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Khintchine types of translated coordinate hyperplanes

by

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1. Introduction

1.1. General setting. The central object of study in simultaneous metric Diophantine approximation is the set

$$\mathcal{W}_d(\psi) = \{\boldsymbol{x} \in \mathbb{R}^d : \|q\boldsymbol{x} - \boldsymbol{p}\|_\infty < \psi(q) \text{ for infinitely many } (\boldsymbol{p}, q) \in \mathbb{Z}^d \times \mathbb{N}\}$$

of *ψ-approximable vectors* in $\mathbb{R}^d$, where $\psi : \mathbb{N} \to \mathbb{R}^+ \cup \{0\}$ is a given map, which we call an *approximating function* if it is non-increasing. In words, $\mathcal{W}_d(\psi)$ is the set of d-tuples of real numbers that can be rationally approximated simultaneously, meaning with common denominator, at the "rate" given by ψ, with infinitely many different denominators. For $\tau \in \mathbb{R}^+$ we denote $\mathcal{W}_d(q \mapsto q^{-\tau}) = \mathcal{W}_d(\tau)$. The supremum of all $\tau \in \mathbb{R}^+ \cup \{\infty\}$ such that $\boldsymbol{x} \in \mathcal{W}_d(\tau)$ is called the *Diophantine type* of $\boldsymbol{x}$, and if it is ∞, then $\boldsymbol{x}$ is called *Liouville*. The Liouville numbers form a set of Hausdorff dimension 0 in $\mathbb{R}$.

1.2. Foundational results. The seminal result on Diophantine approximation, *Dirichlet's Theorem* (c. 1840), guarantees that if $\psi(q) \geq q^{-1/d}$, then $\mathcal{W}_d(\psi)$ is all of $\mathbb{R}^d$. On the other hand, a standard argument using the Borel–Cantelli Lemma shows that if $\psi(q) \leq q^{-1/d-\varepsilon}$ for some $\varepsilon > 0$, then $m_d(\mathcal{W}_d(\psi)) = 0$, where m_d is Lebesgue measure on $\mathbb{R}^d$. One may guess that the difference lies in the convergence or divergence of the integral of ψ^d. Indeed, *Khintchine's Theorem* (1926) set the foundation for simultaneous metric Diophantine approximation by making this dichotomy precise [Kh].

Khintchine's Theorem (1926). *Let ψ be an approximating function, and $d \in \mathbb{N}$. Then*

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$$m_d(\mathcal{W}_d(\psi)) = \begin{cases} \text{NULL} & \textit{if } \sum_{q=1}^{\infty} \psi(q)^d < \infty, \\ \text{FULL} & \textit{if } \sum_{q=1}^{\infty} \psi(q)^d = \infty. \end{cases}$$

As regards the Lebesgue measure of $\mathcal{W}_d(\psi)$, Khintchine's Theorem tells us the whole story. Of course, there are other measures, and notions of size, that one may consider. *Jarník's Theorem* (1931) provides a similar dichotomy for Hausdorff measures of $\mathcal{W}_d(\psi)$.

Later, Gallagher [Ga] extended Khintchine's Theorem in the following sense.

Gallagher's Theorem (1965). *If $d \geq 2$, then Khintchine's Theorem is also true for functions $\psi : \mathbb{N} \to \mathbb{R}^+ \cup \{0\}$ that are not monotone.*

Remark. Gallagher's Theorem is one of the main tools here. We use it in the proofs of Theorems 1–4. (See §2.7.)

1.3. Current directions. One of the major trends is developing the theory of rational approximations and "Khintchine types" for manifolds embedded in $\mathbb{R}^d$. A manifold $\mathcal{M} \subset \mathbb{R}^d$ is said to be of *Khintchine type for divergence* if whenever ψ is an approximating function such that $\sum_{q\in\mathbb{N}} \psi(q)^d$ diverges, almost every point on $\mathcal{M}$ is ψ-approximable. On the other hand, it is said to be of *Khintchine type for convergence* if whenever ψ is an approximating function such that $\sum_{q\in\mathbb{N}} \psi(q)^d$ converges, almost no point on $\mathcal{M}$ is ψ-approximable. If it is both, it is of *Khintchine type.*

Recently, Beresnevich, Dickinson, and Velani have shown that any analytic non-degenerate (meaning curved enough that no part of it is contained in any hyperplane) submanifold of $\mathbb{R}^d$ is of Khintchine type for divergence [BDV, B]. Vaughan and Velani showed that non-degenerate planar curves are of Khintchine type for convergence [VV].

1.4. Our focus. This article is about the *degenerate* case. Far from deviating from all hyperplanes, the manifolds we consider here *are* hyperplanes. Specifically, we investigate questions related to the following general problem:

> *Describe the set of rationally approximable points in the fiber over a given fixed coordinate in Euclidean space.*

For instance, suppose ψ is an approximating function such that $\sum \psi(q)^d$ diverges, say $\psi(q) = (q \log q)^{-1/d}$. Fix $x \in \mathbb{R}$. Dirichlet's Theorem guarantees that x is ψ-approximable. But our ψ decays quite slowly, so we may expect that almost every point $(x, x_2, \ldots, x_d) \in \mathbb{R}^d$ in the fiber over x is also ψ-approximable. Our first result, Theorem 1, confirms this for $d \geq 3$. On the other hand, if we had chosen ψ such that $\sum \psi(q)^d$ converges, then it would make sense to expect the opposite statement: almost no points

$(x, x_2, \dots, x_d) \in \mathbb{R}^d$ are ψ-approximable, with $x \in \mathbb{R}$ fixed. We find in Theorem 5 that this is sometimes true, sometimes not.

All of our results (presented in §2) are of a similar flavor. Namely, they are steps toward the more general and distant goal of bringing the theory of Khintchine types to the setting of affine subspaces in $\mathbb{R}^n$. Ultimately, one would like to be able to state a condition on an approximating function ψ that is equivalent to almost all points on a subspace being ψ-approximable. We only manage to do this for certain hyperplanes (see Theorem (b)). The rest of our results are sufficient conditions for the "almost all" or "almost no" cases.

2. Results

2.1. Divergence results for prototypical approximating functions. We have a number of results for the divergence situation, which for illustrative purposes we state in order of increasing generality of approximating functions. The first holds for $\psi(q) = (q \log q)^{-1/d}$.

THEOREM 1. *Let $d \geq 3$ and $\psi(q) = (q \log q)^{-1/d}$. Then*

$$m_{d-1}(\mathcal{W}_d(\psi) \cap (\{x\} \times \mathbb{R}^{d-1})) = \text{FULL} \quad \textit{for every } x \in \mathbb{R}.$$

From Theorem 1 we immediately deduce the same statement for $\psi(q) = (q \log \dots \log q)^{-1/d}$, because this function dominates $(q \log q)^{-1/d}$. Slightly more challenging are approximating functions of the form

$$\psi_{s,d}(q) = \left(\frac{1}{q(\log q)(\log \log q) \dots (\log \dots \log q)} \right)^{1/d}$$

where $s \in \mathbb{N}$ is the length of the last string of logarithms. For these we are able to prove the following.

THEOREM 2. *Let $d \geq 3$ and $s \in \mathbb{N}$. Then*

$$m_{d-1}(\mathcal{W}_d(\psi_{s,d}) \cap (\{x\} \times \mathbb{R}^{d-1})) = \text{FULL}$$

for any x whose Diophantine type is greater than d, and any x whose regular *Diophantine type is greater than* 1.

The *regular Diophantine type* of $x \in \mathbb{R}$ is the supremum of $\tilde{\sigma} \in [1, \infty)$ such that rational approximations $|x - p/q| < q^{-(1+\tilde{\sigma})}$ appear with positive lower asymptotic density in the sequence $\{q_n\}_{n \geq 0}$ of continuants of x. In simpler words, the regular Diophantine type of a number is the maximal rate at which it can be rationally approximated, not just infinitely often, but also with some frequency.

REMARK (On Khintchine's transference principle). We will present a proof of Theorem 1 that holds for all non-Liouville x, and a proof of Theorem 2 that holds for non-Liouville x with regular Diophantine type greater than 1. The remaining cases are covered by Khintchine's transference principle, which implies that if $x \in \mathbb{R}$ has Diophantine type greater than d, then *every* point on $\{x\} \times \mathbb{R}^{d-1}$ has Diophantine type greater than $1/d$. In particular, if ψ is an approximating function that eventually dominates $q^{-(1+\varepsilon)/d}$ for every $\varepsilon > 0$, then every point on $\{x\} \times \mathbb{R}^{d-1}$ is ψ-approximable.

Remark. Theorem 2 is actually a corollary of a more general theorem (Theorem 30) that holds for more fibers, but has a more technical statement. Both theorems are still true for uncountably many numbers *not* satisfying their assumptions, including uncountably many numbers of *any* Diophantine type and regular Diophantine type 1, and *every* number of Diophantine type at most the golden ratio regardless of regular Diophantine type. Such fibers are accounted for in Theorem 3 below.

2.2. Divergence result for approximating functions satisfying the divergence condition. In the next theorem we name fibers on which the desired "almost everywhere" assertion can be made, provided only that the approximating function ψ is such that $\sum \psi(q)^d$ diverges. Among these are all fibers over base-points of Diophantine type less than the golden ratio (or, with an additional restriction, less than two), and an uncountable set of fibers over base-points of any given Diophantine type.

Theorem 3. *Let $d \geq 3$. If ψ is an approximating function such that $\sum_{q\in\mathbb{N}} \psi(q)^d$ diverges, then*

$$m_{d-1}(\mathcal{W}_d(\psi) \cap (\{x\} \times \mathbb{R}^{d-1})) = \text{FULL}$$

for:

- (a) *Any $x \in \mathbb{Q}$ (even if $d = 2$).*
- (b) *Any $x \in \mathbb{R} \setminus \mathbb{Q}$ with the positive density property* (*see Definition* 14), *including but not restricted to:*
  - *Any $x \notin \mathcal{W}_1(\varphi)$ where $\varphi = (1 + \sqrt{5})/2$.*
  - *Any $x \notin \mathcal{W}_1(2)$ for which there exists $R \geq 1$ such that eventually whenever a partial quotient of x exceeds R, its continuants at least double before the next partial quotient exceeding R.*
- (c) *Uncountably many numbers of any Diophantine type.*

Remark. The subpoints in part (b) come from Proposition 15.

One may ask whether Theorem 3 holds for non-monotonic functions. A simple observation shows that it cannot: after fixing $x \in \mathbb{R} \setminus \mathbb{Q}$, consider the function $\psi(q) = \|qx\|$, where $\|\cdot\|$ denotes distance to the nearest integer. Then $\sum \psi(q)^d$ diverges, yet we can never have $\|qx\| < \psi(q)$, so the entire fiber over x is missing from $\mathcal{W}_d(\psi)$.

2.3. Divergence result for approximating functions all of whose convergent subseries have zero density. As to the question of whether the result of Theorem 3 holds for fibers other than those fitting into parts (a), (b), or (c), we have the following theorem, which gives a sufficient condition on the approximating function ψ for the result to hold on *all* fibers. Recall that the density $\mathrm{d}(A)$ of a set $A \subseteq \mathbb{N}$ is given by the limit

$$\mathrm{d}(A) = \lim_{N\to\infty} |A \cap [1, N]| / N$$

when it exists. When the limit does not exist, we can still define the lower density $\underline{\mathrm{d}}(A)$ and the upper density $\overline{\mathrm{d}}(A)$ by the lim inf and lim sup, respectively.

