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Hyperbolicity and integral points off divisors in subgeneral position in projective algebraic varieties

by

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1. Introduction. First of all, we recall some basic notions in Diophantine geometry. For details we refer the readers to [5], [8] and [16].

Let k be a number field. Let $v : k \to [0, \infty)$ be a valuation on k. For each $x \in k$, denote by $|x|_v$ or $v(x)$ the absolute value of x with respect to v.

We denote by M_k a set of representatives of all equivalence classes of nontrivial valuations over k. Let M_k^∞ be the subset of M_k consisting of all archimedean valuations, and M_k^0 be the subset consisting of all nonarchimedean valuations. Then M_k^∞ is finite and $M_k = M_k^\infty \cup M_k^0$.

For each valuation v, we denote by $\overline{k}_v$ the algebraic closure of k_v, and we extend v to $\overline{k}_v$. We also denote by $\overline{k}$ the algebraic closure of k.

Let S be a finite subset of M_k such that $M_k^\infty \subset S$. We set

$$\mathcal{O}_S = \{x \in k \mid |x|_v \le 1 \ \forall v \in M_k \setminus S\}.$$

Then $\mathcal{O}_S$ is a ring, called *the ring of S-integers* of k. A point $x = (x_1, \dots, x_n) \in k^n$ is said to be an *S-integral point* if $x_i \in \mathcal{O}_S$ for all $1 \le i \le n$.

We now recall *the product formula* which is an important fact in Diophantine geometry.

Theorem 1.1. *Let k be a number field. Then for each equivalence class $v \in M_k$ there exists a valuation $\|\cdot\|_v \in v$ such that*

$$\prod_{v \in M_k} \|x\|_v = 1 \quad \textit{for all } x \in k \setminus \{0\}.$$

From now on, we fix for each $v \in M_k$ a representative element $\|\cdot\|_v$ such that the product formula is satisfied.

For each $x = [x_0 : \dots : x_n] \in \mathbb{P}^n(k)$, the *relative height* and the *absolute height* of x are defined

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$$H_k(x) = \prod_{v \in M_k} \max\{\|x_j\|_v \mid 0 \leq j \leq n\}, \qquad h(x) = \frac{1}{[k:\mathbb{Q}]} \log H_k(x).$$

Let k, M_k, S be as above. Let D be a divisor on a nonsingular variety V. Extend $\|\cdot\|_v$ to an absolute value on the algebraic closure $\overline{k_v}$. Then a *local Weil function* for D relative to v is a function $\lambda_{D,v} : V(\overline{k_v}) \setminus |D| \to \mathbb{R}$ such that if D is represented locally by (f) on an open set U, then

$$\lambda_{D,v}(P) = -\frac{1}{[k:Q]} \log \|f(P)\|_v + \alpha(P),$$

where $\alpha(P)$ is a continuous function on $U(\overline{k_v})$. By choosing embeddings $k \to \overline{k_v}$ and $\overline{k} \to \overline{k_v}$, we may also think of $\lambda_{D,v}$ as a function on $V(k) \setminus |D|$ or $V(\overline{k}) \setminus |D|$. Concerning basic notions and properties of global Weil functions for D over k we refer to [5, Chapter 10, Secs. 1 and 2]. A *global Weil function* for D over k is a collection $\{\lambda_{D,v}\}$ of local Weil functions, for $v \in M_k$, where the α_v above satisfy certain reasonable boundedness conditions as v varies.

Now, we give the definition of (S, D)-integral points.

Fix a number field k. Let $\mathcal{O}_S$ be the ring of S-integers of k. A point $P \in \mathbb{A}^n(k)$ is called an *S-integral point* if all its coordinates are S-integers. Similarly, an affine variety $V \subset \mathbb{A}^n$ defined over k inherits a notion of integral point from the definition for $\mathbb{A}^n$. Now let V be a projective variety and D be a very ample effective divisor on V, and let $1 = x_0, x_1, \ldots, x_n$ be a basis for $\mathcal{L}(D)$. Then $P \mapsto (x_1(P), \ldots, x_n(P))$ defines an embedding of $V \setminus D$ into $\mathbb{A}^n$. We say P is an *(S, D)-integral point* if $x_i(P) \in \mathcal{O}_S$ for all i. We note that any point P in $V(k) \setminus D$ can be an (S, D)-integral point for some basis of $\mathcal{L}(D)$. Thus we let integrality be a property of the set of points. This is a natural concept in light of the following lemma.

