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On the Riesz means of $\frac{n}{\phi(n)}$ – III

by

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1. Introduction. Investigating the growth (or decay) of the absolute value of the error term of the summatory function of an arithmetical function is a classical question in number theory. Many results on such interesting questions are available in the literature (for some of them, the readers may refer to [4, Chapter 14]). Let $\phi(n)$ denote the Euler totient function which is defined to be the number of positive integers $\le n$ that are coprime to n. Let us write

$$\sum_{n\le x}\frac{1}{\phi(n)} = A(\log x + B) + E_0^*(x), \tag{1.1}$$

$$\sum_{n\le x}\frac{n}{\phi(n)} = Ax - \log x + E_1^*(x), \tag{1.2}$$

where

$$A = \frac{315\zeta(3)}{2\pi^4}, \qquad B = \gamma_0 - \sum_p \frac{\log p}{p^2-p+1}. \tag{1.3}$$

Here $\zeta(s)$ and γ_0 denote the Riemann zeta-function and the Euler–Mascheroni constant respectively. The sum defining B extends over all primes p. In [6, p. 184], E. Landau proved that

$$E_0^*(x) \ll \frac{\log x}{x} \tag{1.4}$$

as $x\to\infty$. Using a theorem of Walfisz based on Weyl's inequality, R. Sitaramachandrarao [15] established (by elementary methods) that

$$E_0^*(x) \ll \frac{(\log x)^{2/3}}{x} \tag{1.5}$$

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as $x \to \infty$. In another paper [16], R. Sitaramachandrarao studied the discrete average and integral average of these error terms $E_j^*(x)$ for $j = 0, 1$. In particular, he proved by elementary methods that

$$\int_1^x E_1^*(t)\,dt = -\frac{D}{2}x + O(x^{4/5}), \tag{1.6}$$

where

$$D = \gamma_0 + \log(2\pi) + \sum_p \frac{\log p}{p(p-1)}. \tag{1.7}$$

As a consequence of (1.2) and (1.6) (see [16, Remark 4.1]), he derived that

$$\begin{aligned} \sum_{n\le x} \frac{n}{\phi(n)}(x-n) &= \int_1^x \Big(\sum_{n\le u} \frac{n}{\phi(n)}\Big)\,du \\ &= \frac{A}{2}x^2 - \frac{1}{2}x\log x + \frac{1-D}{2}x + O(x^{4/5}). \end{aligned} \tag{1.8}$$

Equivalently, he established that the first Riesz mean satisfies the asymptotic relation

$$\sum_{n\le x} \frac{n}{\phi(n)}\Big(1-\frac{n}{x}\Big) = \frac{A}{2}x - \frac{1}{2}\log x + \frac{1-D}{2} + O(x^{-1/5}). \tag{1.9}$$

If we denote the error term of the first Riesz mean related to the arithmetic function $n/\phi(n)$ in (1.9) by $E_1(x)$, then a *conjecture of Sitaramachandrarao* (see [16, Remark 4.1]) is that

$$E_1(x) \ll \frac{1}{x^{3/4-\delta}} \tag{1.10}$$

for every small fixed positive δ.

The aim of this article is to establish an improved upper bound for the absolute value of the error term of the general kth Riesz mean related to the arithmetic function $n/\phi(n)$ for any positive integer $k \ge 2$. More precisely, we write (for any integer $k \ge 1$)

$$\frac{1}{k!}\sum_{n\le x} \frac{n}{\phi(n)}\Big(1-\frac{n}{x}\Big)^k = M_k(x) + E_k(x) \tag{1.11}$$

where $M_k(x)$ is the main term which is of the form $M_k(x) = c_1(k)x + c_2(k)\log x + c_3(k)$, with $c_1(k), c_2(k), c_3(k)$ certain specific constants that depend only on k, and $E_k(x)$ is the error term of the sum under investigation.

We recall here a general conjecture proposed in [13]:

CONJECTURE. *For every integer $k \ge 1$,*

$$E_k(x) \ll \frac{1}{x^{3/4-\delta}} \tag{1.12}$$

for any small fixed positive constant δ, and the implied constant is independent of k.

