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Sequences with bounded l.c.m. of each pair of terms II

by

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1. Introduction. Let A_x be a set of positive integers with the least common multiple of each pair of terms not exceeding x and $|A_x|$ being the largest. In 1951, P. Erdős [5] (see also Guy [7]) proposed the following problem: what is the value of $|A_x|$? It is known that

$$\sqrt{\frac{9}{8}x}+O(1)\le |A_x|\le \sqrt{4x}+O(1).$$

For a proof see Erdős [6]. Choi [3] improved the upper bound to $1.638\sqrt{x}$, and later [4] to $1.43\sqrt{x}$. Let B_x be the union of the set of positive integers not exceeding $\sqrt{x/2}$ and the set of even integers between $\sqrt{x/2}$ and $\sqrt{2x}$. It is clear that the least common multiple of each pair of terms of B_x does not exceed x. By calculation we have

$$|B_x|=\sqrt{\frac{9}{8}x}+O(1).$$

Chen [1] gave an asymptotic formula for $|A_x|$ and showed that A_x is almost the same as B_x, that is,

$$|A_x\setminus B_x|=o(\sqrt{x}).$$

In particular,

$$|A_x|=|B_x|+o(\sqrt{x})=\sqrt{\frac{9}{8}x}+o(\sqrt{x}).$$

In this paper we study the asymptotic formula for $|A_x|$ and give the following explicit bound on the remainder for $|A_x|$:

THEOREM. *Let x be a large real number. Then*

$$|A_x|=\sqrt{\frac{9}{8}x}+R(x),$$

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where

$$-2 \le R(x) \le 45\sqrt{\frac{x}{\log x}}\log\log x.$$

REMARK. The constant 45 can be improved.

For $R(x)$ we have the following conjecture.

CONJECTURE. $R(x) \to \infty$ *as* $x \to \infty$.

2. Preliminary lemmas. In this section we give several lemmas.

LEMMA 1 (Brun's pure sieve [8, P.50(2.14)]). *Let $\mathcal{A}$ be a finite sequence of integers and let $|\mathcal{A}|$ denote the number of terms of the sequence. Let $\mathcal{P}$ be a set of primes and $\overline{\mathcal{P}}$ be its complement with respect to the set of all primes. For any real number $z \ge 2$, let*

$$P(z) := \prod_{p<z,\, p\in\mathcal{P}} p.$$

Define the sieving function by

$$S(\mathcal{A};\mathcal{P},z) := |\{a : a \in \mathcal{A},\ (a, P(z)) = 1\}|.$$

For every square-free positive integer d, let

$$|\mathcal{A}_d| := |\{a : a \in \mathcal{A},\ a \equiv 0 \pmod d\}|.$$

Let $\omega(d)$ be a multiplicative function and X be a close approximation to $|\mathcal{A}|$. Define the remainder term by

$$r_d := |\mathcal{A}_d| - \frac{\omega(d)}{d}X \qquad (u(d) \ne 0).$$

Let $A_0,\ A_1$ and λ be positive numbers. If

$$|r_d| \le \omega(d) \qquad \text{if } u(d) \ne 0,\ (d,\overline{\mathcal{P}}) = 1,$$
$$\omega(p) \le A_0, \qquad 0 \le \omega(p)/p \le 1 - A_1^{-1},$$
$$0 < \lambda e^{1+\lambda} \le 1,$$
$$\sum_{p<z}\frac{1}{p} \le 1 + \log\log z,$$

then

$$S(\mathcal{A};\mathcal{P},z) = XW(z)\{1 + \theta(\lambda e^{1+\lambda})^{(A_0A_1/\lambda)(\log\log z+1)}\} + \theta'\Big(1 + \sum_{p<z}\omega(p)\Big)^{(A_0A_1/\lambda)(\log\log z+1)}$$

for some θ, θ' with $|\theta| \le 1,\ |\theta'| \le 1$.