Lemma 1.2 ([16, Lemma 1.4.1]). *Let D be a very ample effective divisor on V. Let R be a subset of $V(k) \setminus |D|$. Then the following are equivalent:*

(i) *R is a set of (S, D)-integral points on V.*
(ii) *There exists a global Weil function $\lambda_{D,v}$ and constants c_v for each $v \in M_k \setminus S$, such that almost all $c_v = 0$ and for all $P \in R$, all $v \in M_k \setminus S$ and all embeddings of $\overline{k}$ in $\overline{k_v}$,*

$$\lambda_{D,v}(P) \leq c_v.$$

Corollary 1.3 ([16, Lemma 1.4.2]). *The notion of (S, D)-integrality is independent of the multiplicities of the components of D.*

The lemma motivates a more general definition of integrality:

Definition 1.4. Let D be an effective divisor on V and let R be a subset of $V(\overline{k}) \setminus |D|$. Then R is an *(S, D)-integral set of points* if there exists a global Weil function satisfying condition (ii) of Lemma 1.2.

The problem of integral points has a long history, going back to A. Thue [15], C. L. Siegel [14], S. Lang [5], P. Vojta [16], G. Faltings [3] and others. The classical theorem of Thue–Siegel says that $\mathbb{P}^1 \setminus \{3 \text{ distinct points}\}$ has finitely many integral points.

In 1991, M. Ru and P.-M. Wong [12] estimated the dimensions of integral points in the case of $\mathbb{P}^n \setminus \{2n+1 \text{ hyperplanes in general position}\}$.

In 2008, A. Levin [6] generalized the above theorem of Ru–Wong to the case of $\mathbb{P}^n \setminus \{r \text{ hyperplanes in } s\text{-subgeneral position}\}$. Namely, he proved the following.

Theorem A ([6, Corollary 3A]). *Let $\mathcal{H}$ be a set of hyperplanes in $\mathbb{P}^n$ defined over a number field k. Suppose that the intersection of any $s+1$ distinct hyperplanes in $\mathcal{H}$ is empty. Let $r = \sharp\mathcal{H}$. Suppose $r > s$. Then for every number field $K \supset k$ and $S \subset M_K$, for all sets R of S-integral points on $\mathbb{P}^n \setminus |\mathcal{H}|$,*

$$\dim R \leq \left[\frac{s}{r-s}\right].$$

In particular, if $r > 2s$, then all such R are finite. Furthermore, if the hyperplanes in $\mathcal{H}$ are in general position $(s = n)$, then the above bound is achieved by some R.

In 1993, M. Ru [11] estimated the dimensions of integral points in the case of $\mathbb{P}^n \setminus \{2n+1 \text{ hypersurfaces in general position}\}$. Namely, he proved the following.

Theorem B ([11, Theorem 5]). *Let k be a number field. Let $\{D_i\}_{i=1}^q$ be a finite family of (irreducible) hypersurfaces in $\mathbb{P}^n_{\overline{k}}$ in general position and set $D = \sum_{i=1}^q D_i$. Then every set of (S,D)-integral points is contained in a finite union of subvarieties of $\mathbb{P}^n_{\overline{k}}$ of dimension $2n-q+1$.*

Working in a different direction, J. Noguchi and J. Winkelmann [8, Secs. 4.9 and 9.7] showed the finiteness of the set of integral points of $V \setminus \{l \text{ hypersurfaces in general position}\}$, where V is a projective algebraic variety over k. Namely, they showed

Theorem C ([8, Theorem 9.7.6]). *Let V be a projective algebraic variety over k. Let $\{D_i\}_{i=1}^l$ be a finite family of ample divisors on V in general position and set $D = \sum_{i=1}^l D_i$. Assume that $l \geq 2\dim V + r(\{D_i\})$, or more strongly that $l \geq 2\dim V + \operatorname{rank}_{\mathbb{Z}} \mathrm{NS}(V)$, where $\mathrm{NS}(V)$ is the Néron–Severi group of V. Then every (S,D)-integral subset is finite.*

Here we denote by $r(\{D_i\})$ the rank of the subgroup $\sum_i \mathbb{Z}\cdot c_1(D_i)$ generated by $c_1(D_i) \in H^2(M,\mathbb{Z}), 1 \leq i \leq l$, in $H^2(M,\mathbb{Z})$ (see [8, Notation 4.9.1, p. 157]).

Note that in Theorem C, we have $l > 2\dim V + 1$ in general, and there are no estimations of dimensions for sets of (S, D)-integral points in V.

A natural question arises: *How to generalize Theorems A, B and C to the case where the hypersurfaces are in N-subgeneral position in an* (*irreducible*) *projective algebraic variety V over k?*

It seems that some key techniques in the proofs of Theorems A, B and C could not be used for the above question. The first main purpose of this paper is to give a complete answer to the above question. To state the first result, we recall the following.