THEOREM 4. *Let $d \geq 3$. If ψ is an approximating function such that every convergent subseries $\sum_{q\in A} \psi(q)^d$ has asymptotic density $\mathrm{d}(A) = 0$, then*

$$m_{d-1}(\mathcal{W}_d(\psi) \cap (\{x\} \times \mathbb{R}^{d-1})) = \text{FULL} \quad \text{for all } x \in \mathbb{R}.$$

For example, the approximating function $\psi(q) = cq^{-1/d}$, where $c > 0$, satisfies the requirement that all convergent subseries of $\sum \psi(q)^d$ have asymptotic density 0. Therefore, almost every point on every $d - 1$-dimensional fiber of $\mathbb{R}^d$ is ψ-approximable. Of course, in the case $c = 1$ we already knew this (and *more*) from Dirichlet's Theorem. But when we allow any $c \in (0, 1)$, Theorem 4 reflects the fact that *badly approximable vectors*—vectors $\boldsymbol{x} \in \mathbb{R}^d$ for which there exists $c := c(\boldsymbol{x}) > 0$ such that $\|q\boldsymbol{x} - \boldsymbol{p}\|_\infty \geq cq^{-1/d}$ for all $(\boldsymbol{p}, q) \in \mathbb{Z}^d \times \mathbb{N}$—do not overpopulate any hyperplanes.

2.4. Convergence result. The next result deals with the convergence situation. Given ψ such that $\sum_{q\in\mathbb{N}} \psi(q)^d$ converges, we would like to assert that *almost no* points on the fiber $\{x\} \times \mathbb{R}^{d-1}$ are ψ-approximable. Again, we are able to make the desired statement for certain fibers, but not for others, depending on the Diophantine type of the base-point.

THEOREM 5. *Let $d \geq 2$. If ψ is an approximating function such that $\sum_{q\in\mathbb{N}} \psi(q)^d$ converges, then*

$$m_{d-1}(\mathcal{W}_d(\psi) \cap (\{x\} \times \mathbb{R}^{d-1})) = \text{NULL}$$

for:

(a) $\begin{cases} \textit{No } x \in \mathbb{Q} & \textit{if } \sum_{q\in\mathbb{N}} \psi(q)^{d-1} \textit{ diverges.} \\ \textit{Every } x \in \mathbb{R} & \textit{if it converges.} \end{cases}$

(b) *Any $x \in \mathbb{R} \setminus \mathbb{Q}$ with the bounded ratio property* (*see Definition* 16), *including but not restricted to:*

- *Any x of Diophantine type less than $\varphi = (1+\sqrt{5})/2$.*
- *Any x of Diophantine type less than 2 for which there exists $R \geq 1$ such that eventually whenever a partial quotient of x exceeds R, its continuants at least double before the next partial quotient exceeding R.*

Remark. The subpoints in (b) are Proposition 17. In part (a), when $\sum_{q\in\mathbb{N}} \psi(q)^{d-1}$ diverges, we get FULL instead of NULL.

We were unaware during submission of this manuscript that Theorem 5(b) actually follows from [Gh, Theorem 1.6] in the work of A. Ghosh ([1]). He describes "dual" approximability properties of points on hyperplanes when the approximating function gives a convergent series. After applying Khintchine's transference principle, one finds that Ghosh's result implies in particular that coordinate hyperplanes in $\mathbb{R}^d$, translated perpendicularly by a distance of Diophantine type $< d$, are of Khintchine type for convergence.

His methods come from dynamics on homogeneous spaces. Specifically, the approximability properties of a point in $\mathbb{R}^d$ are related to the behavior of an associated flow orbit in the space of unimodular lattices in $\mathbb{R}^{d+1}$. Whether the orbit diverges into the cusp, and at what rate, determines the Diophantine type of the point in $\mathbb{R}^d$ (see [KM]). Ghosh's work comes from a growing family of results exploiting the connections between homogeneous dynamics and Diophantine approximation, and its most immediate ancestor is a paper [Kl03] of Kleinbock on extremality of affine subspaces of $\mathbb{R}^d$, relevant in §2.6.

Our arguments for Theorem 5 are very elementary by comparison.

2.5. A repackaging in terms of Khintchine types. We can state Theorems 3 and 5 more succinctly by using the terminology of Khintchine types.

In the following statements, "perpendicular translate of a coordinate hyperplane" means a coordinate hyperplane that has been translated by a vector perpendicular to it.

Theorem (a). *Perpendicular translates of coordinate hyperplanes in $\mathbb{R}^d$ (where $d \geq 2$) by rational numbers are of Khintchine type for divergence, but not for convergence.*

Theorem (b). *Perpendicular translates of coordinate hyperplanes in $\mathbb{R}^d$ (where $d \geq 3$) by numbers with the bounded ratio property are of Khintchine type. In fact they are of Khintchine type for convergence even when $d = 2$.*

([1]) We thank the reviewer for bringing this paper to our attention.

Theorem (c). *Uncountably many perpendicular translates of coordinate hyperplanes in $\mathbb{R}^d$ (where $d \geq 3$) by numbers of any given Diophantine type are of Khintchine type for divergence among approximating functions dominating any given.*

2.6. Extremality corollaries. There is a weaker notion than Khintchine type for convergence, called "extremality." A manifold $\mathcal{M} \subset \mathbb{R}^d$ is *extremal* if for every approximating function such that $\psi(q) \leq q^{-(1+\delta)/d}$ for some $\delta > 0$, almost no point on $\mathcal{M}$ is ψ-approximable.

The idea of extremality dates back to a 1932 conjecture of Mahler, that Veronese curves are extremal. These are curves of the form

$$(x, x^2, x^3, \ldots, x^d) \subset \mathbb{R}^d.$$

Mahler's conjecture was settled by Sprindžuk [Sp] in 1964, and this led to a great deal of research into the extremality of curves, and in general manifolds, embedded in $\mathbb{R}^d$. In the 1980s Sprindžuk conjectured that any nondegenerate analytic submanifold of $\mathbb{R}^d$ is extremal, and this was eventually settled by Kleinbock and Margulis [KM] in 1998, even without analyticity. (For a manifold that is not analytic, the non-degeneracy condition must be stated somewhat more carefully, in terms of the linear span of the partial derivatives of the manifold's parametrizing functions. See [KM].)

Theorem 5 yields some corollaries on extremality of certain translated hyperplanes (*degenerate* manifolds). They were already known (and can be read off from [Kl03, Theorem 1.3]), but we list them for the sake of completeness.

The following corollary is an immediate consequence of Theorem 5(b).

Corollary 6. *Perpendicular translates of coordinate hyperplanes in $\mathbb{R}^d$, $d \geq 2$, by numbers with the bounded ratio property are extremal.*

From our proofs we will also be able to read off the following two corollaries, also listing translated coordinate hyperplanes that are extremal, this time according to their Diophantine type.

Corollary 7. *Any perpendicular translate of a coordinate hyperplane by a number of Diophantine type $\varphi = (1+\sqrt{5})/2$ or less is extremal.*

Corollary 8. *Any perpendicular translate of a coordinate hyperplane by a number of Diophantine type* 2 *or less, for which there exists $R \geq 1$ such that eventually whenever a partial quotient of x exceeds R, its continuants at least double before the next partial quotient exceeding R, is extremal.*

Remark. Notice that in these corollaries the bounds on Diophantine type are not strict, whereas in Theorem 5 (or, really, Proposition 17) they are.

Remark. As we mentioned above, these corollaries already follow from the work of Kleinbock, which tells us exactly which hyperplanes are extremal and which are not. In fact, even more is known. Notice that to say that a submanifold is extremal is to say that almost every point on it is of Diophantine type $1/d$. It turns out that even if a subspace is *not* extremal, almost all of its points still share a common Diophantine type, as do almost all the points on any non-degenerate submanifold of that subspace (where non-degeneracy in this case is determined with respect to the subspace). Details of this, and formulas for these Diophantine types, can be found in [Kl08, Z].

2.7. On the proofs. Our strategy for Theorems 1–5 is to arrive at a point where we can apply either Khintchine's Theorem or Gallagher's Theorem to a hyperplane in $\mathbb{R}^d$.

Given an approximating function ψ and a point $x \in \mathbb{R}$, we define a new function

$$\bar{\psi}(q) := \begin{cases} \psi(q) & \text{if } \|qx\| < \psi(q), \\ 0 & \text{if not,} \end{cases}$$

where $\|\cdot\|$ denotes distance to the nearest integer, and we examine the sum

$$\sum_{q=1}^{\infty} \bar{\psi}(q)^{d-1}. \tag{2.1}$$

If $d-1 \geq 2$, we can apply Gallagher's Theorem to the fiber $\{x\} \times \mathbb{R}^{d-1}$ and the non-monotonic function $\bar{\psi}$, to prove that

$$m_{d-1}(\mathcal{W}_{d-1}(\bar{\psi})) = m_{d-1}(\mathcal{W}_d(\psi) \cap (\{x\} \times \mathbb{R}^{d-1}))$$

is either NULL or FULL, depending on whether (2.1) converges or diverges.

All of the effort in all of our "divergence" results is in proving the divergence of (2.1) in different scenarios. Our strategy is centered around showing that the intersection of the set

$$\mathcal{Q}(x, \psi) = \{q \in \mathbb{N} : \|qx\| < \psi(q)\}$$

with an interval $[M, N]$ grows quickly and steadily as the length $N - M$ grows. For this it is most natural to think in terms of circle rotations. We develop an argument based on the Three Gaps Theorem. (See §3.5.)

For our "convergence" results, we try to show that (2.1) converges. Here we do not even need Gallagher, as the monotonicity condition in Khintchine's Theorem is really only relevant to the divergence part. It is well-known that the convergence part is an easy consequence of the Borel–Cantelli Lemma, and holds even when ψ is not monotone. This is why Theorem 5 holds for $d \geq 2$.

Finally, we point out that although we do need $d \geq 3$ in order to apply Gallagher's Theorem in our divergence results, it is not the *only* reason we make the assumption. Lemma 19 in §5 also requires it.

3. Mathematical preliminaries

3.1. Asymptotic notation. We use the following notation:

- $\ll$ means "less than or equal to a positive multiple of."
- $\asymp$ means "$\ll$ and $\gg$."
- $<^*, =^*$, and $\leq^*$ mean "eventually less than," "eventually equal to," or "eventually less than or equal to," respectively.
- $\lesssim$ means "less than or asymptotically equal to."
- $\sim$ means "$\lesssim$ and $\gtrsim$," i.e. "asymptotically equal to."

3.2. Continued fractions. For $x \in \mathbb{R} \setminus \mathbb{Q}$, let

$$x = [a_0; a_1, a_2, a_3, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

be the simple continued fraction expansion of x, let $\{p_k/q_k\}_{k\in\mathbb{N}}$ be its convergents, and $\eta_k = |q_k x - p_k|$ the associated differences. The continuants $\{q_k\}$ follow the recursion $q_k = a_k q_{k-1} + q_{k-2}$ and therefore grow at least exponentially fast. Every $m \in \mathbb{N}$ has a unique representation as $m = rq_k + q_{k-1} + s$ where $1 \leq r \leq a_{k+1}$ and $0 \leq s < q_k$.