LEMMA 2. *For any positive number $z \geq 2$ we have*

$$\sum_{p \leq z} \frac{1}{p} \leq 0.9 + \log\log z.$$

This follows from a result of Rosser–Schoenfeld [9]

$$\sum_{p \leq x} \frac{1}{p} < \log\log x + B + \frac{1}{(\log\log x)^2}, \qquad x > 1,$$

where $B = 0.26149\ldots$. It can also be deduced from Chen–Sun [2, Lemma 3].

LEMMA 3. *Let $x \geq e^e$, M be an integer with $M \geq 5$ and a_i, b_i $(1 \leq i \leq t \leq M/5)$ be integers with $(a_i, b_i) = 1$ $(1 \leq i \leq t)$. Let c_1 be a real number and $\varepsilon = (2M\log\log x)^{-1}$. If each prime factor of $\prod_{i=1}^{t}(a_i n + b_i)$ exceeds M for any integer n, then there exists an integer $k \in (c_1, c_1 + 2x^{1/2-\varepsilon})$ such that each prime factor of $\prod_{i=1}^{t}(a_i k + b_i)$ exceeds x^ε/M.*

Proof. If $x^\varepsilon/M \leq M$, then the assertion is trivial. Now we assume that $x^\varepsilon/M > M$. We employ Brun's pure sieve. Set

$$\mathcal{A} = \Big\{ \prod_{i=1}^{t}(a_i k + b_i) : k \in (c_1, c_1 + 2x^{1/2-\varepsilon}) \Big\},$$

$$z = x^\varepsilon/M, \qquad X = 2x^{1/2-\varepsilon}, \qquad A_0 = M/5, \qquad A_1 = 5/4, \qquad \lambda = 1/4.$$

Let $\mathcal{P}$ be the set of all primes, and $\omega(p)$ the number of solutions of

$$\prod_{i=1}^{t}(a_i n + b_i) \equiv 0 \pmod{p}.$$

Then we have $|r_d| \leq \omega(d)$ if $\mu(d) \neq 0$, $\omega(p) \leq t \leq M/5 = A_0$, $0 \leq \omega(p)/p \leq 1/5 = 1 - A_1^{-1}$ and $0 < \lambda e^{1+\lambda} \leq 1$. Since $z = x^\varepsilon/M > M \geq 5$, we obtain

$$1 + \sum_{p<z} \omega(p) \leq 1 + \frac{M}{5}\sum_{p<z} 1 \leq Mz.$$

Thus by Lemmas 1 and 2 we have

$$\begin{aligned} S(\mathcal{A}; \mathcal{P}, z) &= XW(z)(1 + \theta(\lambda e^{1+\lambda})^{(A_0A_1/\lambda)(\log\log z+1)}) \\ &\quad + \theta'\Big(1 + \sum_{p<z}\omega(p)\Big)^{(A_0A_1/\lambda)(\log\log z+1)} \\ &\geq 2x^{1/2-\varepsilon}\prod_{p<z}\left(1 - \frac{\omega(p)}{p}\right)\cdot\left(1 - \left(\frac{e^{5/4}}{4}\right)^{M(\log\log z+1)}\right) \\ &\quad - (Mz)^{M(1+\log\log z)} \\ &\geq x^{1/2-\varepsilon}\prod_{M<p<z}\left(1 - \frac{M}{5p}\right) - (Mz)^{M(1+\log\log z)}. \end{aligned}$$

It is sufficient to prove that the right hand side is positive. This is equivalent to proving

$$\frac{1}{2}\log x > \varepsilon\log x + \sum_{M<p<z}\log\left(1-\frac{M}{5p}\right)^{-1} + M(1+\log\log z)\log(Mz).$$

Since

$$\log(Mz) = \varepsilon\log x,$$
$$\log\log z < \log\log x + \log\varepsilon < \log\log x - 2$$

and

$$\sum_{M<p<z}\log\left(1-\frac{M}{5p}\right)^{-1} \le \frac{2}{5}M\sum_{M<p<z}\frac{1}{p} \le \frac{2}{5}M\int_M^z \frac{1}{t}\,dt$$
$$\le \frac{2}{5}M\log z \le \frac{2}{5}M\varepsilon\log x,$$

we have

$$\varepsilon\log x + \sum_{M<p<z}\log\left(1-\frac{M}{5p}\right)^{-1} + M(1+\log\log z)\log(Mz)$$
$$\le \frac{1}{5}M\varepsilon\log x + \frac{2}{5}M\varepsilon\log x + M(\log\log x - 1)\varepsilon\log x$$
$$< M\varepsilon\log x\log\log x = \frac{1}{2}\log x.$$

This completes the proof of Lemma 3.