DEFINITION 1.5. Let k be a number field and V be an irreducible subvariety of dimension n of $\mathbb{P}^m_{\overline{k}}$. Let $N \geq n$. A family of hypersurfaces $D_1, \dots, D_q$ of $\mathbb{P}^m_{\overline{k}}$ is said to be in *N-subgeneral position* in V if $\bigcap_{j=0}^{N} D_{i_j} \cap V(\overline{k}) = \emptyset$ for all tuples $q \geq i_N > i_{N-1} > \cdots > i_0 \geq 1$.

In this paper, we always assume that k is a number field, M_k is the set of all nonequivalent valuations of k and $S \subset M_k$ is a finite set containing all the archimedean valuations. We now state the main theorem of this paper.

THEOREM 1.6. *Let V be an irreducible algebraic subvariety of dimension n of $\mathbb{P}^m_{\overline{k}}$ and $D_1, \dots, D_q$ be hypersurfaces in $\mathbb{P}^m_{\overline{k}}$ in N-subgeneral position on $V (q > N \geq n)$. Assume that $D = \bigcup_{i=1}^{q} D_i$. Then every set of (S, D)-integral points is contained in an algebraic subvariety W of V such that*

$$\dim W \leq \frac{N}{q - N}.$$

When D_i are hyperplanes in N-subgeneral position in $\mathbb{P}^m$, Theorem 1.6 implies Theorem A. When $N = n$, we get Theorems B and C. Theorem 1.6 also implies the finiteness of the set of integral points off divisors in subgeneral position in a projective algebraic variety $V \subset \mathbb{P}^m_{\overline{k}}$.

COROLLARY 1.7. *Let the notation be as above.*

(i) *Assume that $q \geq 2N + 1$ and $N \geq n$. Then every set of (S, D)-integral points is finite.*
(ii) *Assume that $q \geq N + n + 1$ and $n \leq N < 2(n + 1)$. If D intersects any irreducible rational curve in $V(\overline{k})$ in* (*at least*) *three points, then every (S, D)-integral point set of V is finite. However, if there is an irreducible rational curve in $V(\overline{k})$ such that D only intersects this curve in* (*at most*) *two points, then generally the assertion is not true.*

As is well known, there are deep interactions between Kobayashi hyperbolicity and Diophantine approximation. In 1974, S. Lang conjectured the following.

Lang's conjecture. *Let F be an algebraic number field and V a projective algebraic variety. Assume that for some embedding $F \hookrightarrow \mathbb{C}$, the complex manifold $V_{\mathbb{C}}$ given by V is Kobayashi hyperbolic. Then $V(F)$ is a finite set.*

Motivated by Lang's conjecture, the complete hyperbolicity of the complement of divisors in general position in a projective algebraic variety $V \subset \mathbb{P}^M_{\mathbb{C}}$ has been studied by several authors (see M. Ru [11] and Noguchi–Winkelmann [8] and references therein for related subjects). For instance, Noguchi and Winkelmann showed the following.

Theorem D ([8, Theorem 7.3.4]). *Let $\{D_i\}_{i=1}^l$ be a family of ample divisors on a projective algebraic variety X of dimension n, which is in general position. Assume that $l \geq 2n + r(\{D_i\}_{i=1}^l)$, or more strongly that $l \geq 2n + \operatorname{rank}_{\mathbb{Z}} \operatorname{NS}(V)$, where $\operatorname{NS}(V)$ is the Néron–Severi group of V. Then the open variety $X \setminus \operatorname{Supp} \sum_{i=1}^l D_i$ is complete hyperbolic and hyperbolically embedded into X.*

The second main purpose of this paper is to show the complete hyperbolicity of the complement of divisors in N-subgeneral position in a projective algebraic variety $V \subset \mathbb{P}^M_{\mathbb{C}}$. Namely, we will prove the following.

Theorem 1.8. *Let V be an algebraic subvariety of dimension n of $\mathbb{P}^m_{\mathbb{C}}$. Let $\{D_i\}_{i=1}^q$ be a family of hypersurfaces of $\mathbb{P}^m_{\mathbb{C}}$ in N-subgeneral position in V $(q > N \geq n)$. Let W be a subvariety of V such that there is a nonconstant holomophic curve $f : \mathbb{C} \to W \setminus \bigcup_{W \not\subset D_i} D_i$ with Zariski dense image. Then*

$$\dim W \leq \frac{N}{q-N}.$$

In particular, if $q \geq 2N+1$, then $V \setminus \bigcup_{i=1}^q D_i$ is complete hyperbolic and hyperbolically embedded into V.