We take this opportunity to introduce a notation that we use throughout the paper. Given $x = [a_0; a_1, a_2, \dots] \in \mathbb{R} \setminus \mathbb{Q}$ and a fixed $R \geq 0$, let

$$\{k_m := k_m^{x,R}\}_{m\geq 0}$$

be the sequence of indices where $a_{k_m+1} > R$, starting with the conventional $k_0 = -1$. Let $\Delta k_m := k_{m+1} - k_m$.

We will use the following simple lemma.

LEMMA 9. *Let $\{F(n)\}_{n\in\mathbb{N}} := \{1, 1, 2, 3, 5, \dots\}$ be the Fibonacci sequence. Then*

$$q_{k+n} \geq F(n+1) q_k \quad \textit{for all } k, n \in \mathbb{N}.$$

Proof. By the recursive relations between continuants, we have

$$\begin{aligned} q_k &\geq q_{k-1} + q_{k-2} \geq 2q_{k-2} + q_{k-3} \geq 3q_{k-3} + 2q_{k-4} \geq 5q_{k-4} + 3q_{k-5} \\ &\geq 8q_{k-5} + 5q_{k-6} \geq \cdots \geq F(n+1) q_{k-n} + F(n) q_{k-n-1} \end{aligned}$$

for any $n < k$, which implies the result. ∎

In general this lemma may not give a very strong bound. We only use the particular case $q_{k_{m+1}} \geq F(\Delta k_m)\, q_{k_m+1}$. For an upper bound we have Lemma 10 below.

3.3. Diophantine type and growth of continuants. Recall that for $\sigma \in [1,\infty)$, we define

$$\mathcal{W}_1(\sigma) = \{x \in \mathbb{R} : |x - p/q| < 1/q^{1+\sigma} \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{N}\}.$$

It is a standard fact that the convergents of $x \in \mathbb{R} \setminus \mathbb{Q}$ satisfy

$$\frac{1}{q_n(q_n + q_{n+1})} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}},$$

and therefore

$$x \in \mathcal{W}_1(\sigma) \;\Rightarrow\; q_n^\sigma < 2q_{n+1} \text{ for infinitely many } n$$

and

$$q_n^\sigma < q_{n+1} \text{ for infinitely many } n \;\Rightarrow\; x \in \mathcal{W}_1(\sigma).$$

In particular, the Diophantine type of x is the supremum over $\sigma \in [1,\infty)$ such that $q_n^\sigma < q_{n+1}$ for infinitely many $n \in \mathbb{N}$.

Conversely, $x \notin \mathcal{W}_1(\sigma)$ implies that $q_{n+1} \leq^* q_n^\sigma$. We may equivalently define the Diophantine type of x as the infimum of $\sigma \in [1,\infty)$ for which $q_n^\sigma \gg q_{n+1}$ as $n \to \infty$.

Lemma 10. *If $x \notin \mathcal{W}_1(\sigma)$, then*

$$q_{k_m+1} \leq^* (R+1)^{\sigma \Delta k_{m-1} + \sigma^2 \Delta k_{m-2} + \cdots + \sigma^m \Delta k_0} \leq (R+1)^{\sigma^m k_m}$$

for any $R \geq 1$.

Proof. Since $x \notin \mathcal{W}_1(\sigma)$, we have

$$\begin{aligned} q_{k_m+1} &\leq^* q_{k_m}^\sigma \leq (R+1)^{\sigma \Delta k_{m-1}} q_{k_{m-1}+1}^\sigma \\ &\leq^* \cdots \leq^* (R+1)^{\sigma \Delta k_{m-1} + \sigma^2 \Delta k_{m-2} + \cdots + \sigma^m \Delta k_0} \leq (R+1)^{\sigma^m k_m}. \quad \blacksquare \end{aligned}$$

3.4. Types of Diophantine types. A number $x \in \mathbb{R}$ belongs to the set $\mathcal{W}_1(\sigma)$ of σ-approximable numbers if there are infinitely many rational approximations to x with denominator q satisfying $\|qx\| < q^{-\sigma}$. In view of the approximating properties of convergents, this can be expressed as

$$\mathcal{W}_1(\sigma) = \{x : q_n^\sigma < q_{n+1} \text{ for infinitely many } n \in \mathbb{N}\},$$

where $\{q_n\}$ are the continuants of x. It is useful to refine this definition further by making a distinction between numbers $x \in \mathcal{W}_1(\sigma)$ for which these approximating q's appear often, and those for which the q's appear seldom.

Example/Definition (Uniform Diophantine type). Perhaps the most natural way to define "frequent approximability" is to require that eventually *all* continuants satisfy the growth condition. We may call

$$\mathcal{W}_1^{\mathrm{uni}}(\sigma) = \{x : q_n^\sigma < q_{n+1} \text{ for all sufficiently large } n \subseteq \mathbb{N}\}$$

the set of *uniformly σ-approximable* numbers. Notice that this means, in particular, that the set of continuants satisfying the growth condition has density 1 as a subsequence of $\{q_n\}_{n\geq 0}$. The following definition relaxes this.

Example/Definition (Regular Diophantine type). Another natural notion of frequent approximability is captured by the set of *regularly σ-approximable* numbers:

$$\mathcal{W}_1^{\mathrm{reg}}(\sigma) = \left\{x : q_{n_j}^\sigma < q_{n_j+1} \text{ for some s.p.l.a.d. } \{n_j\} \subseteq \mathbb{N}\right\}$$

where s.p.l.a.d. stands for "sequence of positive lower asymptotic density." It is obvious that $\mathcal{W}_1^{\mathrm{uni}}(\sigma) \subset \mathcal{W}_1^{\mathrm{reg}}(\sigma) \subset \mathcal{W}_1(\sigma)$. Notice that $\mathcal{W}_1^{\mathrm{reg}}(1) = \mathbb{R}$, because all continuants satisfy $q_n < q_{n+1}$. We define the *regular Diophantine type* of x to be the supremum over $\sigma \in [1, \infty)$ such that $x \in \mathcal{W}_1^{\mathrm{reg}}(\sigma)$.

Actually, we will work with a more permissive set.

Example/Definition (Essential Diophantine type). We define the set of *essentially σ-approximable* numbers to be

$$\mathcal{W}_1^{\mathrm{ess}}(\sigma) = \{x \in \mathbb{R} : \text{there exists } R \geq 0 \text{ for which } q_{k_{m_j}}^\sigma < q_{k_{m_j}+1} \\ \text{on some s.p.l.a.d. } \{m_j\} \subseteq \mathbb{N}\}.$$

The containments $\mathcal{W}_1^{\mathrm{uni}}(\sigma) \subset \mathcal{W}_1^{\mathrm{reg}}(\sigma) \subset \mathcal{W}_1^{\mathrm{ess}}(\sigma) \subset \mathcal{W}_1(\sigma)$ are clear. Again, any number x is an element of $\mathcal{W}_1^{\mathrm{ess}}(1)$, and we define its *essential Diophantine type* to be the supremum over $\sigma \in [1, \infty)$ where $x \in \mathcal{W}_1^{\mathrm{ess}}(\sigma)$.

3.5. Three Gaps Theorem. For any $x \in \mathbb{R}$ and $m \in \mathbb{N}$ the set $\{qx + \mathbb{Z}\}_{q=1}^m \subset \mathbb{R}/\mathbb{Z}$ cuts the circle $\mathbb{R}/\mathbb{Z}$ into arcs of at most three different lengths; this is known as the Three Gaps Theorem.

For $m \in \mathbb{N}$, write

$$m = rq_k + q_{k-1} + s$$

where $1 \leq r \leq a_{k+1}$ and $0 \leq s < q_k$ as in §3.2, and let $r_x(m+1)$ denote the ratio of the longest gap length to the shortest gap length in the trajectory $\{qx + \mathbb{Z}\}_{q=1}^{m+1} \subset \mathbb{R}/\mathbb{Z}$. Then for $m \in \mathbb{N}$,

$$r_x(m+1) = \begin{cases} \epsilon + \dfrac{\eta_{k+2}}{\eta_{k+1}} + a_{k+2} & \text{if } r = a_{k+1}, \\ \epsilon + \dfrac{\eta_{k+1}}{\eta_k} + (a_{k+1} - r) & \text{if } r < a_{k+1}, \end{cases} \tag{3.1}$$

where $\epsilon = 1$ unless $s = q_k - 1$, in which case $\epsilon = 0$. (See [MK].)

4. Sequences with bounded gap ratios. Formula (3.1) shows that r_x is always bounded if and only if x is badly approximable. On the other hand, for any $R \geq 1$ it is easy to generate a sequence

$$\{L_n := L_n^R := L_n^{x,R}\} \subseteq \mathbb{N}$$

such that the ratios $r_x(L_n)$ are bounded by R for all n, regardless of the continued fraction expansion of x. The reason for doing this is to control the density of points on partial orbits of x of *length* L_n.

LEMMA 11. *Let $x \in \mathbb{R}\setminus\mathbb{Q}$. Suppose the gap ratio for $\{qx+\mathbb{Z}\}_{q=1}^{L} \subset \mathbb{R}/\mathbb{Z}$ is bounded by R, and $L \geq 2$. Then for any $q_0 \in \mathbb{N}$,*

$$\frac{1}{RL} < \ell_{\min} < \frac{1}{L} < \ell_{\max} < \frac{R}{L}$$

where $\ell_{\min}$ and $\ell_{\max}$ are the minimum and maximum arc-lengths into which the set $\{qx+\mathbb{Z}\}_{q=q_0+1}^{q_0+L}$ cuts the circle.

Proof. The L points of $\{qx+\mathbb{Z}\}_{q=1}^{L}$ partition the circle $\mathbb{R}/\mathbb{Z}$ into L intervals. Let $\ell_{\min}$ and $\ell_{\max}$ be the shortest and the longest lengths of these intervals. Assuming x is irrational and $L \geq 2$, we have $\ell_{\min} < 1/L < \ell_{\max}$. (Of course, if $L = 1$, then $\ell_{\min} = \ell_{\max} = 1$, no matter what x is.) By the ratio bound, $\ell_{\max} \leq R\ell_{\min}$. Putting the two inequalities together gives the desired system of inequalities, which is of course unchanged by a rotation by $q_0 x$. ∎

LEMMA 12. *Let $R \geq 0$. Then $r_x(L) \leq 2 + R$ exactly when L belongs to some block*

$$\{q_k - Rq_{k-1}, \ldots, q_\ell\} \subseteq \mathbb{N}$$

of consecutive integers, where $k = 0$ or $a_k > R$, and $\ell \geq k$ indexes the next time $a_{\ell+1} > R$ again. (If it never happens again, we interpret this as $\ell = \infty$ and $q_\infty = \infty$.)

Proof. This follows simply by invoking (3.1). We can list all $m \in \mathbb{N}$ that result in bounded gap ratios, and find that $r_x(m+1) \leq 2+R$ exactly when

$$m \in \begin{cases} \{q_k - Rq_{k-1} - 1, \ldots, q_k - 1\} & \text{for some } a_k > R, \\ \{q_k, \ldots, q_{k+1} - 1\} & \text{for some } a_{k+1} \leq R. \end{cases}$$

Concatenating these blocks and setting $L = m + 1$ gives the lemma. ∎

REMARK. A consequence of this lemma that is interesting in itself (and probably known to experts) is that the continuants $\{q_n\}_{n=0}^{\infty}$ are *exactly* the times when the gap ratios for $\{qx\}_{q=1}^{q_n}$ are bounded by 2.