Lemma 4. *Let $x \ge e^e$, $x^\varepsilon \ge M^2$ and M be an integer with $M \ge 5$. Let D be an integer with $0 < |D| \le (Mx)^{M(M-5)/100}$ and with each prime factor of D exceeding x^ε/M, where $\varepsilon = (2M\log\log x)^{-1}$. Then*

$$|\{a : a \in (0,\sqrt{Mx}], (a,D) > 1\}| \le \frac{1}{25}M^{9/2}x^{1/2-\varepsilon}\log\log x.$$

Proof. Let $|D| = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ be the prime factorization of $|D|$. Then

$$r\log\frac{x^\varepsilon}{M} \le \sum_{i=1}^r \log p_i \le \log|D| \le \frac{M(M-5)}{100}(\log x + \log M).$$

For $x^{\varepsilon/2} \ge M$ we have

$$\frac{M(M-5)}{100}\log M < \frac{M}{25}\log x.$$

Thus

$$\frac{1}{2}r\varepsilon\log x \le \frac{M^2}{100}\log x,$$

that is,

$$r \le \frac{1}{25}M^3\log\log x.$$

Hence

$$\sum_{\substack{a\in(0,\sqrt{Mx}]\\(a,D)>1}} 1 \le \sum_{i=1}^{r} \sum_{\substack{a\in(0,\sqrt{Mx}]\\p_i|a}} 1 \le \sum_{i=1}^{r} \frac{\sqrt{Mx}}{p_i} \le r\sqrt{Mx}\,\frac{M}{x^{\varepsilon}}$$
$$\le \frac{1}{25}\,M^{9/2}x^{1/2-\varepsilon}\log\log x.$$

This completes the proof of Lemma 4.

For an interval $I=(a,b]$, let

$$|I\sqrt{x}\cap A_x| = \alpha(I)|I|\sqrt{x},$$

where $|X|$ denotes the number of elements of X or the length of an interval X. Let $\mathcal{I}=\{I_1,\ldots,I_l\}$ be a set of pairwise disjoint intervals with $I_i=(a_i,b_i]$, $0<a_1<\cdots<a_l$ and b_l^2 being an integer. Let

$$\alpha_i=\alpha(I_i), \qquad M=5b_l^2.$$

Suppose that $|I_i|\ge M^{-3/2}$ $(i=1,\ldots,l)$.

LEMMA 5. *Let $x\ge e^e$, $x^{\varepsilon}\ge M^5$, and $\varepsilon=(2M\log\log x)^{-1}$. Let r_{ij} $(j=1,\ldots,k_i,\ i=1,\ldots,l)$ be distinct integers with*

$$|r_{ij}-r_{uv}|\le a_i a_u.$$

Then

$$\sum_{i=1}^{l} k_i\alpha_i \le 1+\frac{3}{25}\,M^{9/2}x^{-\varepsilon/2}\log\log x.$$

Proof. We follow the proof of Chen [1, Lemma 3].

Let $K=\sum_{i=1}^{l}k_i$. If $K=0$ or 1, then by the definition of α_i the assertion is true. In the following we assume that $K\ge 2$. Let $\delta=x^{-\varepsilon/2}$ and let

$$I_i(t)=(a_i+t\delta, a_i+(t+1)\delta].$$

For the (index) set

$$\{t_{ij}: 0\le t_{ij}\le |I_i|/\delta-1,\ t_{ij}\in\mathbb{Z},\ j=1,\ldots,k_i,\ i=1,\ldots,l\}$$

we first show that

$$\Big|\bigcup_{i,j}(I_i(t_{ij})\sqrt{x}\cap A_x)\Big| \le \delta\sqrt{x}+\frac{2}{25}\,M^{9/2}x^{1/2-\varepsilon}\log\log x.$$