In [13], we proved:

THEOREM A. *Let $x \geq x_0$ where x_0 is a sufficiently large positive number. For any integer $k \geq 1$,*

$$E_k(x) \ll \frac{1}{x^{1/2-\delta}} \tag{1.13}$$

for any small fixed positive constant δ, and the implied constant is independent of k.

Later, we refined certain arguments of [13], and with some extra inputs we established in [14]:

THEOREM B. *Let $x \geq x_0$ where x_0 is a sufficiently large positive number. Let c^* be any real number ≥ 10. For any integer $k \geq 1$,*

$$E_1(x) \ll \frac{(\log x)^{5/4}(\log\log x)}{x^{1/2}}, \tag{1.14}$$

and for any integer $k \geq 2$,

$$E_k(x) \ll \frac{\max(4^k, c^{*2/3+\epsilon})\ (\log x)}{x^{c^*k-1}} + c^{*1/2}\,\frac{x^{-1/2}(\log x)^{1/4}(\log\log x)}{e^k}, \tag{1.15}$$

where the implied constants are independent of k.

Theorems A and B use some ideas from [11] and [12]. For a related work see also [5].

The aim of this article is to prove:

THEOREM 1 (unconditional). *Let $x \geq x_0$ where x_0 is a sufficiently large positive number and k is any integer ≥ 2. Then there exists a computable constant c such that*

$$E_k(x) \ll \frac{x^{-1/2}}{k}\ \exp\left(\frac{-c(\log x)^{1/3}}{(\log\log x)^{1/3}}\right).$$

In support of the above conjecture, we also prove:

THEOREM 2 (conditional). *Let $x \geq x_0$ where x_0 is a sufficiently large positive number and k is any integer ≥ 2. Then, on the assumption of the Riemann Hypothesis, the inequality*

$$E_k(x) \ll \frac{x^{-3/4+\delta}}{k}$$

holds for any small positive constant δ.

Remark. By choosing $c^* = 10$ in Theorem B, from (1.15) it is not difficult to see that the estimate

$$E_k(x) \ll \frac{x^{-1/2}(\log x)^{1/4}(\log\log x)}{e^k}$$

holds uniformly for $2 \le k \le A_1 \log x$ for some effective positive constant A_1. It is also not difficult to see from the proof of Theorem B (see [14]) that for all integers $k \ge 2$,

$$E_k(x) \ll x^{-1/2}.$$

It is plain that Theorem B is better than Theorem 1 when $A_2(\log x)^{1/3}/(\log\log x)^{1/3} \le k \le A_3 \log x$, whereas Theorem 1 provides a stronger upper bound estimate for instance when $2 \le k \le A_2(\log x)^{1/3}/(\log\log x)^{1/3}$ and $k \ge (\log x)^{1+\epsilon}$.

The constants $c_1(k)$, $c_2(k)$ and $c_3(k)$ of the main term $M_k(x)$ were already determined explicitly in [13]. It should be noted that although the conjecture is still far from being resolved, Theorem 2 reveals that the conjecture is true on the assumption of the Riemann Hypothesis with the implied k dependence given explicitly. It is important to note that even if one assumes the Riemann Hypothesis, we are unable to draw any stronger conclusion towards the conjecture for $E_1(x)$.

2. Notation and conventions. Throughout the paper, $s = \sigma + it$, the parameters T and x are sufficiently large real numbers, and k is an integer ≥ 2.

δ and ϵ always denote sufficiently small fixed positive constants.

As usual, $\zeta(s)$ denotes the Riemann zeta-function and γ_0 is Euler–Mascheroni constant.

The implied constants may depend on ϵ and δ, and we do not mention this fact explicitly.

The letters A, B, C and a, b, c with or without subscripts denote absolute effective constants.

3. Some lemmas

Lemma 3.1 ([14, Lemma 3.1]). *For* $\Re s > 1$,

$$F(s) := \sum_{n=1}^{\infty} \frac{n}{\phi(n) n^s} = \zeta(s)\zeta(s+1)\frac{\zeta(4s+4)}{\zeta(2s+2)} h(s)$$

where

$$h(s) := \prod_p \left(1 + \frac{1}{p^{s+2}} \frac{1}{(1-1/p)} - \frac{1}{p^{2s+3}(1-1/p)} + \frac{1}{p^{3s+4}(1-1/p)} - \frac{1}{p^{4s+4}(1-1/p)}\right)$$

with $h(s)$ absolutely and uniformly convergent in any compact set in the half-plane $\Re s \geq -3/4 + \delta$ for any fixed small positive δ.