If we form the sequence $\{k_m := k_m^{x,R}\}$ and set

$$B_{m+1} = B_{m+1}^{x,R} = [q_{k_m+1} - Rq_{k_m}, q_{k_{m+1}}] \cap \mathbb{N}, \tag{4.1}$$

Lemma 12 implies that our sequence of $2+R$-bounded gap ratios is the concatenation $\{L_n\} = \{B_1, B_2, \dots\}$. If the sequence $\{k_m\}$ terminates at k_t, then

$$B_{t+1} = [q_{k_t+1} - Rq_{k_t}, \infty) \cap \mathbb{N}.$$

This happens only if $x \in \mathbb{R} \setminus \mathbb{Q}$ is badly approximable; conversely, if x is badly approximable, we can choose $R \geq 0$ large enough that this happens.

4.1. Calculations based on (4.1)**.** It will be useful to keep certain measurements of $B_m^{x,R}$ in mind. First, the length of the block B_{m+1} is

$$|B_{m+1}| = q_{k_{m+1}} - q_{k_m+1} + Rq_{k_m} + 1. \tag{4.2}$$

If the sequence $\{k_m\}$ terminates at k_t, then we can obviously consider $|B_{t+1}|$ to be infinite.

Let $\{\omega_m\}_{m=1}^{\infty}$ be the sequence such that $L_{\omega_m} = q_{k_m}$ is the right end-point of the block B_m. Then ω_m is the sum of the lengths of the blocks $B_1, \dots, B_m$, which, by (4.2), is

$$\omega_m = \sum_{n=0}^{m-1} (q_{k_{n+1}} - q_{k_n+1} + Rq_{k_n} + 1).$$

Let α_m be the index for the left end-point L_{α_m} of the block B_m, so that $\alpha_m = \omega_{m-1} + 1$ for all $m \in \mathbb{N}$, and $\alpha_1 = 1$.

The distance between consecutive blocks B_{m+1} and B_m is

$$B_{m+1} - B_m := \min B_{m+1} - \max B_m = q_{k_m+1} - (R+1)q_{k_m}.$$

The following lemma describes the sum Σ_{B_m} of the elements in the block B_m.

Lemma 13. *We have*

$$\Sigma_{B_m} \sim \tfrac{1}{2}(q_{k_m}^2 - (q_{k_{m-1}+1} - Rq_{k_{m-1}})^2).$$

In particular,

$$q_{k_m} q_{k_{m-1}} \ll \Sigma_{B_m} \ll q_{k_m} q_{k_m-1} \quad \text{for all } m \in \mathbb{N}.$$

Also,

$$q_{k_m}^2 \ll \Sigma_{B_m} \ll q_{k_m} q_{k_m-1} \quad \text{whenever} \quad q_{k_m} \geq 2q_{k_{m-1}+1}$$

(i.e. if $a_{k_m} > 1$ or $k_m - k_{m-1} > 2$).

Proof. Block sums are given by the formula

$$\begin{aligned}\Sigma_{B_m} &= \tfrac{1}{2}(q_{k_m} + q_{k_{m-1}+1} - Rq_{k_{m-1}})(q_{k_m} - q_{k_{m-1}+1} + Rq_{k_{m-1}} + 1)\\ &\sim \tfrac{1}{2}(q_{k_m}^2 - (q_{k_{m-1}+1} - Rq_{k_{m-1}})^2).\end{aligned}$$

If $k_m \neq k_{m-1}+1$,

$$\begin{aligned}\frac{\Sigma_{B_m}}{q_{k_m}q_{k_m-1}} &\sim \frac{q_{k_m}^2-(q_{k_{m-1}+1}-Rq_{k_{m-1}})^2}{2q_{k_m}q_{k_m-1}}\\ &= \frac{1}{2}\left(\frac{q_{k_m}}{q_{k_m-1}}-\frac{q_{k_{m-1}+1}^2}{q_{k_m}q_{k_m-1}}-R^2\frac{q_{k_{m-1}}^2}{q_{k_m}q_{k_m-1}}+2R\frac{q_{k_{m-1}+1}q_{k_{m-1}}}{q_{k_m}q_{k_m-1}}\right)\\ &\leq \frac{1}{2}((a_{k_m}+1)+2R) \ll 1\end{aligned}$$

because $a_{k_m} \leq R$ in this case. On the other hand, if $k_m = k_{m-1}+1$, then

$$\begin{aligned}\frac{\Sigma_{B_m}}{q_{k_m}q_{k_m-1}} &\sim \frac{q_{k_m}^2-(q_{k_m}-Rq_{k_{m-1}})^2}{2q_{k_m}q_{k_m-1}}\\ &= \frac{2Rq_{k_m}q_{k_{m-1}}-R^2q_{k_{m-1}}^2}{2q_{k_m}q_{k_m-1}} \ll 1,\end{aligned}$$

which establishes the upper bound.

For the lower bound, first suppose that $k_{m-1}+1 = k_m$. In this case

$$q_{k_m}q_{k_m-1} = (a_{k_m}q_{k_m-1}+q_{k_m-2})q_{k_m-1} \geq (R+1)q_{k_m-1}^2,$$

so that

$$\begin{aligned}\frac{q_{k_m}^2-(q_{k_{m-1}+1}-Rq_{k_{m-1}})^2}{q_{k_m}q_{k_m-1}} &= \frac{2Rq_{k_m}q_{k_m-1}-R^2q_{k_m-1}^2}{q_{k_m}q_{k_m-1}}\\ &= 2R-\frac{R^2q_{k_m-1}^2}{q_{k_m}q_{k_m-1}} = 2R-\frac{R^2}{R+1} > R,\end{aligned}$$

proving $\Sigma_{B_m} \gg q_{k_m}q_{k_m-1} = q_{k_m}q_{k_{m-1}}$ in this case.

If $k_{m-1}+2 = k_m$ and $a_{k_m} = 1$ then

$$\begin{aligned}\Sigma_{B_m} &= \tfrac{1}{2}(q_{k_m}+q_{k_{m-1}+1}-Rq_{k_{m-1}})(q_{k_m}-q_{k_{m-1}+1}+Rq_{k_{m-1}}+1)\\ &= \tfrac{1}{2}(2q_{k_{m-1}+1}-(R-1)q_{k_{m-1}})((R+1)q_{k_{m-1}}+1)\\ &= (R+1)q_{k_{m-1}+1}q_{k_{m-1}}-\tfrac{1}{2}(R+1)(R-1)q_{k_{m-1}}^2\\ &\quad + q_{k_{m-1}+1}-\tfrac{1}{2}(R-1)q_{k_{m-1}}.\end{aligned}$$

Dividing by $q_{k_{m-1}+1}q_{k_{m-1}}$ gives

$$\begin{aligned}&\frac{\Sigma_{B_m}}{q_{k_{m-1}+1}q_{k_{m-1}}}\\ &\quad= (R+1)-\frac{(R+1)(R-1)q_{k_{m-1}}^2}{2q_{k_{m-1}+1}q_{k_{m-1}}}+\frac{q_{k_{m-1}+1}}{q_{k_{m-1}+1}q_{k_{m-1}}}-\frac{(R-1)q_{k_{m-1}}}{2q_{k_{m-1}+1}q_{k_{m-1}}}\\ &\quad\sim (R+1)-\frac{(R+1)(R-1)q_{k_{m-1}}}{2q_{k_{m-1}+1}} \geq (R+1)-\frac{(R-1)}{2} \gg 1.\end{aligned}$$

which proves $\Sigma_{B_m} \gg q_{k_m-1}q_{k_m-2}$. But in this case we have $q_{k_m} = q_{k_m-1} + q_{k_m-2} \leq 2q_{k_m-1}$, so we have proved $\Sigma_{B_m} \gg q_{k_m}q_{k_m-2} = q_{k_m}q_{k_m-1}$.

In the remaining cases we have $k_{m-1}+1 \neq k_m$ and there is some integer $A \geq 2$ such that

$$q_{k_m} \geq Aq_{k_{m-1}+1} + q_{k_{m-1}}.$$

We write

$$q_{k_m}^2 - (q_{k_{m-1}+1} - Rq_{k_{m-1}})^2 = \frac{A^2-1}{A^2}q_{k_m}^2 + \frac{1}{A^2}q_{k_m}^2 - (q_{k_{m-1}+1} - Rq_{k_{m-1}})^2$$

and proceed to bound

$$\begin{aligned}
&\frac{A^2-1}{A^2}q_{k_m}^2 + \frac{1}{A^2}q_{k_m}^2 - (q_{k_{m-1}+1} - Rq_{k_{m-1}})^2 \\
&\geq \frac{A^2-1}{A^2}q_{k_m}^2 + \left(q_{k_{m-1}+1} + \frac{1}{A}q_{k_{m-1}}\right)^2 \\
&\quad - \left(q_{k_{m-1}+1} + \frac{1}{A^2}q_{k_{m-1}} - \left(R + \frac{1}{A}\right)q_{k_{m-1}}\right)^2 \\
&= \frac{A^2-1}{A^2}q_{k_m}^2 + 2\left(R+\frac{1}{A}\right)\left(q_{k_{m-1}+1} + \frac{1}{A}q_{k_{m-1}}\right)q_{k_{m-1}} - \left(R+\frac{1}{A}\right)^2 q_{k_{m-1}}^2 \\
&\geq \frac{A^2-1}{A^2}q_{k_m}^2 + \left(R+\frac{1}{A}\right)^2 q_{k_{m-1}}^2 \gg q_{k_m}^2
\end{aligned}$$

because $A \geq 2$. ∎

4.2. Positive density property. The following definition is relevant to our "divergence" results.

Definition 14 (Positive density property). We say $x \in \mathbb{R} \setminus \mathbb{Q}$ has the *positive density property* if there exists $R \geq 1$ such that

$$\limsup_{m\to\infty} \frac{L_{\alpha_m}^R}{\Sigma_{\alpha_m}^R} < 1.$$

An intuitive interpretation is that a number with positive density property has blocks $B_m := B_m^{x,R}$ that are not too far away from each other.

Proposition 15. *The number $x \in \mathbb{R}\setminus\mathbb{Q}$ has the positive density property if and only if*

$$q_{k_m+1} - Rq_{k_m} \ll \sum_{\ell=1}^{m}(q_{k_\ell}^2 - (q_{k_{\ell-1}+1} - Rq_{k_{\ell-1}})^2)$$

as $m \to \infty$. In particular,

- *any $x \notin \mathcal{W}_1(\varphi)$, and*
- *any $x \notin \mathcal{W}_1(2)$ for which there exists $R \geq 1$ such that $q_{k_m} \geq 2q_{k_{m-1}+1}$ for all but finitely many $m \in \mathbb{N}$*

has the positive density property.

Proof. The positive density property is the requirement that there is some $\delta < 1$ such that

$$\frac{L^R_{\alpha_{m+1}}}{\Sigma^R_{\alpha_{m+1}}} = \frac{q_{k^R_m+1} - Rq_{k^R_m}}{q_{k^R_m+1} - Rq_{k^R_m} + \Sigma^R_{\omega_m}} \le \delta$$

for all sufficiently large m. This is equivalent to $q_{k_m+1} - Rq_{k_m} \ll \Sigma_{\omega_m}$, which by Lemma 13 is equivalent to

$$q_{k_m+1} - Rq_{k_m} \ll \sum_{\ell=1}^{m} (q^2_{k_\ell} - (q_{k_{\ell-1}+1} - Rq_{k_{\ell-1}})^2).$$

In particular,

$$q_{k_m+1} - Rq_{k_m} \ll \Sigma_{B_m} \tag{4.3}$$

is sufficient.