To do this we consider the set

$$\Delta(a)=\bigcup_{i,j}\{M!l_{ij}+r_{ij}+a\},$$

where l_{ij} are integers which will be determined later such that

$$(1)\qquad (a_i+t_{ij}\delta)\sqrt{x}\le M!l_{ij}+r_{ij}\le (a_i+t_{ij}\delta)\sqrt{x}+2x^{1/2-\varepsilon}$$

for $j = 1, \dots, k_i$ and $i = 1, \dots, l$. For convenience we rewrite $\Delta(0)$ as

$$\Delta(0) = \{M!l_1 + r_1, M!l_2 + r_2, \dots, M!l_K + r_K\}.$$

Since $M = 5b_l^2$, we have $a_i a_u < b_i b_u \le M/5$ $(i, u = 1, \dots, l)$. Then by the conditions of Lemma 5 we have

$$|r_i - r_j| < M/5, \qquad i, j = 1, \dots, K,$$

whence $K \le M/5$. Now we take l_1 satisfying (1). Suppose that we have chosen $l_1, \dots, l_u$ $(u < K)$. By Lemma 3 there exists an l_{u+1} satisfying (1) such that each prime factor of

$$\prod_{i=1}^{u} \left(\frac{M!}{r_{u+1} - r_i} \, l_{u+1} - \frac{M!}{r_{u+1} - r_i} \, l_i + 1 \right)$$

exceeds x^{ε}/M. Thus by induction we have determined all l_u $(1 \le u \le K)$. Let

$$D = \prod_{1 \le v < u \le K} \left(\frac{M!}{r_u - r_v} \, l_u - \frac{M!}{r_u - r_v} \, l_v + 1 \right).$$

Then each prime factor of D exceeds x^{ε}/M. By (1) and since $K \le M/5$ we have

$$\begin{aligned} |D| &\le \prod_{1 \le v < u \le K} |M!l_u + r_u - M!l_v - r_v| \\ &\le (2b_l\sqrt{x})^{K(K-1)/2} \le (\sqrt{M}x)^{M(M-5)/50}. \end{aligned}$$

Thus by Lemma 4 we have

$$|\{a : (a, D) > 1,\ a \in (0, b_l\sqrt{x}]\}| \le \frac{1}{25} \, M^{9/2} x^{1/2-\varepsilon} \log\log x.$$

Let

$$B = \Big\{ a : a \in \bigcup_{i=1}^{l} (I_i\sqrt{x} \cap \mathbb{Z}),\ (a, D) = 1 \Big\}.$$

If $a \in (0, \delta\sqrt{x}]$ and

$$M!l_u + r_u + a \in B, \qquad M!l_v + r_v + a \in B,$$

then for $u \ne v$ we have

$$\begin{aligned} (M!l_u + r_u + a, M!l_v + r_v + a) &= (M!l_u + r_u + a, M!(l_v - l_u) + r_v - r_u) \\ &= (M!l_u + r_u + a, r_v - r_u) \le |r_u - r_v|. \end{aligned}$$

Thus for $a \in (0, \delta\sqrt{x}]$ with

$$\begin{aligned} &M!l_{ij} + r_{ij} + a \in B, \\ &M!l_{uv} + r_{uv} + a \in B, \quad (i-u)^2 + (j-v)^2 \ne 0, \end{aligned}$$

by (1) and the conditions of Lemma 5 we have

$$\begin{aligned}
&\text{l.c.m.}\{M!l_{ij}+r_{ij}+a, M!l_{uv}+r_{uv}+a\}\\
&\qquad = \frac{(M!l_{ij}+r_{ij}+a)(M!l_{uv}+r_{uv}+a)}{(M!l_{ij}+r_{ij}+a, M!l_{uv}+r_{uv}+a)}\\
&\qquad > \frac{(a_i+t_{ij}\delta)(a_u+t_{uv}\delta)x}{|r_{ij}-r_{uv}|} \ge \frac{(a_i+t_{ij}\delta)(a_u+t_{uv}\delta)x}{a_i a_u} \ge x.
\end{aligned}$$