LEMMA 3.2 ([13, Lemma 3.2] or [3, p. 31, Theorem B]). *Let k be an integer ≥ 1. Let c and y be any positive real numbers, and $T \geq T_0$ where T_0 is sufficiently large. Then*

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s(s+1)\cdots(s+k)}\,ds = \begin{cases} \frac{1}{k!}\left(1-\frac{1}{y}\right)^k + O(4^k y^c/T^k) & \text{if } y \geq 1, \\ O(1/T^k) & \text{if } 0 < y \leq 1. \end{cases}$$

LEMMA 3.3 ([17, p. 116] or [4, pp. 8–12]). *The Riemann zeta-function $\zeta(s)$ is extended as a meromorphic function in the whole complex plane $\mathbb{C}$ having a simple pole at $s = 1$ with residue 1, and it satisfies the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$ where*

$$\chi(s) = \frac{\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)}{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)}.$$

Also, in any bounded vertical strip, using Stirling's formula, we get

$$\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-1/2} e^{i(t+\pi/4)}(1+O(t^{-1}))$$

as $|t| \to \infty$. Thus, in any bounded vertical strip,

$$|\chi(s)| \asymp t^{1/2-\sigma}(1+O(t^{-1}))$$

as $|t| \to \infty$.

LEMMA 3.4 ([4, p. 143, Theorem 6.1]). *There is an absolute constant $C > 0$ such that $\zeta(s) \neq 0$ for*

$$\sigma \geq 1 - C(\log t)^{-2/3}(\log\log t)^{-1/3} \qquad (t \geq t_0).$$

LEMMA 3.5 ([4, pp. 144 and 310] or [17, pp. 134–137]). *For $|t| \geq 2$ and $\sigma \geq 1 - C(\log t)^{-2/3}(\log\log t)^{-1/3}$, we have*

$$\zeta(\sigma+it) \ll (\log t)^{2/3}(\log\log t)^{1/3}, \qquad 1/\zeta(\sigma+it) \ll (\log t)^{2/3}(\log\log t)^{1/3}.$$

LEMMA 3.6 ([17, p. 141, Theorem 7.2(A)]). *We have*

$$\int_1^T |\zeta(\sigma+it)|^2\,dt \ll T\min\left(\frac{1}{2\sigma-1}, \log T\right)$$

uniformly for $1/2 \leq \sigma \leq 2$.

LEMMA 3.7 ([17, p. 337, equations 14.2.5 and 14.2.6]). *Assuming the Riemann Hypothesis,*

$$\zeta(s) = O(t^{\epsilon}) \quad \text{and} \quad 1/\zeta(s) = O(t^{\epsilon})$$

for every $\sigma \geq 1/2 + \delta$ and $t \geq t_0$ where t_0 is a sufficiently large number.

4. Proof of Theorem 1. Let $\epsilon(T) = (C/100)(\log T)^{-2/3}(\log\log T)^{-1/3}$ where C is as in Lemma 3.4. From Lemma 3.2, with $c = 1 + \frac{1}{\log x}$ and writing $F(s) := \zeta(s)\zeta(s+1)\frac{\zeta(4s+4)}{\zeta(2s+2)}h(s)$, we obtain

$$S := \frac{1}{k!}\sum_{n\le x}\frac{n}{\phi(n)}\Big(1-\frac{n}{x}\Big)^k \tag{4.1}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\frac{x^s}{s(s+1)(s+2)\cdots(s+k)}\,ds$$

$$= \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s)\frac{x^s}{s(s+1)\cdots(s+k)}\,ds + O\Big(\frac{4^k x^c\log x}{T^k}\Big).$$

Now move the line of integration to $\Re s = \alpha := -1/2 - \epsilon(T)$. In the rectangular contour formed by the line segments joining the points $c-iT$, $c+iT$, $\alpha+iT$, $\alpha-iT$ and $c-iT$ in the counter-clockwise sense, we observe that $s=1$ is a simple pole and $s=0$ is a double pole of the integrand, thus we get the main term from the sum of the residues coming from the poles $s=1$ and $s=0$, namely $c_1(k)x + c_2(k)\log x + c_3(k)$. We note that