2. Integral points off divisors in subgeneral position in projective algebraic varieties. We now recall the following lemmas.

Lemma 2.1 ([16, Lemma 1.4.5]). *Let S be a finite set of valuations of k containing all the archimedean valuations. Let k' be a finite extension of k. Let S' be the set of valuations of k' lying over valuations of S. Assume D is an effective divisor on V. Then $I \subset V(\overline{k})$ is a set of (S, D)-integral points if and only if it is a set of (S', D)-integral points.*

Lemma 2.2 ([16, Lemma 1.4.6]). *Let I be an (S, D)-integral set of points on V and let f be a rational function with no poles outside of D. Then there is a nonzero constant $b \in k$ such that $bf(P)$ is S-integral for all $p \in I$.*

Lemma 2.3 (Unit lemma). *Let k be a number field and n a positive integer. Let Λ be a finitely generated subgroup of k^*. Then all but finitely* *many solutions of the equation*

$$u_0 + u_1 + \cdots + u_n = 1, \qquad u_i \in \Lambda \ \forall i,$$

satisfy an equation of the form $\sum_{i \in I} u_i = 0$*, where* I *is a proper nonempty subset of* $\{0, \ldots, n\}$*.*

LEMMA 2.4 ([4, Chapter I, Theorem 7.2]). *Let* V *be a closed irreducible algebraic subvariety of* $\mathbb{P}^m_{\overline{k}}$ *of dimension* $n \geq 1$ *and let* D *be a hypersurface. Then either* $V \subset D$*, or the intersection* $X = V \cap D$ *is nonempty and every component of* X *has dimension* $n-1$*.*

LEMMA 2.5. *Let* V *be a closed (irreducible) algebraic variety in* $\mathbb{P}^m_{\overline{k}}$ *of dimension* $n \geq 1$*,* $N \geq n$ *and* $D_1, \ldots, D_{2N+1}$ *be hypersufaces in* N*-subgeneral position in* V*. Then there exists a subset* $\{i_1, \ldots, i_{n+2}\}$ *of* $\{1, \ldots, 2N+1\}$ *such that we can choose one irreducible component* X_j *of each* $V \cap D_{i_j}$ $(j = 1, \ldots, n+2)$ *in such a way that* $X_1, \ldots, X_{n+2}$ *are distinct. Moreover if* $t_1 < \cdots < t_s$*, then* $X_{t_1} \not\subseteq \bigcup_{i=2}^{s} D_{t_i}$*.*

Proof. Denote by A^i_j $(1 \leq i \leq m_j)$ the irreducible components of $V \cap D_j$ $(1 \leq j \leq 2N+1)$. It is easy to see that there exists $j_1 \in \{1, \ldots, 2N+1\}$ such that $V \not\subseteq D_{j_1}$. By Lemma 2.4, we have $\dim A^i_{j_1} = n-1$ for each $1 \leq i \leq m_{j_1}$. In particular, $\dim A^1_{j_1} = n-1$. Similarly, we can take D_{j_2} such that $A^1_{j_1} \not\subseteq D_{j_2}$. This implies that every component of $A^1_{j_1} \cap D_{j_2}$ has dimension $n-2$. Since $A^1_{j_1} \cap D_{j_2} \subset V \cap D_{j_2} = \bigcup_{i=1}^{m_{j_2}} D^i_{j_2}$, we can find i such that $D^i_{j_2}$ contains an irreducible component of $A^1_{j_1} \cap D_{j_2}$. By setting $A^1_{j_2} = D^i_{j_2}$, we have $\dim\{A^1_{j_1} \cap A^1_{j_2}\} = n-2$, where $\{Y\}$ denotes any irreducible component of the projective algebraic variety Y of maximal dimension. Set $X_1 = A^1_{j_1}$ and $X_2 = A^1_{j_2}$. Then $X_1 \neq X_2$. Note that $2N+1-i > N$ for each $i \leq n \leq N$.

By repeating the above process, for each $1 \leq i \leq n$, we can select D_{j_i} such that

$$\dim\left\{A^1_{j_i} \cap \{A^1_{j_{i-1}} \cap \{\cdots \cap A^1_{j_1}\} \cdots\}\right\} = n - i.$$