If $k_{m-1} + 1 = k_m$, the sufficient condition (4.3) becomes

$$q_{k_m+1} - Rq_{k_m} \ll q_{k_m} q_{k_m-1},$$

for which it is sufficient that

$$\frac{q_{k_m+1}}{q_{k_m} q_{k_m-1}} \ll 1.$$

We will have this comparison whenever $x \notin \mathcal{W}_1(\varphi)$.

On the other hand, if $k_{m-1}+1 \ne k_m$, then Lemma 13 allows us to consider two cases: either $\Delta k_{m-1} := k_m - k_{m-1} = 2$ and $a_{k_m} = 1$, or $q_{k_m} \ge 2q_{k_{m-1}+1}$. In the first case, (4.3) becomes

$$q_{k_m+1} - Rq_{k_m} \ll q_{k_m} q_{k_{m-1}} = q_{k_m} q_{k_m-2},$$

and for this it is sufficient that

$$\frac{q_{k_m+1}}{q_{k_m} q_{k_m-2}} \ll 1.$$

If $x \notin \mathcal{W}_1(\varphi)$, then

$$\frac{q_{k_m+1}}{q_{k_m} q_{k_m-2}} \ll \frac{q^{\varphi}_{k_m}}{q_{k_m} q^{1/\varphi}_{k_m-1}},$$

and since $q_{k_m} = q_{k_m-1} + q_{k_m-2} \le 2q_{k_m-1}$ in this case,

$$\frac{q^{\varphi}_{k_m}}{q_{k_m} q^{1/\varphi}_{k_m-1}} \ll \frac{q^{\varphi}_{k_m}}{q^{1+1/\varphi}_{k_m}} \ll 1,$$

as desired.

In the case of $q_{k_m} \ge 2q_{k_{m-1}+1}$, the last part of Lemma 13 implies that

$$q_{k_m+1} - Rq_{k_m} \ll q^2_{k_m}$$

is sufficient for (4.3) to hold. This is satisfied whenever $x \notin \mathcal{W}_1(2)$. In particular, if $x \notin \mathcal{W}_1(\varphi)$, then (4.3) holds, which proves the first point in the proposition. This last paragraph also proves the second point. ∎

4.3. Bounded ratio property. The following property is slightly stronger than the positive density property. It is relevant to our "convergence" results.

Definition 16 (Bounded ratio property). We say that $x \in \mathbb{R} \setminus \mathbb{Q}$ has the *bounded ratio property* if there exists a bound $R \geq 1$ such that

$$\sum_{m\in\mathbb{N}} \frac{B_{m+1}^R - B_m^R}{\Sigma_{\omega_m}^R} < \infty.$$

This is equivalent to

$$\sum_{m\in\mathbb{N}} \frac{L_{\alpha_{m+1}}}{\Sigma_{\omega_m}} < \infty.$$

Again, having the bounded ratio property means that the jumps between the blocks B_m are not too high.

The following proposition gives numbers with the bounded ratio property, based on Diophantine type.

Proposition 17 (Numbers with the bounded ratio property).

- *Every number of Diophantine type less than* $\varphi = (1 + \sqrt{5})/2$ *has the bounded ratio property.*
- *Every number of Diophantine type less than* 2 *for which there is some* $R \geq 1$ *such that* $q_{k_m} \geq 2q_{k_{m-1}+1}$ *for all but finitely many* $m \in \mathbb{N}$ *has the bounded ratio property.*

Remark. Notice that these are not the same numbers listed in Proposition 15. There, we require (for example) that $x \notin \mathcal{W}_1(\varphi)$, whereas here we are requiring that $x \notin \mathcal{W}_1(\sigma)$ for some $\sigma < \varphi$, which is slightly stronger.

Proof of Proposition 17. For the first assertion, let $\sigma < \varphi$ be such that $x \notin \mathcal{W}_1(\sigma)$. By Lemma 13 we have

$$\frac{L_{\alpha_{m+1}}}{\Sigma_{\omega_m}} \ll \frac{q_{k_m+1}}{q_{k_m} q_{k_m-1}}$$

unless $\Delta k_{m-1} = 2$ and $a_{k_m} = 1$, and in turn

$$\ll \frac{q_{k_m+1}}{q_{k_m} q_{k_m-1}} \ll q_{k_m}^{\sigma-1-1/\sigma}.$$

On the other hand, if $\Delta k_{m-1} = 2$ and $a_{k_m} = 1$, then $q_{k_m} \asymp q_{k_m-1}$, so

$$\frac{L_{\alpha_{m+1}}}{\Sigma_{\omega_m}} \ll \frac{q_{k_m+1}}{q_{k_m} q_{k_m-2}} \ll \frac{q_{k_m}^{\sigma}}{q_{k_m-1}^{1+1/\sigma}} \ll q_{k_m}^{\sigma-1-1/\sigma},$$

as above. Now, the sum $\sum_{m\in\mathbb{N}} q_{k_m}^{\sigma-1-1/\sigma}$ converges because $\sigma-1-1/\sigma<0$. Therefore, x has the bounded ratio property.

For the second assertion, let $\sigma<2$ and let $x\notin\mathcal{W}_1(\sigma)$ be such that $q_{k_m}\geq 2q_{k_{m-1}+1}$ for all but finitely many $m\in\mathbb{N}$, for some $R\geq 1$. By Lemma 13,

$$\frac{L_{\alpha_{m+1}}}{\Sigma_{\omega_m}} \ll \frac{q_{k_m+1}}{q_{k_m}^2} \ll q_{k_m}^{\sigma-2},$$

and the sum $\sum_{m\in\mathbb{N}} q_{k_m}^{\sigma-2}$ diverges because $\sigma-2<0$. Therefore x has the bounded ratio property, and the proposition is proved. ∎

5. Some counting lemmas. The counting Lemmas 18 and 20 below give bounds on $|\mathcal{Q}(x,\psi)\cap[M,N]|$ when M and N come from our bounded ratio sequences $\{L_n\}$.

Lemma 18. *Let $\{L_n\}$ be a sequence of R-bounded gap ratios for $x\in\mathbb{R}\setminus\mathbb{Q}$, and define $\Sigma_n=L_1+\cdots+L_n$. If ψ is an approximating function such that $L_n\psi(\Sigma_n)\geq R$ for n sufficiently large, then*

$$\sum_{\substack{q=\Sigma_n+1\\ q\in\mathcal{Q}(x,\psi)}}^{\Sigma_{n+1}} 1 \gg L_{n+1}\,\psi(\Sigma_{n+1}) \qquad \text{as } n\to\infty,$$

where $\mathcal{Q}(x,\psi)=\{q\in\mathbb{N}:\|qx\|<\psi(q)\}$ is the set of denominators that ψ-approximate x in $\mathbb{R}$.

Proof. We bound below

$$\sum_{\substack{q=\Sigma_n+1\\ q\in\mathcal{Q}(x,\psi)}}^{\Sigma_{n+1}} 1 \geq |\{qx\}_{q=\Sigma_n+1}^{\Sigma_{n+1}}\cap[0,\psi(\Sigma_{n+1}))| \geq \left\lfloor\frac{\psi(\Sigma_{n+1})}{\ell_{\max}}\right\rfloor,$$

which by Lemma 11 we can bound below by

$$\left\lfloor \psi(\Sigma_{n+1})\cdot\frac{L_{n+1}}{R}\right\rfloor \gg L_{n+1}\,\psi(\Sigma_{n+1})$$

as $n\to\infty$, because we have assumed that $L_n\psi(\Sigma_n)\geq R$ eventually. ∎

The next lemma will allow us to assume without loss of generality that ψ satisfies the conditions of Lemma 18.

Lemma 19. *Let $\{L_n\}$ be a sequence of R-bounded gap ratios for $x\in\mathbb{R}\setminus\mathbb{Q}$, and define $\Sigma_n=L_1+\cdots+L_n$. Let ψ be an approximating function. There is an approximating function $\tilde\psi\geq\psi$ such that $L_n\,\tilde\psi(\Sigma_n)\geq R$ and*

$$\sum_{q\in\mathcal{Q}(x,\tilde\psi)}\tilde\psi(q)^{d-1}=\infty \;\Rightarrow\; \sum_{q\in\mathcal{Q}(x,\psi)}\psi(q)^{d-1}=\infty$$

for any $d\geq 3$.

Proof. Let φ be the approximating function defined by $\varphi(q) = RL_n^{-1}$ where $q \in (\Sigma_{n-1}, \Sigma_n]$ and define $\tilde{\psi}(q) := \max\{\psi(q), \varphi(q)\}$. Let $A = \{q : \psi(q) \geq \varphi(q)\}$ and $B = \mathbb{N} \setminus A$. Then

$$\sum_{q \in \mathcal{Q}(x,\tilde{\psi})} \tilde{\psi}(q)^{d-1} = \sum_{q \in A \cap \mathcal{Q}(x,\psi)} \psi(q)^{d-1} + \sum_{q \in B \cap \mathcal{Q}(x,\varphi)} \varphi(q)^{d-1}. \tag{5.1}$$

The second sum is bounded by

$$\sum_{q \in \mathcal{Q}(x,\varphi)} \varphi(q)^{d-1} = \sum_{n \in \mathbb{N}} |\mathcal{Q}(x,\varphi) \cap (\Sigma_{n-1}, \Sigma_n]| \left(\frac{R}{L_n}\right)^{d-1}. \tag{5.2}$$

By Lemma 11,

$$|\mathcal{Q}(x,\varphi) \cap (\Sigma_{n-1}, \Sigma_n]| < \frac{2R}{L_n} / \ell_{\min} < \frac{2R}{L_n} \Big/ \frac{1}{L_n} = 2R,$$

so we can bound (5.2) by

$$2R^d \sum_{n \in \mathbb{N}} \left(\frac{1}{L_n}\right)^{d-1},$$

which converges as long as $d - 1 > 1$. Now formula (5.1) shows that if $\sum_{q \in \mathcal{Q}(x,\tilde{\psi})} \tilde{\psi}(q)^{d-1}$ diverges, then so does $\sum_{q \in \mathcal{Q}(x,\psi)} \psi(q)^{d-1}$. ∎

Remark. Besides our repeated applications of Gallagher's Theorem, Lemma 19 is the only other place where we need $d \geq 3$. Notice that the sum $\sum_{n \in \mathbb{N}} L_n^{-1}$ can diverge, for example, if x is badly approximable.

The following lemma should be compared with Lemma 18.