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s)\frac{x^s}{s(s+1)\cdots(s+k)}\,ds \tag{4.2}$$

$$= \frac{1}{2\pi i}\Big\{\int_{-1/2-\epsilon(T)+iT}^{c+iT}\cdots + \int_{-1/2-\epsilon(T)-iT}^{-1/2-\epsilon(T)+iT}\cdots + \int_{c-iT}^{-1/2-\epsilon(T)-iT}\cdots\Big\}$$
$$+ \text{sum of the residues.}$$

Let $T \ge T_0$ where T_0 is a sufficiently large real number. The left vertical line segment contributes the quantity

$$Q_{1,k} := \frac{1}{2\pi}\int_{-T}^{T} F(\alpha+it)\frac{x^{\alpha+it}}{(\alpha+it)(\alpha+1+it)\cdots(\alpha+k+it)}\,dt$$

$$= \frac{1}{2\pi}\Big(\int_{|t|\le T_0} + \int_{T_0<|t|\le T}\Big)\frac{x^{\alpha+it}\zeta(\alpha+it)\zeta(\alpha+1+it)\frac{\zeta(4\alpha+4+4it)}{\zeta(2\alpha+2+2it)}h(\alpha+it)}{(\alpha+it)(\alpha+1+it)\cdots(\alpha+k+it)}\,dt$$

$$\ll \frac{x^\alpha}{(k-1)!} + x^\alpha\int_{T_0<|t|\le T}\frac{t^{(1/2-\alpha)}|\zeta(1-\alpha+it)|t^{1/2-\alpha-1}|\zeta(-\alpha+it)|}{t^{k+1}}$$
$$\times\Big|\frac{\zeta(4\alpha+4+4it)}{\zeta(2\alpha+2+2it)}\Big|\,|h(\alpha+it)|\,dt$$

$$\ll \frac{x^\alpha}{(k-1)!} + x^\alpha\int_{T_0<|t|\le T} t^{1+2\epsilon(T)}\Big|\frac{\zeta(1/2+\epsilon(T)+it)}{\zeta(1-2\epsilon(T)+2it)}\Big|\,\frac{dt}{t^{k+1}}.$$

Using the bound in Lemma 3.5 and noticing that $t^{2\epsilon(T)} \ll t^{\epsilon}$ and $|\zeta(\sigma+it)| \ll t^{1/6+\epsilon}$ for $\sigma \geq 1/2$, we obtain

$$Q_{1,k} \ll \frac{x^{-1/2-\epsilon(T)}}{(k-1)!} + x^{-1/2-\epsilon(T)} \int_1^T t^{1/6+2\epsilon} (\log t)^{2/3+\epsilon} \frac{dt}{t^k} \ll \frac{x^{-1/2-\epsilon(T)}}{k - 11/6 - 3\epsilon}. \tag{4.3}$$

Now we will estimate the contributions coming from the upper horizontal line (the lower horizontal line is similar).

Lemma 4.1. *Let $T = x^{10}$. Then*

$$Q_2 := \int_{T/2}^{T} \left| \int_{-1/2-\epsilon(T)}^{1+1/\log x} \frac{F(\sigma+it)x^{\sigma+it}}{(\sigma+it)(\sigma+1+it)\cdots(\sigma+k+it)}\, d\sigma \right| dt \ll \frac{2^k}{x^{10(k-1)+1/2}} \exp\left(C_1 \frac{(\log x)^{1/3}}{(\log\log x)^{1/3}} \right). \tag{4.4}$$

Proof. We note that

$$Q_2 \leq \left(\int_{T/2}^{T} \int_{-1/2-\epsilon(T)}^{-1/2} + \int_{T/2}^{T} \int_{-1/2}^{0} + \int_{T/2}^{T} \int_{0}^{1/2} + \int_{T/2}^{T} \int_{1/2}^{1+1/\log x} \right) \left| \frac{F(\sigma+it)x^{\sigma+it}}{(\sigma+it)(\sigma+1+it)\cdots(\sigma+k+it)} \right| d\sigma\, dt$$
$$= I_4 + I_1 + I_2 + I_3 \quad \text{(say)}.$$

