\alpha_1, \ldots, \alpha_n$ (with $n \geq 2$) denote algebraic numbers different from 0 and 1. Let $K = Q(\alpha_1, \ldots, \alpha_n)$ and set $d = [K : Q]$. Define a modified height by the formula

$$h_m(\alpha) = \max\{h(\alpha), |\log \alpha|/d, 1/d\},$$

where $h(\alpha)$ denotes the standard logarithmic Weil height of an algebraic number α.

THEOREM 5.1 (Baker–Wüstholz [3]). *Let $b_1, \ldots, b_n$ be integers such that*

$$\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n \neq 0.$$

Then if $B = \max\{|b_1|, \ldots, |b_n|\} \geq 3$, we have the inequality

$$\log |\Lambda| > -C_1 h_m(\alpha_1) \cdots h_m(\alpha_n) \log B$$

with

$$C_1 = 18(n+1)! n^{n+1} (32d)^{n+2} \log(2nd).$$

Proof of Theorem 1.1. We apply Theorem 5.1 with

$$\alpha_1 = (c-1)/(a-1), \quad \alpha_2 = \alpha^2, \quad \alpha_3 = \beta^2,$$
$$b_1 = 1, \quad b_2 = l_3, \quad b_3 = -k_3,$$

where $\alpha = \sqrt{a} + \sqrt{a-1}$ and $\beta = \sqrt{c} + \sqrt{c-1}$. Then

$$\Lambda = \log \frac{c-1}{a-1} + l_3 \log \alpha^2 - k_3 \log \beta^2.$$

We may take $d = 4$, and

$$h_m(\alpha_1) = \log(c-1) < 2 \log \beta, \quad h_m(\alpha_2) = \log \alpha,$$
$$h_m(\alpha_3) = \log \beta, \quad B = l_3.$$

Therefore by Theorem 5.1 we have

$$\log |\Lambda| > -7.6420496 \cdot 10^{15} \log \alpha \log^2 \beta \log l_3. \tag{28}$$

If $k_2 \neq 3$, by Lemma 3.2, (11) and (28),

$$l_3 < 3.8210248 \cdot 10^{15} \log^3 l_3.$$

It follows that $l_3 < 4.101 \cdot 10^{20}$. Hence by Lemma 3.2 and $l_2 \geq 7$, we get

$$c < 7.1 \cdot 10^{38}.$$

If $k_3 = 3$, similarly, by Lemma 3.3, we obtain

$$c < 1.16 \cdot 10^{59}. \ \blacksquare$$

Proof of Theorem 1.2. We apply Theorem 5.1 with

$$\alpha_1 = \sqrt{a}, \quad \alpha_2 = \alpha, \quad \alpha_3 = \beta,$$
$$b_1 = -1, \quad b_2 = -l_3, \quad b_3 = k_3, \quad n = 3,$$

where $\alpha = x_1 + y_1\sqrt{a}$ and $\beta = y_1 + \sqrt{y_1^2 - 1}$. Then

$$\Lambda = -\tfrac{1}{2}\log a - l_3 \log \alpha + k_3 \log \beta.$$

Take $d = 4$, and

$$h_m(\alpha_1) = \tfrac{1}{2}\log a < \log \alpha, \quad h_m(\alpha_2) = \tfrac{1}{2}\log \alpha,$$
$$h_m(\alpha_3) = \tfrac{1}{2}\log \beta, \quad B = k_3.$$

By Theorem 5.1 we have

$$\log|\Lambda| > -9.56 \cdot 10^{14} \log^2 \alpha \log \beta \log k_3. \tag{29}$$

On the other hand, by (22),

$$\log|\Lambda| < -2k_3 \log \beta + \log 1.5. \tag{30}$$

By Lemma 4.3, (29) and (30) we get

$$k_3 < 4.78 \cdot 10^{14} \log^3 k_3.$$

It follows that $k_3 < 4.43 \cdot 10^{19}$. Hence by Lemma 4.3 and $k_2 \ge 7$, we obtain

$$ay_1^2 < 3.31 \cdot 10^{35}. \ \blacksquare$$

Proof of Theorem 1.3. Similarly, by Lemma 4.2, (29) and (30) and the assumptions we have

$$k_2 < 4.78 \cdot 10^{18} \log^3 k_2.$$

It follows that $k_2 < 8 \cdot 10^{13}$. Hence by Lemma 4.2,

$$a_1 < 6.34 \cdot 10^{2326}.$$

This completes the proof. $\blacksquare$

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