Lemma 20. *Let $\{L_n\}$ be a sequence of R-bounded gap ratios for $x \in \mathbb{R} \setminus \mathbb{Q}$, and define $\Sigma_n = L_1 + \cdots + L_n$. If ψ is an approximating function, then*

$$\sum_{\substack{q = \Sigma_{n-1} \\ q \in \mathcal{Q}(x,\psi)}}^{\Sigma_n - 1} 1 \ll L_n \psi(\Sigma_{n-1}) \quad \text{as } n \to \infty.$$

Proof. We bound above

$$\sum_{\substack{q = \Sigma_{n-1} \\ q \in \mathcal{Q}(x,\psi)}}^{\Sigma_n - 1} 1 \leq |\{qx\}_{q=\Sigma_{n-1}}^{\Sigma_n - 1} \cap [0, \psi(\Sigma_{n-1}))|,$$

which by Lemma 11 can be bounded by

$$\psi(\Sigma_{n-1}) \Big/ \frac{1}{RL_n} \ll L_n \psi(\Sigma_{n-1}) \quad \text{as } n \to \infty. \ ∎$$

Lemma 21. *If ψ is an approximating function such that $\sum_{q \in \mathbb{N}} \psi(q)^d$ converges, then $\psi(q) \ll q^{-1/d}$.*

Proof. Since ψ is non-increasing, convergence of $\sum_{q\in\mathbb{N}}\psi(q)^d$ is equivalent to convergence of $\sum_{k\in\mathbb{N}}2^k\psi(2^k)^d$, therefore we know that the terms $2^k\psi(2^k)^d$ approach 0, meaning that for any $c>0$, we eventually have $\psi(2^k)^d < c\cdot 2^{-k}$, so $\psi(q) \ll q^{-1/d}$ on the sequence $\{2^k\}_{k\in\mathbb{N}}$ with some implied constant $C>0$. Now, every q is between some 2^k and the next one, so

$$2^k < q \le 2^{k+1} \quad \text{and} \quad \psi(2^{k+1}) \le \psi(q) < \psi(2^k).$$

Combining these and our previous observations we find

$$\psi(q)^d < \psi(2^k)^d \le C\cdot 2^{-k} \le C\cdot 2/q,$$

and we have shown $\psi(q) \ll q^{-1/d}$ with implied constant $(2C)^{1/d}$. ∎

6. Proofs of divergence results. In this section, we work with approximating functions ψ with $\sum_{q\in\mathbb{N}}\psi(q)^d$ diverging. Our goal is to determine when we can guarantee the divergence of

$$\sum_{q\in\mathcal{Q}(x,\psi)} \psi(q)^{d-1} \tag{6.1}$$

so that we can apply Gallagher's extension of Khintchine's Theorem to the hyperplane passing through $x\in\mathbb{R}$. To this end, define a subset $A(x,R)\subseteq\mathbb{N}$ as the concatenation $A(x,R)=\{A_1^{(x,R)},A_2^{(x,R)},\ldots\}$ of blocks

$$A_\ell^{(x,R)} = [\varSigma_{\alpha_\ell}, \varSigma_{\omega_\ell}-1]\cap\mathbb{N}.$$

We prove the following lemma.

Lemma 22. *Let $x\in\mathbb{R}\setminus\mathbb{Q}$. If there exists a number $R\ge 1$ such that*

$$\sum_{q\in A(x,R)} \psi(q)^d \tag{6.2}$$

diverges, then (6.1) *diverges.*

Proof. We write partial sums of (6.1) along $\{\varSigma_N\}$ as

$$\sum_{\substack{q=1\\ q\in\mathcal{Q}(x,\psi)}}^{\varSigma_N} \psi(q)^{d-1} = \sum_{n=0}^{N-1} \sum_{\substack{q=\varSigma_n+1\\ q\in\mathcal{Q}(x,\psi)}}^{\varSigma_{n+1}} \psi(q)^{d-1},$$

where $\varSigma_N = L_1+\cdots+L_N$, and $\varSigma_0=0$. Since ψ is non-increasing we can bound this below by

$$\sum_{n=0}^{N-1} \psi(\varSigma_{n+1})^{d-1} \sum_{\substack{q=\varSigma_n+1\\ q\in\mathcal{Q}(x,\psi)}}^{\varSigma_{n+1}} 1,$$

and Lemma 19 allows us to assume without loss of generality that $L_n\psi(\Sigma_n) \geq 2 + R$, so that we can apply Lemma 18 to see that the above is

$$\gg \sum_{n=1}^{N} L_n\psi(\Sigma_n)^d.$$

Rewriting along the subsequence $\{\omega_m - 1\}$,

$$\sum_{n=1}^{\omega_m-1} L_n\,\psi(\Sigma_n)^d = \sum_{\ell=0}^{m}\sum_{n=\alpha_\ell}^{\omega_\ell-1} L_n\,\psi(\Sigma_n)^d + \sum_{\ell=0}^{m-1} L_{\omega_\ell}\,\psi(\Sigma_{\omega_\ell})^d,$$

we can safely ignore the second sum because it converges as $m \to \infty$. Since $L_{n+1} = L_n + 1$, except when $n = \omega_\ell$,

$$\sum_{\ell=0}^{m}\sum_{n=\alpha_\ell}^{\omega_\ell-1} L_n\psi(\Sigma_n)^d \gg \sum_{\ell=0}^{m}\sum_{n=\alpha_\ell}^{\omega_\ell-1} L_{n+1}\,\psi(\Sigma_n)^d \geq \sum_{\ell=0}^{m}\sum_{n=\alpha_\ell}^{\omega_\ell-1}\sum_{q=\Sigma_n}^{\Sigma_{n+1}-1} \psi(q)^d$$
$$= \sum_{\ell=0}^{m}\sum_{q=\Sigma_{\alpha_\ell}}^{\Sigma_{\omega_\ell}-1} \psi(q)^d,$$

and taking $m \to \infty$, we have bounded (6.1) below by (6.2), which implies the result. ∎

The challenge now is to determine when we can find $R \geq 1$ such that (6.2) diverges.

6.1. Proofs of Theorems 3 and 4. Since $\sum_{q\in\mathbb{N}} \psi(q)^d$ diverges, it is sufficient to find $A(x,R)$ with positive lower asymptotic density in $\mathbb{N}$.

LEMMA 23. *We have*

$$\underline{\mathrm{d}}(A(x,R)) > 0 \;\Leftrightarrow\; \limsup_{m\to\infty} \frac{L^R_{\alpha_m}}{\Sigma^R_{\alpha_m}} < 1,$$

that is, $A(x,R)$ has positive lower asymptotic density for some $R \geq 1$ if and only if $x \in \mathbb{R}\setminus\mathbb{Q}$ has the positive density property.

Proof. Since $A(x,R)$ is made up of blocks of consecutive integers, the lower asymptotic density is achieved by computing along the subsequence corresponding to the points just before the left end-points of each block. That is,

$$\underline{\mathrm{d}}(A(x,R)) = \liminf_{m\to\infty} \frac{\sum_{\ell\leq m}|A_\ell|}{\min A_{m+1} - 1} = \liminf_{m\to\infty} \frac{\sum_{\ell\leq m}(\Sigma_{\omega_\ell} - \Sigma_{\alpha_\ell})}{\Sigma_{\alpha_{m+1}} - 1}$$
$$= \liminf_{m\to\infty} \frac{\Sigma_{\alpha_{m+1}} - \sum_{\ell\leq m+1} L_{\alpha_\ell}}{\Sigma_{\alpha_{m+1}} - 1}$$

$$= \liminf_{m\to\infty} \frac{\Sigma_{\alpha_{m+1}} - (L_{\alpha_{m+1}} + L_{\alpha_m} + \cdots + L_{\alpha_0})}{\Sigma_{\alpha_m+1} - 1}$$
$$= 1 - \limsup_{m\to\infty} \frac{L_{\alpha_{m+1}} + L_{\alpha_m} + \cdots + L_{\alpha_0} - 1}{\Sigma_{\alpha_{m+1}} - 1},$$

and so $\underline{\mathrm{d}}(A(x,R)) > 0$ if and only if

$$\limsup_{m\to\infty} \frac{L_{\alpha_{m+1}} + L_{\alpha_m} + \cdots + L_{\alpha_0}}{\Sigma_{\alpha_{m+1}}} < 1;$$

but

$$\lim_{m\to\infty} \frac{L_{\alpha_m} + L_{\alpha_{m-1}} + \cdots + L_{\alpha_0}}{\Sigma_{\alpha_{m+1}}} = 0,$$

so we have proved the claim. ∎

Since divergent series diverge along subseries of positive lower asymptotic density, this lemma all but solves the problem for fibers over points with the positive density property. The following lemma shows that, at least for some approximating functions, one can deal with fibers over base-points that do not have the positive density property.

Lemma 24. *For any $x \in \mathbb{R} \setminus \mathbb{Q}$ and $R \geq 1$ the set $A(x,R)$ has positive upper asymptotic density.*

Proof. Since $A(x,R)$ is made up of blocks of consecutive integers, the upper asymptotic density is achieved by computing along the subsequence corresponding to the right end-points of each block. That is,

$$\overline{\mathrm{d}}(A(x,R)) = \limsup_{m\to\infty} \frac{\sum_{\ell\le m}|A_\ell|}{\max A_m} = \limsup_{m\to\infty} \frac{\sum_{\ell\le m}(\Sigma_{\omega_\ell} - \Sigma_{\alpha_\ell})}{\Sigma_{\omega_m} - 1}$$
$$= \limsup_{m\to\infty} \frac{\Sigma_{\omega_m} - \sum_{\ell\le m} L_{\alpha_\ell}}{\Sigma_{\omega_m} - 1}$$
$$= \limsup_{m\to\infty} \frac{\Sigma_{\omega_m} - (L_{\alpha_m} + L_{\alpha_{m-1}} + \cdots + L_{\alpha_0})}{\Sigma_{\omega_m} - 1}$$
$$= 1 - \liminf_{m\to\infty} \frac{L_{\alpha_m} + L_{\alpha_{m-1}} + \cdots + L_{\alpha_0} - 1}{\Sigma_{\omega_m} - 1},$$

and so $\overline{\mathrm{d}}(A(x,R)) > 0$ if and only if

$$\liminf_{m\to\infty} \frac{L_{\alpha_m} + L_{\alpha_{m-1}} + \cdots + L_{\alpha_1} + L_{\alpha_0}}{\Sigma_{\omega_m}} < 1;$$

but this is always the case. ∎

We can now prove that almost every point on every fiber is ψ-approximable if $\sum_A \psi(q)^d$ diverges for every $A \subseteq \mathbb{N}$ with positive upper asymptotic density.

Proof of Theorem 4. Lemma 24 tells us that for any $x \in \mathbb{R} \setminus \mathbb{Q}$ and $R \geq 1$, the set $A(x, R)$ has positive upper asymptotic density. By assumption, then, (6.2) diverges. Therefore, by Lemma 22, so does (6.1). Since $d - 1 \geq 2$, Gallagher's Theorem applies to the hyperplane $\{x\} \times \mathbb{R}^{d-1}$ and the approximating function ψ. ∎

These density considerations only give *sufficient* conditions for divergence, and the following lemma serves to show that they are not necessary.

LEMMA 25. *For any $R \geq 1$, there are uncountably many $x \in \mathbb{R} \setminus \mathbb{Q}$ of any given Diophantine type such that* (6.2) *diverges.*

Proof. We offer a construction. Fix $R \geq 1$. Let

$$\Psi(m) := \sum_{\ell=0}^{m} \sum_{q=\Sigma_{\alpha_\ell}}^{\Sigma_{\omega_\ell}-1} \psi(q)^d$$

be a partial sum of (6.2). The sequence $\{\Psi(m)\}$ is increasing, and we can make

$$\Psi(m) - \Psi(m-1) = \sum_{q=\Sigma_{\alpha_m}}^{\Sigma_{\omega_m}-1} \psi(q)^d$$

as large as we wish by choosing $\Delta k_{m-1} - 1 := k_m - k_{m-1} - 1$ arbitrarily large, so we can make $\Psi(m) \to \infty$ simply by prescribing $\{k_m\}$.

To see that we can achieve any Diophantine type, we observe that at each step, after having chosen k_m so that $\Psi(m) - \Psi(m-1)$ has the desired size, we are free to choose a_{k_m+1} without affecting $\Psi(m)$. Therefore we can ensure that any given $\sigma \in [1, \infty)$ is the infimum of $\tau \in \mathbb{R}$ satisfying $q_{k_m+1} \ll q_{k_m}^{\tau}$ as $m \to \infty$. ∎

We are now prepared to prove the following theorem, from which Theorem 3 immediately follows.