We set $X_i = A^1_{j_i}$ $(1 \leq i \leq n)$. These are irreducible and distinct. Moreover, each X_i is an irreducible component of $V \cap D_{j_i}$. By our choice, $\{A^1_{j_n} \cap \{A^1_{j_{n-1}} \cap \{\cdots \cap A^1_{j_1}\} \cdots\}\}$ is nonempty; pick x_0 in this set. Since at most N of the D_j $(1 \leq j \leq 2N+1)$ can intersect at x_0, we can find a $D_{j_{n+1}}$ such that $x_0 \notin D_{j_{n+1}}$. Select $y_0 \in D_{j_{n+1}} \cap V$. At most N of the D_j $(1 \leq j \leq 2N+1)$ can intersect at y_0. The total number of hypersufaces that intersect either at x_0 or at y_0 is at most $2N$. Therefore, there exists $D_{j_{n+2}}$ such that $\{x_0, y_0\} \cap D_{j_{n+2}} = \emptyset$. Denote by X_{n+1} the irreducible component of $V_{j_{n+1}}$ containing y_0, and by X_{n+2} any irreducible component of $V \cap D_{j_{n+2}}$. It is obvious that $X_j \neq X_i$ for all $1 \leq i < j \leq n$. Since $x_0 \in X_i$ $(1 \leq i \leq n)$ and x_0 belongs to neither X_{n+1} nor X_{n+2}, we have $X_i \neq X_j$ $(1 \leq i \leq n;\ n+1 \leq j \leq n+2)$.

Furthermore, since X_{n+1} contains y_0, while X_{n+2} does not, it follows that $X_{n+1} \neq X_{n+2}$. In summary, $X_1, \dots, X_{n+2}$ are distinct.

For the "moreover" part, by the irreducibility of X_{t_1}, suppose for contradiction that there exists $2 \leq i \leq s$ such that $X_{t_1} \subset D_{t_i}$. Again by the construction above, we have

$$\bigl\{A^1_{j_{t_{i-1}}} \cap \{A^1_{j_{t_{i-2}}} \cap \{\cdots \cap A^1_{j_{t_1}}\}\cdots\}\bigr\} \subset A^1_{j_{t_1}} = X_{t_1}$$

and

$$\bigl\{A^1_{j_{t_{i-1}}} \cap \{A^1_{j_{t_{i-2}}} \cap \{\cdots \cap A^1_{j_{t_1}}\}\cdots\}\bigr\} \not\subseteq D_{j_{t_i}}$$

This is impossible. ∎

Lemma 2.6. *Let V be an irreducible algebraic subvariety of dimension n of $\mathbb{P}^m_{\overline{k}}$ and $D_1, \dots, D_q$ be hypersurfaces in $\mathbb{P}^m_{\overline{k}}$ in N-subgeneral position in V. Set $D = \bigcup_{i=1}^q D_i$. Assume that $q \geq 2N+1$. Then every set of (S, D)-integral points is finite.*

Proof. Let J be a set of (S, D)- integral points of $V(k)$. Assume that the hypersufaces $D_1, \dots, D_q$ are defined by $P_1, \dots, P_q$ respectively, which are homogeneous polynomials in $n+1$ variables with coefficients in $\overline{k}$. By Lemma 2.1, without loss of generality, we may assume that the coefficients of P_i $(1 \leq i \leq q)$ are in k.

Claim. *For every (irreducible) algebraic subvariety U of V of dimension p defined over k, $J \cap U$ is contained in a finite union of proper closed subvarieties of U.*

Indeed, by the assumption, $D_1, \dots, D_q$ are in N-subgeneral position over U. By Lemma 2.5, there exist $p+2$ distinct (irreducible) hypersufaces $X_1, \dots, X_{p+2}$ in $U(\overline{k})$ such that each X_i $(1 \leq i \leq p+2)$ is an irreducible component of $U(\overline{k}) \cap D_{j_i}$. Set $Q_i = P_{j_i}^{d/d_{j_i}}$ $(1 \leq i \leq p+2)$, where $d = \gcd(d_{j_1}, \dots, d_{j_{p+2}})$. Then the function field of $U(\overline{k})$ has transcendence degree p, and hence the rational functions $Q_2/Q_1, \dots, Q_{p+2}/Q_1$ on $U(\overline{k})$ are algebraically dependent, so there exists a polynomial T with coefficients in $\overline{k}$ such that

$$T(Q_2/Q_1, \dots, Q_{p+2}/Q_1) = 0$$

identically on $U(\overline{k})$. By using the norm $N_k^{k'}$ of T, where k' is a finite extension of k that contains all coefficients of T, we may assume that the coefficients of T are in k. Thus,

$$\sum_{i=1}^{l} c_i T_i/T_0 = 1,$$

where $c_i \in k^*$ and each $T_0, \dots, T_l$ is a monomial in $\{Q_2/Q_1, \dots, Q_{p+2}/Q_1\}$. We can choose T with l minimal. Since Q_i/Q_1 $(2 \leq i \leq p+2)$ are rational functions of U and have no poles outside D, this implies that there exists $a_i \in k^*$ such that for every $x \in J \cap U$,

$$a_i Q_i(x)/Q_1(x) \in \mathcal{O}_S.$$

By the same argument, there exists $b_i \in k^*$ such that

$$b_i Q_1(x)/Q_i(x) \in \mathcal{O}_S$$

for all $x \in J \cap U$. Set

$$\begin{aligned} A &= \{a_i, b_i, c_j \mid 2 \le i \le p+2\,, 1 \le j \le l\}, \\ S' &= \{v \in M_k \mid \exists a \in A \text{ such that } \|a\|_v \neq 1\}. \end{aligned}$$