So $|\Delta(a)\cap B\cap A_x|\le 1$. Since (see (1))

$$\begin{aligned}
I_i(t_{ij})\sqrt{x}\cap\mathbb{Z} &\subseteq ((M!l_{ij}+r_{ij}, M!l_{ij}+r_{ij}+\delta\sqrt{x}]\\
&\qquad \cup ((a_i+t_{ij}\delta)\sqrt{x}, (a_i+t_{ij}\delta)\sqrt{x}+2x^{1/2-\varepsilon}]) \cap\mathbb{Z}\\
&\subseteq \Big(\bigcup_{0<a\le\delta\sqrt{x}}\{M!l_{ij}+r_{ij}+a\}\Big)\\
&\qquad \cup (((a_i+t_{ij}\delta)\sqrt{x}, (a_i+t_{ij}\delta)\sqrt{x}+2x^{1/2-\varepsilon}]\cap\mathbb{Z}),
\end{aligned}$$

we have

$$\begin{aligned}
\bigcup_{i,j}(I_i(t_{ij})\sqrt{x}\cap\mathbb{Z}) &\subseteq \Big(\bigcup_{0<a\le\delta\sqrt{x}}\bigcup_{i,j}\{M!l_{ij}+r_{ij}+a\}\Big)\\
&\qquad \cup \Big(\bigcup_{i,j}(((a_i+t_{ij}\delta)\sqrt{x}, (a_i+t_{ij}\delta)\sqrt{x}+2x^{1/2-\varepsilon}]\cap\mathbb{Z})\Big)\\
&\subseteq \Big(\bigcup_{0<a\le\delta\sqrt{x}}\Delta(a)\Big)\\
&\qquad \cup \Big(\bigcup_{i,j}(((a_i+t_{ij}\delta)\sqrt{x}, (a_i+t_{ij}\delta)\sqrt{x}+2x^{1/2-\varepsilon}]\cap\mathbb{Z})\Big).
\end{aligned}$$

Hence

$$\begin{aligned}
&\Big|\bigcup_{i,j}(I_i(t_{ij})\sqrt{x}\cap A_x\cap B)\Big|\\
&\qquad \le \delta\sqrt{x}+\sum_{i,j}2x^{1/2-\varepsilon} \le \delta\sqrt{x}+2Kx^{1/2-\varepsilon} \le \delta\sqrt{x}+\frac{2}{5}Mx^{1/2-\varepsilon},
\end{aligned}$$

whence

$$\begin{aligned}
&\Big|\bigcup_{i,j}(I_i(t_{ij})\sqrt{x}\cap A_x)\Big|\\
&\qquad \le \Big|\bigcup_{i,j}(I_i(t_{ij})\sqrt{x}\cap A_x\cap B)\Big| + |\{a : a\in(0, b_l\sqrt{x}],\ (a,D)>1\}|
\end{aligned}$$

$$\le \delta\sqrt{x} + \frac{2}{5}\, Mx^{1/2-\varepsilon} + \frac{1}{25}\, M^{9/2}x^{1/2-\varepsilon}\log\log x$$
$$\le \delta\sqrt{x} + \frac{2}{25}\, M^{9/2}x^{1/2-\varepsilon}\log\log x.$$

Since $I_1, \ldots, I_l$ are pairwise disjoint, we have

$$(2)\quad \Big|\bigcup_{i\le l-1}\bigcup_{j}(I_i(t_{ij})\sqrt{x}\cap A_x)\Big| + \Big|\bigcup_{j=1}^{k_l}(I_l(t_{lj})\sqrt{x}\cap A_x)\Big| \le \delta\sqrt{x} + \frac{2}{25}\, M^{9/2}x^{1/2-\varepsilon}\log\log x.$$