THEOREM 26. *Let $d \geq 3$. If ψ is an approximating function such that $\sum_{q\in\mathbb{N}} \psi(q)^d$ diverges, then*

$$\sum_{q \in \mathcal{Q}(x,\psi)} \psi(q)^{d-1} = \infty$$

for:

(a) *Any $x \in \mathbb{Q}$.*
(b) *Any $x \in \mathbb{R} \setminus \mathbb{Q}$ with the positive density property.*
(c) *Uncountably many $x \in \mathbb{R} \setminus \mathbb{Q}$ of any given Diophantine type.*

Proof. (a) In the case of rational $x = a/b$, the set $\mathcal{Q}(x, \psi)$ contains the arithmetic sequence $\{kb\}_{k\in\mathbb{N}}$. Then

$$\sum_{q\in\mathcal{Q}(x,\psi)} \psi(q)^{d-1} \gg \sum_{k\in\mathbb{N}} b\psi(kb)^{d-1} \gg \sum_{q\in\mathbb{N}} \psi(q)^{d-1} = \infty.$$

We have used here that ψ is non-increasing. (Notice that this does not require $d \geq 3$.)

(b) If $x \in \mathbb{R}\setminus\mathbb{Q}$ has the positive density property, then Lemma 23 implies that there is some $R \geq 1$ such that $A(x,R)$ has positive lower asymptotic density. This implies that (6.2) diverges, which by Lemma 22 implies that (6.1) diverges.

(c) By Lemma 25, there are uncountably many $x \in \mathbb{R}\setminus\mathbb{Q}$ of any given Diophantine type such that (6.2) diverges, and again Lemma 22 implies that (6.1) diverges. ∎

Proof of Theorem 3. Apply Gallagher's Theorem to fibers over the basepoints in Theorem 26. ∎

6.2. Proofs of Theorems 1 and 2. By the discussion in §2.7, the proof of Theorem 1 reduces to the following lemma.

LEMMA 27. *If $x \in \mathbb{R}\setminus\mathbb{Q}$ is not Liouville, then* (6.2) *diverges for $\psi(q) = (q\log q)^{-1/d}$.*

Proof. If x has the positive density property, then there is some $R \geq 1$ for which $A(x,R)$ has positive lower asymptotic density, by Lemma 23. This implies that (6.2) diverges.

On the other hand, if x does not have the positive density property, then

$$\limsup_{m\to\infty} \frac{L^R_{\alpha_m}}{\Sigma^R_{\alpha_m}} = 1$$

no matter which $R \geq 1$ we choose. Therefore, after fixing some $R \geq 1$, there is some sequence $\{m_j\} \subseteq \mathbb{N}$ where the limit superior is achieved, which means that on this sequence we have $L_{\alpha_{m_j}} \sim \Sigma_{\alpha_{m_j}}$. The partial sums of (6.2) are then bounded by

$$\sum_{q=\Sigma_{\alpha_m}}^{\Sigma_{\omega_m}-1} \psi(q)^d \geq \int_{\Sigma_{\alpha_m}}^{\Sigma_{\omega_m}} \frac{1}{q\log q} = \log\frac{\log \Sigma_{\omega_m}}{\log \Sigma_{\alpha_m}};$$

but

$$\frac{\log \Sigma_{\omega_{m_j}}}{\log \Sigma_{\alpha_{m_j}}} \sim \frac{\log \Sigma_{\omega_{m_j}}}{\log L_{\alpha_{m_j}}} \overset{\text{Lem. 13}}{\gtrsim} \frac{\log q_{k_{m_j}-1} q_{k_{m_j-1}}}{\log q_{k_{m_j-1}+1}} \geq 1 + \frac{1}{\sigma} \quad \text{as } j\to\infty,$$

where $\sigma \in [1,\infty)$ is such that $x \notin \mathcal{W}_1(\sigma)$. This implies that there is some $\delta > 0$ such that

$$\sum_{q=\Sigma_{\alpha_m}}^{\Sigma_{\omega_m}-1} \psi(q)^d \geq \delta$$

infinitely often. (We can take any $\delta < \log(1+1/\sigma)$.) Hence (6.2) diverges. ∎

Proof of Theorem 1. We can now apply the divergence part of Gallagher's Theorem to any fiber over a non-Liouville base-point. The fibers over Liouville base-points are covered by Khintchine's transference principle, in view of the remark at the end of §2.1. ∎

The next two lemmas combine to form Theorem 30, which is more general than Theorem 2. Note that for $s \in \mathbb{N}$, we use $\log^s$ to denote the logarithm iterated s times.

LEMMA 28. *If $x \in \mathbb{R} \setminus \mathbb{Q}$ is not Liouville and there are $\varepsilon > 0$ and $R \geq 1$ such that $\frac{\log^{s-1} \Delta k_m}{\log^{s-1} k_m} \geq 1 + \varepsilon$ on a sequence of m's, then* (6.2) *diverges for the approximating function $\psi_{s,d}$.*

Proof. Comparing sums to integrals we have

$$\sum_{q \in A_{m+1}} \psi_{s,d}(q) \geq \log \frac{\log^s \Sigma_{\omega_{m+1}}}{\log^s \Sigma_{\alpha_{m+1}}},$$

and we will show that this expression is bounded below by $\log(1+\varepsilon)$ on the sequence where $\frac{\log^{s-1} \Delta k_m}{\log^{s-1} k_m} \geq 1 + \varepsilon$.

On that sequence, we have

$$\frac{\log^s \Sigma_{\omega_{m+1}}}{\log^s \Sigma_{\alpha_{m+1}}} \overset{\text{Lem. 13}}{\gtrsim} \frac{\log^s q_{k_{m+1}}^2}{\log^s \max\{q_{k_m} q_{k_m - 1}, q_{k_m+1}\}} \overset{\text{Lem. 9}}{\gtrsim} \frac{\log^s (F(\Delta k_m)\, q_{k_m+1})}{\log^s q_{k_m+1}}$$
$$\overset{\text{Lem. 10}}{\gtrsim} \frac{\log^s (F(\Delta k_m)\, q_{k_m+1})}{\log^s (R+1)^{\sigma^m k_m}} \gtrsim \frac{\log^{s-1} (\Delta k_m + \log q_{k_m+1})}{\log^{s-1} k_m} \gtrsim 1 + \varepsilon.$$

Therefore (6.2) diverges. ∎

LEMMA 29. *If $x \in \mathbb{R} \setminus \mathbb{Q}$ is not Liouville, has essential Diophantine type greater than* 1, *and $\Delta k_m \leq^* k_m$ for some $R \geq 1$, then there is a positive lower asymptotic density sequence $\{\ell_j\} \subseteq \mathbb{N}$ on which*

$$\int_{\Sigma_{\alpha_\ell}}^{\Sigma_{\omega_\ell}} \psi_{s,d}(t)^d \, dt \gg \psi_{s-2,d}(\ell)^d,$$

where $\psi_{s,d}(q) = (q \log q \log^2 q \ldots \log^s q)^{-1/d}$. Therefore, (6.2) *diverges.*

Proof. Let $1 < \tilde{\sigma} < \sigma < \infty$ be such that $x \in \mathcal{W}_1^{\mathrm{ess}}(\tilde{\sigma}) \setminus \mathcal{W}_1(\sigma)$. We first show that

$$\int_{\Sigma_{\alpha_\ell}}^{\Sigma_{\omega_\ell}} \psi_{1,d}(t)^d \, dt = \log \frac{\log \Sigma_{\omega_\ell}}{\log \Sigma_{\alpha_\ell}} \gg 1 \tag{6.3}$$

on a sequence $\{\ell_j\} \subseteq \mathbb{N}$ of positive lower asymptotic density, by showing that there is some $\varepsilon > 0$ such that $\frac{\log \Sigma_{\omega_\ell}}{\log \Sigma_{\alpha_\ell}} \gtrsim 1 + \varepsilon$ on a sequence of ℓ's of positive lower asymptotic density.

By Lemma 13 we have $\Sigma_{\omega_{m+1}} \gg q_{k_{m+1}} q_{k_m}$ and $\Sigma_{\alpha_{m+1}} \ll q_{k_m} q_{k_m-1} + q_{k_m+1}$, and because $x \in \mathcal{W}_1^{\mathrm{ess}}(\tilde{\sigma})$ there is a sequence $\{m_j\} \subseteq \mathbb{N}$ of positive lower asymptotic density such that $q_{k_{m_j}}^{\tilde{\sigma}} < q_{k_{m_j}+1}$. We now have

$$\frac{\log \Sigma_{\omega_{m_j+1}}}{\log \Sigma_{\alpha_{m_j+1}}} \gtrsim \frac{\log q_{k_{m_j+1}} q_{k_{m_j}}}{\log(q_{k_{m_j}} q_{k_{m_j}-1} + q_{k_{m_j}+1})} \gtrsim \frac{\log q_{k_{m_j+1}} q_{k_{m_j}}}{\log \max\{q_{k_{m_j}} q_{k_{m_j}-1}, q_{k_{m_j}+1}\}}.$$

Whenever $q_{k_{m_j}} q_{k_{m_j}-1} \le q_{k_{m_j}+1}$, this becomes

$$\frac{\log q_{k_{m_j+1}} q_{k_{m_j}}}{\log q_{k_{m_j}+1}} = \frac{\log q_{k_{m_j+1}}}{\log q_{k_{m_j}+1}} + \frac{\log q_{k_{m_j}}}{\log q_{k_{m_j}+1}} \ge 1 + \frac{1}{\sigma}.$$

And whenever $q_{k_{m_j}} q_{k_{m_j}-1} \ge q_{k_{m_j}+1}$, we get

$$\frac{\log q_{k_{m_j+1}} q_{k_{m_j}}}{\log q_{k_{m_j}} q_{k_{m_j}-1}} > \frac{\log q_{k_{m_j}}^{1+\tilde{\sigma}}}{\log q_{k_{m_j}}^2} = \frac{1+\tilde{\sigma}}{2} > 1$$

because $\tilde{\sigma} > 1$. The sequence $\{\ell_j\}$ in the previous paragraph is $\ell_j = m_j + 1$, and we have proved

$$\frac{\log \Sigma_{\omega_{\ell_j}}}{\log \Sigma_{\alpha_{\ell_j}}} \gtrsim 1 + \varepsilon$$

with any fixed

$$0 < \varepsilon < \min\left\{\frac{1}{\sigma}, \frac{\tilde{\sigma} - 1}{2}\right\},$$

and this establishes the comparison (6.3).

We now show that

$$\int_{\Sigma_{\alpha_{\ell_j}}}^{\Sigma_{\omega_{\ell_j}}} \psi_{2,d}(t)^d \, dt \gg \psi_{0,d}(\ell_j). \tag{6.4}$$

Evaluating the integral gives

$$\int_{\Sigma_{\alpha_{\ell_j}}}^{\Sigma_{\omega_{\ell_j}}} \psi_{2,d}(t)^d \, dt = \log \frac{\log\log \Sigma_{\omega_{\ell_j}}}{\log\log \Sigma_{\alpha_{\ell_j}}} \ge \log\left(1 + \frac{\log(1+\varepsilon)}{\log\log \Sigma_{\alpha_{\ell_j}}}\right).$$

Lemma 10 and the assumption that $\Delta k_m \le^* k_m$ imply

$$\log\log \Sigma_{\alpha_{\ell_j}} \ll \ell_j,$$

and recalling the fact that $\log(1+t) \sim t$ as $t \to 0$, we have (6.4).