We observe that using Lemma 4.1 of [14] we get

$$I_1 + I_2 + I_3 \ll \frac{2^k (\log x)^{2/3+\epsilon}}{x^{10(k-1)+1/2}}.$$

Now we will estimate I_4. We note that

$$I_4 := \int_{T/2}^{T} \int_{-1/2-\epsilon(T)}^{-1/2} \left| \zeta(\sigma+it)\zeta(\sigma+1+it) \frac{\zeta(4\sigma+4+4it)}{\zeta(2\sigma+2+2it)} \times \frac{x^{\sigma+it} h(\sigma+it)}{(\sigma+it)(\sigma+1+it)\cdots(\sigma+k+it)} \right| d\sigma\, dt.$$

We observe that

$$\left| \frac{1}{\zeta(2\sigma+2+2it)} \right| \ll (\log t)^{2/3+\epsilon} \quad \text{for } -1/2-\epsilon(T) \leq \sigma \leq -1/2.$$

From the functional equation of $\zeta(s)$ (Lemma 3.3) and using the Cauchy–

Schwarz inequality along with Lemma 3.6, we get

$$\begin{aligned} I_4 &\ll \int_{-1/2-\epsilon(T)}^{-1/2} x^\sigma \int_{T/2}^{T} t^{1/2-\sigma} t^{1/2-\sigma-1} \left| \frac{\zeta(1-s)\zeta(-s)}{\zeta(2s+2)} \right| dt\, d\sigma \\ &\ll \frac{2^k (\log T)^{2/3+\epsilon}}{T^{k+1}} \int_{-1/2-\epsilon(T)}^{-1/2} \left(\frac{x}{T^2}\right)^\sigma \int_{T/2}^{T} |\zeta(-\sigma-it)|\, dt\, d\sigma \\ &\ll \frac{2^k (\log T)^{2/3+\epsilon}}{T^{k+1}} \frac{T(\log T)^{1/2}((x/T^2)^{-1/2} + (x/T^2)^{-1/2-\epsilon(T)})}{|\log(x/T^2)|} \\ &\ll \frac{2^k (\log T)^{7/6+\epsilon}}{T^k} \frac{(x/T^2)^{-1/2}}{|\log(x/T^2)|} \big(1 + (x/T^2)^{-\epsilon(T)}\big). \end{aligned}$$

Now fixing $T = x^{10}$, we obtain

$$\begin{aligned} I_4 &\ll \frac{2^k (\log x)^{7/6+\epsilon}}{x^{10(k-1)} x^{1/2} \log x} (1 + x^{19\epsilon(T)}) \\ &\ll \frac{2^k}{x^{10(k-1)+1/2}} \exp\left(19 c_1 \frac{\log x}{(\log x)^{2/3} (\log\log x)^{1/3}}\right) \\ &\ll \frac{2^k}{x^{10(k-1)+1/2}} \exp\left(C_1 \frac{(\log x)^{1/3}}{(\log\log x)^{1/3}}\right) \end{aligned} \tag{4.5}$$

where C_1 is some effective positive constant. Clearly $I_4 \gg I_1 + I_2 + I_3$. Hence the lemma. ∎

Recall that $T := x^{10}$. Let

$$G(t) := \int_{-1/2-\epsilon(T)}^{1+1/\log x} \frac{F(\sigma+it)\ x^{\sigma+it}}{(\sigma+it)(\sigma+1+it)\cdots(\sigma+k+it)}\, d\sigma.$$

Then by Lemma 4.1, there exists a $T^* \in [T/2, T]$ such that $|G(T^*)|$ is minimum and

$$\begin{aligned} |G(T^*)| &\ll \frac{1}{T} \frac{2^k}{x^{10(k-1)+1/2}} \exp\left(C_1 \frac{(\log x)^{1/3}}{(\log\log x)^{1/3}}\right) \\ &\ll \frac{2^k}{x^{10k+1/2}} \exp\left(C_1 \frac{(\log x)^{1/3}}{(\log\log x)^{1/3}}\right). \end{aligned}$$