Now let $K_l = [|I_l|/\delta]$. Since $|I_l|/\delta \ge M^{-3/2}x^{\varepsilon/2} \ge M \ge K \ge k_l$, we have $K_l \ge k_l$. Suppose that $k_l \ge 1$. Then by (2) we have

$$\binom{K_l}{k_l}\Big|\bigcup_{i\le l-1}\bigcup_{j}(I_i(t_{ij})\sqrt{x}\cap A_x)\Big| + \binom{K_l-1}{k_l-1}\Big|\bigcup_{0\le t\le |I_l|/\delta-1}(I_l(t)\sqrt{x}\cap A_x)\Big| \le \binom{K_l}{k_l}\delta\sqrt{x} + \binom{K_l}{k_l}\frac{2}{25}\, M^{9/2}x^{1/2-\varepsilon}\log\log x.$$

Since

$$\binom{K_l}{k_l} = \frac{K_l}{k_l}\binom{K_l-1}{k_l-1} \le \frac{|I_l|}{k_l\delta}\binom{K_l-1}{k_l-1},$$

we have

$$\Big|\bigcup_{i\le l-1}\bigcup_{j}(I_i(t_{ij})\sqrt{x}\cap A_x)\Big| + \frac{k_l\delta}{|I_l|}\Big|\bigcup_{0\le t\le |I_l|/\delta-1}(I_l(t)\sqrt{x}\cap A_x)\Big| \le \delta\sqrt{x} + \frac{2}{25}\, M^{9/2}x^{1/2-\varepsilon}\log\log x.$$

By (2) this inequality is also true for $k_l = 0$. Noting that

$$\Big|\bigcup_{0\le t\le |I_l|/\delta-1}(I_l(t)\sqrt{x}\cap A_x)\Big| = |I_l\sqrt{x}\cap A_x| - \theta_l\delta\sqrt{x} \quad (0\le\theta_l\le 2)$$
$$= \alpha_l|I_l|\sqrt{x} - \theta_l\delta\sqrt{x},$$

we have

$$\Big|\bigcup_{i\le l-1}\bigcup_{j}(I_i(t_{ij})\sqrt{x}\cap A_x)\Big| + k_l\alpha_l\delta\sqrt{x}$$
$$\le \delta\sqrt{x} + \theta_l\,\frac{k_l\delta^2}{|I_l|}\,\sqrt{x} + \frac{2}{25}\, M^{9/2}x^{1/2-\varepsilon}\log\log x$$
$$\le \delta\sqrt{x} + 2k_lM^{3/2}\delta^2\sqrt{x} + \frac{2}{25}\, M^{9/2}x^{1/2-\varepsilon}\log\log x.$$

Continuing this procedure we have

$$\begin{aligned}\sum_{i=1}^{l} k_i\alpha_i\delta\sqrt{x} &\le \delta\sqrt{x} + 2\Big(\sum_{i=1}^{l} k_i\Big)M^{3/2}\delta^2\sqrt{x} + \frac{2}{25}\,M^{9/2}x^{1/2-\varepsilon}\log\log x\\ &\le \delta\sqrt{x} + 2KM^{3/2}\delta^2\sqrt{x} + \frac{2}{25}\,M^{9/2}x^{1/2-\varepsilon}\log\log x\\ &\le \delta\sqrt{x} + \frac{2}{5}\,M^{5/2}x^{1/2-\varepsilon} + \frac{2}{25}\,M^{9/2}x^{1/2-\varepsilon}\log\log x\\ &\le \delta\sqrt{x} + \frac{3}{25}\,M^{9/2}x^{1/2-\varepsilon}\log\log x.\end{aligned}$$

Therefore

$$\sum_{i=1}^{l} k_i\alpha_i \le 1 + \frac{3}{25}\,M^{9/2}x^{-\varepsilon/2}\log\log x.$$

This completes the proof of Lemma 5.

Similarly to the proof of Lemmas 4–8 in Chen [1] we have the following Lemma 6. The proof is omitted.