Since A is finite, so is S'. Set $S'' = S \cup S'$. Then S'' is finite, $\mathcal{O}_S \subset \mathcal{O}_{S''}$ and $a \in \mathcal{O}^*_{S''}$ for each $a \in A$. Since $\mathcal{O}_{S''}$ is a ring and a is a unit element for every $a \in A$, it follows that both $c_i Q_i(x)/Q_1(x)$ and $Q_1(x)/(c_i Q_i(x))$ are in $\mathcal{O}_{S''}$ for all $x \in J \cap U$. Hence $c_i Q_i(x)/Q_1(x)$ is a unit element in $\mathcal{O}_{S''}$ for each $x \in J \cap U$. Since S'' is finite, $\mathcal{O}^*_{S''}$ is a finitely generated subgroup of k^*. The unit lemma implies that all but finitely many points of $J \cap U$ are contained in some diagonal hypersuface

$$H_I = \Big\{ x \in U \;\Big|\; \sum_{i \in I} c_i T_i(x)/T_0(x) = 0 \Big\},$$

where I is a proper subset of $\{1, \ldots, l\}$. If $H_I(x) = 0$ on $U(\overline{k})$, then we can take $T' = \sum_{i \in I} c_i T_i$ and since I is a proper subset of $\{1, \ldots, l\}$, we get $l' < l$. This contradicts the minimum property of l. If $(T_1(x)/T_0(x), \ldots, T_l(x)/T_0(x))$ belongs to the finite set of exceptional solutions $\{(d_{j_i})_{i=1}^l \mid j = 1, \ldots, s\}$, then $x \in \bigcup_{j=1}^s \{y \in U \setminus D \mid T_1(y) - d_{j_1} T_0(y) = 0\}$. Since $x \in U \setminus D$, we get $T_1(x) \neq 0$, and hence we can eliminate j such that $d_{j_1} = 0$. If $T_1(x) - dT_0(x) = 0$ for all $x \in U(\overline{k})$, and $d \neq 0$, then we may write

$$Q_{i_1}^{\alpha_1} \cdots Q_{i_t}^{\alpha_t} = d Q_{i_{t+1}}^{\alpha_{t+1}} \cdots Q_{i_s}^{\alpha_s}$$

on $U(\overline{k})$. Without loss of generality, we may suppose that

$$i_1 = \min\{i_j \mid j = 1, \ldots, s\}.$$

By Lemma 2.5, we see that $X_{i_1} \not\subseteq \bigcup_{j=t+1}^s D_{i_j}$. Then there exists $x_0 \in X_{i_1} \setminus \bigcup_{j=t+1}^s D_{i_j}$. So we have

$$Q_{i_1}^{\alpha_1}(x_0) \cdots Q_{i_t}^{\alpha_t}(x_0) = d Q_{i_{t+1}}^{\alpha_{t+1}}(x_0) \cdots Q_{i_s}^{\alpha_s}(x_0).$$

Since the right side is nonzero, so is the left side. This is a contradiction proving the Claim.

By induction, we can show that J is contained in a finite union of proper closed subvarieties of dimension i for each $n \ge i \ge 0$. For $i = 0$, this implies that J is a finite set. ∎

We emphasize that the assumption $q \geq 2N+1$ in Lemma 2.6 plays an essential role, because we need to use Lemma 2.5 to construct the sequence $X_1, \ldots, X_{p+2}$. So a natural question is to find conditions on D, V such that we also get $X_1, \ldots, X_{p+2}$ by the same process as in Lemma 2.5. This idea suggests the following lemma.

LEMMA 2.7. *The process in Lemma* 2.5 *is successful if* $n > N/(q-N)$.

Proof. We suppose the contrary.