Hence using horizontal lines of height $\pm T^*$ to move the line of integration in (4.1), we find that the total contribution of the horizontal lines in absolute value is

$$\ll \frac{2^k}{x^{10k+1/2}} \exp\left(C_1 \frac{(\log x)^{1/3}}{(\log\log x)^{1/3}}\right). \tag{4.6}$$

Now collecting the error estimates (4.1), (4.3), (4.6) and noting that $c =$

$1 + 1/\log x$, we get, for $k \geq 2$,

$$(4.7)\quad E_k(x) \ll \frac{4^k x \log x}{T^k} + \frac{x^{-1/2-\epsilon(T)}}{k - 11/6 - 3\epsilon} + \frac{2^k}{x^{10k+1/2}} \exp\left(C_1 \frac{(\log x)^{1/3}}{(\log\log x)^{1/3}}\right)$$
$$\ll \frac{x^{-1/2}}{k - 11/6 - 3\epsilon} \exp\left(-C_2 \frac{(\log x)^{1/3}}{(\log\log x)^{1/3}}\right)$$
$$\ll \frac{x^{-1/2}}{k} \exp\left(-C_2 \frac{(\log x)^{1/3}}{(\log\log x)^{1/3}}\right).$$

Note that the implied constant in $E_k(x)$ is independent of k. This proves Theorem 1 since the exact values of $c_1(k)$, $c_2(k)$ and $c_3(k)$ are already given in [13].

Remark. We note that on the line $\sigma = 1/2$ we have

$$T(\log T)^{1/4} \ll \int_1^T |\zeta(1/2 + it)|\, dt \ll T(\log T)^{1/4}$$

(see for example [8]–[10]). For more general estimations of moments (sometimes unconditional and sometimes assuming the Riemann Hypothesis) of $\int_1^T |\zeta(1/2+it)|^{2k}\, dt$, one may refer to [2] and the recent works [7] or [1]. Using these estimates, one can be very precise in powers of $\log x$ in the estimate of I_4, but we do not need it.

5. Proof of Theorem 2. Throughout this section we will assume the Riemann Hypothesis. Similar to the proof of Theorem 1, here we will take the left vertical line to be $\sigma = -3/4 + \delta =: \beta$. Now the contribution from the left vertical line is

$$(5.1)\qquad Q^*_{1,k} := \frac{1}{2\pi} \int_{-T}^{T} F(\beta + it) \frac{x^{\beta+it}}{(\beta + it)(\beta + 1 + it) \cdots (\beta + k + it)}\, dt$$
$$= \frac{1}{2\pi}\Big(\int_{|t|\le T_0} + \int_{T_0<|t|\le T}\Big) \frac{x^{\beta+it}\zeta(\beta + it)\zeta(\beta + 1 + it)\frac{\zeta(4\beta+4+4it)}{\zeta(2\beta+2+2it)}h(\beta + it)}{(\beta + it)(\beta + 1 + it) \cdots (\beta + k + it)}\, dt$$
$$\ll \frac{x^\beta}{(k-1)!} + x^\beta \int_{T_0<|t|\le T} t^{1/2-\beta}|\zeta(1 - \beta + it)|t^{1/2-\beta-1}|\zeta(-\beta + it)| \times \left|\frac{\zeta(4\beta + 4 + 4it)}{\zeta(2\beta + 2 + 2it)}\right| \frac{dt}{t^{k+1}}$$
$$\ll \frac{x^\beta}{(k-1)!} + x^\beta \int_{T_0<|t|\le T} t^{3/2-2\delta+2\epsilon}\, \frac{dt}{t^{k+1}} \ll \frac{x^{-3/4+\delta}}{k - 3/2 + 2\delta - 3\epsilon}.$$

Now we estimate the contribution from the upper horizontal line.