Lemma 6. *Let*

$$\sum_{i=1}^{l} k_{ij}\alpha_i \le 1 + \frac{3}{25}\,M^{9/2}x^{-\varepsilon/2}\log\log x, \qquad j = 1, \ldots, r,$$

be r relations obtained by using Lemma 5 (not necessarily from the same r_{ij}). Let $\beta_1, \ldots, \beta_l,\ \delta_1, \ldots, \delta_r$ be nonnegative real numbers with

$$\sum_{i=1}^{t}\beta_i \le \sum_{i=1}^{t}\sum_{j=1}^{r}\delta_j k_{ij}, \qquad t = 1, \ldots, l.$$

Then

$$\sum_{i=1}^{l}\beta_i\alpha_i \le \sum_{j=1}^{r}\delta_j + \frac{3}{25}\,M^{9/2}\sum_{j=1}^{r}\delta_j\cdot x^{-\varepsilon/2}\log\log x.$$

3. Proof of the Theorem. We recall some definitions from Chen [1]. Let x be a large real number. Let S be an integer with

$$2^S \le \frac{1}{11}\,\frac{\sqrt{\log x}}{\log\log x} < 2^{S+1}.$$

Let $L = 2^{2S}$, $T = 2LS - 1$, $q = 2^{1/(2L)}$ and

$$I_i = (q^i, q^{i+1}], \qquad i = -T, -T+1, \ldots, T, \qquad M = 5q^{2(T+1)} = 5\cdot 2^{2S}.$$

Lemma 7. $1 - q^{-1} \le 2^{-2S-1}$ *and* $|I_i| \ge M^{-3/2}$.

Proof. Since $q = 2^{1/(2L)}$ we have

$$(q-1)^{-1} = q^{2L-1} + q^{2L-2} + \cdots + q + 1$$

and

$$1 \le q^i < 2 \qquad (0 \le i \le 2L-1).$$

Then

$$2L < (q-1)^{-1} < 4L, \qquad 1 - q^{-1} < \frac{1}{2L} = 2^{-2S-1}.$$

Thus

$$|I_i| = q^i(q-1) \ge q^{-T-1}(q-1) \ge 2^{-S}(4L)^{-1} > M^{-3/2}.$$

This completes the proof of Lemma 7.

LEMMA 8. $x^{-\varepsilon/2} \log\log x \le 2^{-11(S+1)}$.

Proof. It is enough to prove that

$$(11 \log 2)(S+1) + \log\log\log x \le \frac{1}{2}\,\varepsilon \log x.$$

Since

$$2^S \le \frac{1}{11}\,\frac{\sqrt{\log x}}{\log\log x} < 2^{S+1},$$

we have

$$\frac{1}{2}\,\varepsilon \log x = \frac{\log x}{4M \log\log x} = \frac{1}{5}\,\frac{\log x}{2^{2S+2}\log\log x} \ge 5.5 \log\log x,$$

$$\begin{aligned}(11 \log 2)(S+1) + \log\log\log x &\le 5.5 \log\log x - 10 \log\log\log x - 11 \log 5.5 \\ &< 5.5 \log\log x.\end{aligned}$$

This completes the proof of Lemma 8.

For positive real numbers α, β, let

$$\begin{aligned} B(\alpha,\beta) = \{a : a \in \mathbb{Z},\ 1 \le a \le \alpha\beta\} \\ \cup \{a : a \in \mathbb{Z}, -\min\{\alpha\beta, \alpha^{-1}\beta - 1\} \le a \le 0\}, \end{aligned}$$

$$A_{ij} = \begin{cases} B(q^j, q^i) & \text{if } i \ge j, \\ \emptyset & \text{if } i < j. \end{cases}$$

In the following we make the convention that $\sum_{a\in\emptyset} h(a) = 0$ for any function $h(t)$.

Let

$$\begin{aligned}
\alpha &= (10-7\sqrt{2})/32,\\
k_{ij} &= |A_{ij}\setminus A_{(i-1)j}|, \qquad -T\le i\le T,\ -T\le j\le L-1,\\
k_{iL} &= 0 \quad (-T\le i\le T,\, i\ne 0), \qquad k_{0L}=1,\\
\beta_i &= q^i(q-1), \qquad -T\le i\le L-1,\\
\beta_i &= (1+\alpha)q^i(q-1), \qquad L\le i\le T,\\
\delta_j &= q^j(q-1), \qquad -T\le j\le -1,\\
\delta_j &= \frac{1}{2}\,(q-1)(q^j-q^{-j-1}), \qquad 0\le j\le L-1,\\
\delta_L &= 1-q^{-1}.
\end{aligned}$$

By Lemma 8 we have $x^{\varepsilon}\ge M^5$. Similarly to Lemmas 10–12 in Chen [1] we have the following lemmas.