CLAIM. *There exist* $n+1$ *sets* $I_1, \ldots, I_{n+1}$ *such that:*

(i) $I_1, \ldots, I_{n+1}$ *are disjoint subsets of* $\{1, \ldots, q\}$.
(ii) $|I_j| \geq q - N$ *for each* $1 \leq j \leq n+1$.
(iii) *For each* $1 \leq j \leq n+1$ *and* $s, t \in I_j$, $D_s \cap V = D_t \cap V$. *In other words,* $D_s \cap V =: F_j$ *does not depend on* $s \in I_j$ *for each* $1 \leq j \leq n+1$.
(iv) *For each* $1 \leq j \leq n+1$, *there exists an irreducible component* E_j *of* F_j *such that*

$$\dim\big\{E_i \cap \{E_{i-1} \cap \{\cdots \cap E_1\} \cdots\}\big\} = n - i \quad (1 \leq i \leq n).$$

We shall prove the Claim by induction.

For $j = 1$, by Lemma 2.5, there exist $D_{t_1}, \ldots, D_{t_n}$ such that for every $1 \leq i \leq n$, there is an irreducible component W_i of $D_{t_i} \cap V$ such that

$$\dim\big\{W_i \cap \{W_{i-1} \cap \{\cdots \cap W_1\} \cdots\}\big\} = n - i \quad (1 \leq i \leq n).$$

Then $\{W_n \cap \{W_{n-1} \cap \{\cdots \cap W_1\} \cdots\}\}$ is nonempty. Take

$$x_0 \in \big\{W_n \cap \{W_{n-1} \cap \{\cdots \cap W_1\} \cdots\}\big\}.$$

Set $I_1 = \{1 \leq s \leq q \mid \{x_0\} \not\subseteq D_s \cap V\}$. Since at most N of the D_t $(1 \leq t \leq q)$ can intersect at x_0, we have $|I_1| \geq q - N$. We now show that $D_s \cap V \subset D_t \cap V$ for any $s, t \in I_1$. Indeed, suppose that $y_0 \in D_s \cap V$, but $y_0 \notin D_t \cap V$. Then, by choosing $D_{t_{n+1}} = D_s$ and $D_{t_{n+2}} = D_t$, the process in Lemma 2.5 is successful. This is impossible by the assumption. This yields $D_s \cap V = D_t \cap V$ for any $s, t \in I_1$.

For $j = 2$, take an irreducible component E_1 of $F_1 = D_s \cap V, s \in I_1$. Repeating the process in Lemma 2.5, we may find $D_{t_2}, \ldots, D_{t_n}$ and their respective irreducible components $W_2, \ldots, W_n$ such that

$$\dim\big\{W_i \cap \{W_{i-1} \cap \{\cdots \cap E_1\} \cdots\}\big\} = n - i \quad (1 \leq i \leq n).$$

By the same argument, there exists a subset I_2 of $\{1, \ldots, q\}$ such that $|I_2| \geq q - N$ and $D_s \cap V = D_t \cap V$ for all $s, t \in I_2$ and there exists $x_0 \in E_1 \setminus F_2$, where $F_2 = D_s \cap V$ for some $s \in I_2$. So $E_1 \not\subseteq F_2$.

For $j = 3$, take an irreducible component E_2 of F_2 such that $\dim\{E_1 \cap E_2\} = n - 2$. Repeating the process in Lemma 2.5 for $W_1 = E_1$, $W_2 = E_2$ and by the above argument, we can find I_3 satisfying the above conditions. Moreover, $E_2 \cap E_1 \not\subseteq F_3$.

Similarly, we find subsets $I_1, \ldots, I_n$ satisfying the above conditions. At the end, in the same way, we still find I_{n+1} such that $|I_{n+1}| \geq q - N$ and $D_s \cap V = D_t \cap V$ for all $s, t \in I_{n+1}$. So the Claim is proved.

Since (i) and (ii) hold, we now have $q \geq (n+1)(q-N)$, and hence $n \leq N/(q-N)$, a contradiction. ∎

Proof of Theorem 1.6. Use the same argument as in the proof of Lemma 2.6 and apply Lemma 2.7. ∎

Proof of Corollary 1.7. (i) This assertion is Lemma 2.6.

(ii) The first part can be deduced from the Thue–Siegel theorem (see [14], [15]) and Theorem 1.6. For the second part, we consider the following.

EXAMPLE 2.8. Let $k = \mathbb{Q}[\sqrt{2}], V = \{x_3 = 0\} \subset \mathbb{P}^2_{\overline{k}}$ and $D_1 = \{x_1 = 0\}$, $D_2 = \{x_2 = 0\}$, $D_3 = \{x_1^2 - x_3^2 = 0\}$, $D_4 = \{x_2^2 - x_3^2 = 0\}$. Then D_1, D_2, D_3, D_4 are hypersurfaces in 2-subgeneral position in V. Take $J = \{((1+\sqrt{2})^n : 1 : 0) \mid n \in \mathbb{N}\}$. So $J \subset V \setminus D$ and $|J| = \infty$. Since $\{1, x_2/x_1, x_1/x_2\}$ is a base of $\mathcal{L}(D')$, where $D' = D_1 + D_2$, we have the embedding

$$\left(\frac{x_1}{x_2}, \frac{x_2}{x_1}\right) : V \setminus D' \to \mathbb{A}^1.$$

It is easy to see that $(1+\sqrt{2})^n \in \mathcal{O}_k^* \subset \mathcal{O}_S^*$. Hence, J is a set of (S, D)-integral points of V by Corollary 1.3, but J is an infinite set.