Lemma 5.1. *Let $T = x^{10}$. Then*

$$Q_3 := \int_{T/2}^{T} \left| \int_{-3/4+\delta}^{1+1/\log x} \frac{F(\sigma+it)x^{\sigma+it}}{(\sigma+it)(\sigma+1+it)\cdots(\sigma+k+it)}\, d\sigma \right| dt \tag{5.2}$$
$$\ll \frac{T^{\epsilon}}{T^{k}} \frac{T}{x^{1/2}\log x} \frac{T^{1/2-2\delta}}{x^{1/4}-\delta} \ll \frac{2^k}{x^{10(k-3/2-2\epsilon+2\delta)}x^{3/4-\delta}\log x}.$$

Proof. The proof is similar to the proof of Lemma 4.1 but here we take the lower limit for σ to be $-3/4+\delta =: \beta$, and I_1, I_2, I_3 are the same as in Lemma 4.1. In place of I_4, we have I_4^* given by

$$I_4^* := \int_{T/2}^{T} \int_{-3/4+\delta}^{-1/2} \left| \zeta(\sigma+it)\zeta(\sigma+1+it)\frac{\zeta(4\sigma+4+4it)}{\zeta(2\sigma+2+2it)} \times \frac{x^{\sigma+it}\ h(\sigma+it)}{(\sigma+it)(\sigma+1+it)\cdots(\sigma+k+it)} \right| d\sigma\, dt$$
$$\ll \int_{-3/4+\delta}^{-1/2} x^{\sigma} \int_{T/2}^{T} t^{1/2-\sigma} t^{1/2-\sigma-1} \left| \frac{\zeta(1-s)\zeta(-s)}{\zeta(2s+2)} \right| dt\, d\sigma$$
$$\ll \frac{2^k}{T^{k+1}} \int_{-3/4}^{-1/2} \left(\frac{x}{T^2}\right)^{\sigma} \int_{T/2}^{T} t^{2\epsilon}\, dt\, d\sigma \ll \frac{2^k}{T^{k+1}} T^{1+2\epsilon} \frac{(x/T^2)^{-1/2} + (x/T^2)^{-3/4+\delta}}{|\log(x/T^2)|}$$
$$\ll \frac{2^k T^{2\epsilon}}{T^k} \frac{(x/T^2)^{-1/2}}{|\log(x/T^2)|} \big(1 + (x/T^2)^{-1/4+\delta}\big).$$

With $T = x^{10}$, we find that

$$I_4^* \ll \frac{2^k T^{2\epsilon}}{T^k} \frac{T}{x^{1/2}\log x} \frac{T^{1/2-2\delta}}{x^{1/4-\delta}} \ll \frac{2^k}{T^{k-3/2-2\epsilon+2\delta}x^{3/4-\delta}\log x}$$
$$\ll \frac{2^k}{x^{10(k-3/2-2\epsilon+2\delta)}x^{3/4-\delta}\log x}.$$

Clearly $I_4^* \gg I_1 + I_2 + I_3$. This proves the lemma. ∎

Let

$$G_1(t) := \int_{-3/4+\delta}^{1+1/\log x} \frac{F(\sigma+it)x^{\sigma+it}}{(\sigma+it)(\sigma+1+it)\cdots(\sigma+k+it)}\, d\sigma.$$

Then by Lemma 5.1, there exists a $T^* \in [T/2, T]$ such that $|G_1(T^*)|$ is minimum and

$$|G_1(T^*)| \ll \frac{1}{T} \frac{2^k}{x^{10(k-3/2-2\epsilon+2\delta)}x^{3/4-\delta}\log x} \ll \frac{2^k}{x^{10(k-1/2-2\epsilon+2\delta)}} \frac{1}{x^{3/4-\delta}\log x}.$$

Hence using horizontal lines of height $\pm T^*$ to move the line of integration in (5.1), we find that the total contribution of the horizontal lines in absolute value is

$$\ll \frac{2^k}{x^{10(k-1/2-2\epsilon+2\delta)}} \frac{1}{x^{3/4-\delta}\log x}. \tag{5.3}$$

Now collecting the error estimates (4.1), (5.2), (5.3) and noting that $c = 1 + 1/\log x$, we obtain, for $k \geq 2$,

(5.4)

$$E_k(x) \ll \frac{4^k x \log x}{T^k} + \frac{x^{-3/4+\delta}}{k - 3/2 + 2\delta - 3\epsilon} + \frac{1}{x^{10(k-1/2-2\epsilon+2\delta)}} \frac{2^k}{x^{3/4-\delta}\log x}$$
$$\ll \frac{x^{-3/4+\delta}}{k - 3/2 + 2\delta - 3\epsilon}.$$

We can very well take $\epsilon = \delta/100$. This proves Theorem 2.

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