LEMMA 9. *For* $-T\le j\le L$ *we have*

$$\sum_{-T\le i\le T} k_{ij}\alpha_i \le 1+\frac{3}{25}\,M^{9/2}x^{-\varepsilon/2}\log\log x.$$

LEMMA 10. *There exists an* L_0 *such that if* $L\ge L_0$, *then*

$$\sum_{i=-T}^{t}\beta_i \le \sum_{i=-T}^{t}\sum_{j=-T}^{L}\delta_j k_{ij}, \qquad t=-T,-T+1,\dots,T.$$

LEMMA 11. *Let* G *be a positive integer and* $x\ge 2G$. *Then*

$$|(\sqrt{Gx},x]\cap A_x| \le \frac{2\sqrt{x}}{\sqrt{G}}+\log_2\frac{2\sqrt{x}}{\sqrt{G}}.$$

Proof of the Theorem. By Lemmas 6–10 we have

$$\begin{aligned}
&\sum_{-T\le i\le L-1} q^i(q-1)\alpha_i + (1+\alpha)\sum_{L\le i\le T} q^i(q-1)\alpha_i\\
&\quad= \sum_{-T\le i\le T}\beta_i\alpha_i \le \sum_{-T\le j\le L}\delta_j + \frac{3}{25}\,M^{9/2}\sum_{-T\le j\le L}\delta_j\cdot x^{-\varepsilon/2}\log\log x\\
&\quad\le \sqrt{\frac{9}{8}}-q^{-T}+1-q^{-1}+\frac{3}{25}\,M^{9/2}\biggl(\sqrt{\frac{9}{8}}-q^{-T}+1-q^{-1}\biggr)x^{-\varepsilon/2}\log\log x\\
&\quad\le \sqrt{\frac{9}{8}}-q^{-T}+2^{-2S-1}+\frac{9}{25}\,M^{9/2}x^{-\varepsilon/2}\log\log x\\
&\quad\le \sqrt{\frac{9}{8}}-q^{-T}+2^{-2S-1}+\frac{9}{25}\,5^{9/2}\,2^{9S}\,2^{-11S-11}\\
&\quad\le \sqrt{\frac{9}{8}}-q^{-T}+2^{-2S}.
\end{aligned}$$

Hence

$$|(q^{-T}\sqrt{x},\sqrt{2x}]\cap A_x|+(1+\alpha)|(\sqrt{2x},q^{T+1}\sqrt{x}]\cap A_x| \le \sqrt{\frac{9}{8}x}-q^{-T}\sqrt{x}+2^{-2S}\sqrt{x}.$$

So

$$|[1,\sqrt{2x}]\cap A_x|+(1+\alpha)|(\sqrt{2x},2^S\sqrt{x}]\cap A_x|\le\sqrt{\frac{9}{8}x}+2^{-2S}\sqrt{x}.$$

Thus

$$|[1,2^S\sqrt{x}]\cap A_x|\le\sqrt{\frac{9}{8}x}+2^{-2S}\sqrt{x}.$$

Since $2^{2S+2}\le\log x\le x$, by Lemma 11 we have

$$|(2^S\sqrt{x},x]\cap A_x|\le 2^{-S+1}\sqrt{x}+\log_2(2^{-S+1}\sqrt{x}).$$

Thus, for all sufficiently large x we have

$$\begin{aligned}|A_x|&\le\sqrt{\frac{9}{8}x}+2^{-2S}\sqrt{x}+2^{-S+1}\sqrt{x}+\log_2(2^{-S+1}\sqrt{x})\\&\le\sqrt{\frac{9}{8}x}+45\sqrt{\frac{x}{\log x}}\log\log x.\end{aligned}$$

Since $|A_x|\ge|B_x|\ge\sqrt{\frac{9}{8}x}-2$, the proof of the Theorem is complete.

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