In the general case, we claim that for all $s \in \mathbb{N}$,

$$\frac{\log^s \Sigma_{\omega_{\ell_j}}}{\log^s \Sigma_{\alpha_{\ell_j}}} \gtrsim 1 + \frac{\log(1+\varepsilon)}{\log^s \Sigma_{\alpha_{\ell_j}} \log^{s-1} \Sigma_{\alpha_{\ell_j}} \dots \log^2 \Sigma_{\alpha_{\ell_j}}} = 1 + C_s \psi_{s-2,d}(\ell_j)^d$$

where $C_s > 0$. We have already proved the base case. In the inductive step,

$$\frac{\log^s \Sigma_{\omega_{\ell_j}}}{\log^s \Sigma_{\alpha_{\ell_j}}} \gtrsim 1 + \frac{\log(1 + C_{s-1}\psi_{s-3,d}(\ell_j)^d)}{\log^s \Sigma_{\alpha_{\ell_j}}}$$
$$\sim 1 + \frac{C_{s-1}\psi_{s-3,d}(\ell_j)^d}{\log^s \Sigma_{\alpha_{\ell_j}}} = 1 + C_s\psi_{s-2,d}(\ell_j)^d,$$

proving the claim. Evaluating the integral,

$$\int_{\Sigma_{\alpha_{\ell_j}}}^{\Sigma_{\omega_{\ell_j}}} \psi_{s,d}(t)^d\,dt = \log \frac{\log^s \Sigma_{\omega_{\ell_j}}}{\log^s \Sigma_{\alpha_{\ell_j}}}$$
$$\gtrsim \log(1 + C_s\psi_{s-2,d}(\ell_j)^d) \sim C_s\psi_{s-2,d}(\ell_j)^d \gg \psi_{s-2,d}(\ell_j)^d,$$

we have proved the lemma. ∎

THEOREM 30. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$ be non-Liouville. If*

(a) *for some $\varepsilon > 0$ there is an $s \in \mathbb{N}$ such that $\frac{\log^{s-1} \Delta k_m}{\log^{s-1} k_m} \geq 1 + \varepsilon$ for infinitely many $m \in \mathbb{N}$; or*

(b) *the essential Diophantine type of x is greater than 1 and $\Delta k_m \leq^* k_m$,*

then

$$m_{d-1}(\mathcal{W}_d(\psi_{s,d}) \cap (\{x\} \times \mathbb{R}^{d-1})) = \text{FULL}.$$

Proof. Part (a) follows from Lemma 28, and (b) from Lemma 29, both after applying Lemma 22 and Gallagher's Theorem. ∎

Proof of Theorem 2. Any $x \in \mathbb{R} \setminus \mathbb{Q}$ that is not Liouville and has regular Diophantine type greater than 1 satisfies assumption (b) of Theorem 30. For any x of Diophantine type greater than d, the theorem is proved by the remark on Khintchine's transference principle at the end of §2.1. ∎

6.3. Another point of view. Before moving to convergence results, we offer another point of view on what we have done in this section.

The power set $\mathcal{P}(\mathbb{N})$ surjects onto $[0,1]$ by mapping a subset $A \subseteq \mathbb{N}$ to the binary expansion $0.d_1d_2d_3\ldots$, where $d_q = \mathbf{1}_A(q)$ is the indicator of A. In fact, the set $\mathcal{P}^\infty(\mathbb{N})$ of *infinite* subsets of $\mathbb{N}$ can be identified with $(0,1]$ by considering only binary expansions with infinitely many 1's. With this identification $\mathcal{P}^\infty(\mathbb{N}) \cong (0,1]$ in mind, denote by

$$\mathcal{C}(\psi) := \Big\{A \in \mathcal{P}^\infty(\mathbb{N}) : \sum_{q\in A} \psi(q) < \infty\Big\} \subset (0,1]$$

and

$$\mathcal{D}(\psi) := \Big\{A \in \mathcal{P}^\infty(\mathbb{N}) : \sum_{q\in A} \psi(q) = \infty\Big\} = (0,1] \setminus \mathcal{C}(\psi)$$

the sets of convergent and divergent subseries of $\sum_{q\in\mathbb{N}}\psi(q)$, respectively. Šalát offers the following theorem.

THEOREM 31 ([Sa]). *If $\psi : \mathbb{N} \to \mathbb{R}$ is non-increasing and $\sum_{q\in\mathbb{N}}\psi(q)$ diverges, then $\mathcal{C}(\psi) \subset (0,1]$ has Hausdorff dimension* 0.

At the beginning of this section we have explicitly defined a map $A : (\mathbb{R}\setminus\mathbb{Q})\times\mathbb{N} \to (0,1]$ using our bounded ratio sequences $\{L_n^R\}$, and we have spent our effort showing that $A(x,R) \in \mathcal{D}(\psi^d)$ for as many x's as possible, where ψ is some approximating function satisfying $\sum_{q\in\mathbb{N}}\psi(q)^d = \infty$. Theorem 31 says that this amounts to showing that the map A takes values in a set whose complement has Hausdorff dimension 0.

It is tempting to hope that a closer analysis of the properties of the map A will reveal that the preimage of $\mathcal{C}(\psi^d)$ must also have Hausdorff dimension 0. This would prove the following statement:

> *Let $d \geq 3$. If ψ is an approximating function such that $\sum_{q\in\mathbb{N}}\psi(q)^d$ diverges, then*
>
> $$m_{d-1}(\mathcal{W}_d(\psi)\cap(\{x\}\times\mathbb{R}^{d-1})) = \text{FULL} \quad \text{for all } x \in \mathbb{R}\setminus E,$$
>
> *where the (possibly empty) set E of exceptions has Hausdorff dimension zero.*

We can reasonably expect this to be true (even with an empty E). In particular, we have already proved it for the prototypical $\psi(q) = (q\log q)^{-1/d}$, in Theorem 1, and for ψ with the property that any convergent subseries of $\sum\psi(q)^d$ has asymptotic density zero, in Theorem 4.

7. Proofs of convergence results

7.1. Proof of Theorem 5. The following theorem is a counterpart to Theorem 26, and Theorem 5 follows immediately.

THEOREM 32. *Let $d \geq 2$. If ψ is an approximating function such that $\sum_{q\in\mathbb{N}}\psi(q)^d$ converges, then*

$$\sum_{q\in\mathcal{Q}(x,\psi)}\psi(q)^{d-1} < \infty$$

for:

(a) $\begin{cases} \textit{No } x \in \mathbb{Q} & \textit{if } \sum_{q\in\mathbb{N}}\psi(q)^{d-1} \textit{ diverges.} \\ \textit{Every } x \in \mathbb{R} & \textit{if it converges.}\end{cases}$

(b) *Any $x \in \mathbb{R}\setminus\mathbb{Q}$ with the bounded ratio property.*

Proof. (a) Suppose $\sum_{q\in\mathbb{N}}\psi(q)^{d-1}$ diverges. Let $x = a/b$ be a rational number. Then the sequence $\{kb\}_{k\in\mathbb{N}}$ is contained in $\mathcal{Q}(x,\psi)$, so

$$\sum_{q\in\mathcal{Q}(x,\psi)} \psi(q)^{d-1} \ge \sum_{k\in\mathbb{N}} \psi(kb)^{d-1},$$

and this diverges as in the proof of Theorem 26(a). On the other hand, if $\sum_{q\in\mathbb{N}} \psi(q)^{d-1}$ converges, then it is obvious that so does $\sum_{q\in\mathcal{Q}(x,\psi)} \psi(q)^{d-1}$, regardless of whether x is rational or irrational.

(b) We want to show that $\sum_{q\in\mathcal{Q}(x,\psi)} \psi(q)^{d-1}$ converges. Similar to the proof of Theorem 26, we partition partial sums as

$$\sum_{\substack{q=1\\ q\in\mathcal{Q}(x,\psi)}}^{\Sigma_N-1} \psi(q)^{d-1} = \sum_{n=1}^{N} \sum_{\substack{q=\Sigma_{n-1}\\ q\in\mathcal{Q}(x,\psi)}}^{\Sigma_n-1} \psi(q)^{d-1}. \tag{7.1}$$

This is

$$\le \sum_{n=1}^{N} \psi(\Sigma_{n-1})^{d-1} \sum_{\substack{q=\Sigma_{n-1}\\ q\in\mathcal{Q}(x,\psi)}}^{\Sigma_n-1} 1 \ll \sum_{n=1}^{N} L_n\, \psi(\Sigma_{n-1})^{d},$$

by Lemma 20. Now, as before, we have $L_n = L_{n-1}+1$ except when $n=\alpha_m$ for some $m\ge 2$, so the above is

$$\ll \sum_{n=1}^{N} L_{n-1}\,\psi(\Sigma_{n-1})^d + \sum_{n=1}^{N} \psi(\Sigma_{n-1})^d + \sum_{m\in\mathbb{N}} (B_{m+1}-B_m)\,\psi(\Sigma_{\omega_m})^d \tag{7.2}$$

and recalling Lemma 21 we continue with

$$\ll \sum_{n=1}^{N} L_{n-1}\,\psi(\Sigma_{n-1})^d + \sum_{n=1}^{N} \psi(\Sigma_{n-1})^d + \sum_{m\in\mathbb{N}} \frac{B_{m+1}-B_m}{\Sigma_{\omega_m}},$$

which converges for some $R\ge 1$ if x has the bounded ratio property. Therefore, (7.1) converges as $N\to\infty$, as desired. ∎

Proof of Theorem 5. This follows from Theorem 32 in the same way that Theorem 3 follows from Theorem 26. This time, instead of applying Gallagher's Theorem, we use the convergence part of Khintchine's Theorem (or, really, the Borel–Cantelli Lemma) to see that convergence of $\sum_{q\in\mathbb{N}} \bar\psi(q)^{d-1}$ implies that the measure of $\mathcal{W}_{d-1}(\bar\psi)$ is zero. ∎

7.2. Proof of Corollaries 6–8. (This proof should be read as a continuation of the proof of Theorem 32.) On the other hand, if $\psi(q)\le q^{-(1+\delta)/d}$ for some $\delta>0$, then we can continue (7.2) as

$$\ll \sum_{n=1}^{N} L_{n-1}\,\psi(\Sigma_{n-1})^d + \sum_{n=1}^{N} \psi(\Sigma_{n-1})^d + \sum_{m\in\mathbb{N}} \frac{B_{m+1}-B_m}{\Sigma_{\omega_m}^{1+\delta}}.$$

The first two terms converge, so let us look at the last one. Its convergence is equivalent to that of

$$\sum_{m\in\mathbb{N}} \frac{L_{\alpha_{m+1}}}{\Sigma_{\omega_m}^{1+\delta}}, \tag{7.3}$$

so in particular it converges if x has the bounded ratio property, which proves Corollary 6.

But by Lemma 13 we can compare the summand $L_{\alpha_{m+1}}/\Sigma_{\omega_m}^{1+\delta}$ to ratios of continuants. An argument almost identical to that of Proposition 17 will show that (7.3) converges if x meets the same restrictions on Diophantine type as in that proposition. The only difference is that now we have taken the denominators in the calculations to the power $1+\delta$, which allows our restrictions on Diophantine type to be inclusive, rather than exclusive. ∎

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