3. Hyperbolicity of the complement of divisors in subgeneral position in projective algebraic varieties. First of all, we recall the following.

LEMMA 3.1 (Borel lemma). *Let u_i be nonvanishing entire functions satisfying the unit equation*

$$\sum_{i=1}^{n} u_i = 1.$$

Then the image of the entire curve $f = (u_1, \ldots, u_n)$ is contained in a diagonal hyperplane.

LEMMA 3.2 (see [7, Lemma 4.8] and [8, Theorem 7.2.13]). *Let X be a compact complex space and let $\{E_i\}_{i \in I}$ be a family of Cartier hypersurfaces of X. Assume that for every subset $\emptyset \subseteq J \subseteq I$, every holomorphic curve*

$$f : \mathbb{C} \to \bigcap_{j \in J} E_j \setminus \bigcup_{i \in I \setminus J} E_i$$

is reduced to a constant mapping, where $\bigcap_{j \in \emptyset} E_j = X$. Then $X \setminus \bigcup_{i \in I} E_i$ is complete hyperbolic and hyperbolically embedded into X.

Proof of Theorem 1.8. Let $P_1, \dots, P_q$ be homogeneous polynomials defining the hypersurfaces D_j $(1 \le j \le q)$. Without loss of generality, we may assume that P_j $(1 \le j \le q)$ have the same degree. It suffices to prove that if W is an irreducible subvariety of V with $\dim W := p > N/(q-N)$ and $f(\mathbb{C}) \subset W$, then $f(\mathbb{C})$ is contained in a proper subvariety of W. Using Lemma 2.7, we may choose irreducible components $X_1, \dots, X_{p+2}$ of $D_{j_1}, \dots, D_{j_{p+2}}$ as in Lemma 2.5. Set $Q_i = P_{j_i}$. Then $Q_1(f), \dots, Q_{p+2}(f)$ are nonvanishing entire functions. Since the transcendence dimension of the function field of W is p, the rational functions $Q_2/Q_1, \dots, Q_{p+2}/Q_1$ on W are algebraically dependent. Hence there exists a polynomial T with coefficients in $\mathbb{C}$ such that

$$T(Q_2/Q_1, \dots, Q_{p+2}/Q_1) = 0$$

identically on W. Therefore,

$$\sum_{i=1}^{l} c_i T_i/T_0 = 1,$$

where $c_i \neq 0$ and $T_0, \dots, T_l$ are monomials in $Q_2/Q_1, \dots, Q_{p+2}/Q_1$. Set

$$T_i(f) = T_i(Q_2(f)/Q_1(f), \dots, Q_{p+2}(f)/Q_1(f)) \quad (0 \le i \le l).$$

Then $T_i(f)/T_0(f)$ $(1 \le i \le l)$ are nonvanishing entire functions. Using the Borel lemma and in the same way as in the Unit Lemma and by repeating the discussion as in the case of (S, D)-integral points, we get the first claim.

Now we apply Lemma 3.2 to prove the last claim. In fact, assume that J is any subset of $\{1, \dots, q\}$ with $|J| = l$. If $l \ge N+1$, then $\bigcap_{j \in J} D_j \cap V = \emptyset$ by our assumption, and hence the assertion is proved. If $0 \le l \le N$, then by definition, the family $\{D_i\}_{i \in I \setminus J}$ is in $(N-l)$-subgeneral position on $\bigcap_{j \in J} D_j \cap V$. By the above, any holomorphic curve $f : \mathbb{C} \to (\bigcap_{j \in J} D_j \cap V) \setminus \bigcup_{i \in I \setminus J} D_i$ is contained in a proper subvariety W of $\bigcap_{j \in J} D_j \cap V$ such that

$$\dim W \le \frac{N-l}{(q-l)-(N-l)}.$$

Since $q \ge 2N+1$ and $0 \le l \le N$ we have $\dim W \le \frac{N-l}{q-N} < 1$. Therefore $f(\mathbb{C})$ is discrete, and hence f is constant by the connectedness of $f(\mathbb{C})$. The proof is complete. ∎

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