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Number of solutions of certain congruences

by

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1. Introduction and main results. Let $f(x) = x^d + a_1 x^{d-1} + \cdots + a_d$, $a_1, \ldots, a_d \in \mathbb{Z}$, $d \geq 2$, be an irreducible polynomial. Let $N_f(n)$ be the number of solutions x of $f(x) \equiv 0 \pmod{n}$ satisfying $0 \leq x < n$. It is an important problem to study the function $N_f(n)$.

In 1952, Erdős [2] proved the asymptotic formulae

$$\sum_{p \leq x} N_f(p) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right),$$

$$\sum_{p \leq x} \frac{N_f(p)}{p} = \log\log x + c(f) + o(1),$$

and the lower estimate

$$\sum_{n \leq x} N_f(n) \gg x,$$

where p runs over primes, and n runs over integers.

In 2001, Fomenko showed (see formula (4) in [3]) that

$$\sum_{n \leq x} N_f(n) = C(f)x + O\left(\frac{x}{(\log x)^{1/2-\varepsilon}}\right),$$

where

$$C(f) = e^{-\gamma + c(f)} P > 0. \tag{1.1}$$

Here γ is the Euler constant and

$$P = \prod_p e^{-N_f(p)/p}\left(1 + \frac{N_f(p)}{p} + \frac{N_f(p^2)}{p^2} + \cdots\right).$$

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Let L be the splitting field of f over $\mathbb{Q}$ with Galois group $G = \mathrm{Gal}(L/\mathbb{Q})$. If G is Abelian, the field L is called Abelian. In this case we also call $f(x)$ an *Abelian polynomial*. Otherwise we call $f(x)$ a *non-Abelian polynomial*.

In [3] Fomenko proved that for any Abelian polynomial $f(x)$,

$$\sum_{n\le x} N_f(n) = C(f)x + O(x\exp(-B(\log x)^{\beta}))$$

for a certain positive constant B and any fixed $\beta < 3/5$. In addition, Fomenko mentioned in Remark 1 of [3] that for any Abelian polynomial $f(x)$, under the Riemann Hypothesis on Dirichlet L-functions,

$$\sum_{n\le x} N_f(n) = C(f)x + O(x^{1/2+\varepsilon}). \tag{1.2}$$

Recently Kim [8] introduced the Langlands functoriality to this problem and proved the following two results.

(i) For any non-Abelian polynomial $f(x)$ of degree d, unconditionally we have

$$\sum_{n\le x} N_f(n) = C(f)x + O(x^{1-2/(d+4)+\varepsilon}).$$

(ii) For any Abelian polynomial $f(x)$ of degree d, we have

$$\sum_{n\le x} N_f(n) = C(f)x + O(x^{1-3/(d+6)+\varepsilon}).$$

Based on Kim's method, we shall show the following results.

THEOREM 1.1. *For any Abelian polynomial $f(x)$ of degree d, we have*

$$\sum_{n\le x} N_f(n) = \begin{cases} C(f)x + O(x^{1/2+\varepsilon}) & \text{for } d = 2,3,\\ C(f)x + O(x^{1-3/(d+2)+\varepsilon}) & \text{for } 4 \le d \le 11,\\ C(f)x + O(x^{1-3/d+\varepsilon}) & \text{for } d \ge 12,\end{cases}$$

where $C(f)$ is defined in (3.4).

THEOREM 1.2. *For any non-Abelian polynomial $f(x)$ of degree d, unconditionally we have*

$$\sum_{n\le x} N_f(n) = C(f)x + O(x^{1-3/(d+6)+\varepsilon}),$$

where $C(f)$ is defined in (4.2).

2. Preliminaries. Let D denote the discriminant of the polynomial $f(x)$. By Lemma 3 in Erdős [2], $N_f(n)$ is a multiplicative function, and its value at the power of a prime p satisfies

$$N_f(p^\alpha) \le \begin{cases} d & \text{if } p \nmid D, \\ dD^2 & \text{if } p \mid D, \end{cases}$$

where d is the degree of the polynomial f. Then we have

$$N_f(n) \ll d^{\omega(n)} \ll \tau(n)^{\frac{\log d}{\log 2}}, \tag{2.1}$$

where $\omega(n)$ is the number of distinct prime divisors of n, and $\tau(n)$ is the divisor function. Therefore in the half-plane $\operatorname{Re} s > 1$, we can define the L-function associated to $N_f(n)$,

$$L(s) = \sum_{n=1}^{\infty} \frac{N_f(n)}{n^s}, \tag{2.2}$$

where the series is absolutely convergent in this region. Since $N_f(n)$ is multiplicative, for $\operatorname{Re} s > 1$ we can write

$$L(s) = \prod_p \left(1 + \frac{N_f(p)}{p^s} + \frac{N_f(p^2)}{p^{2s}} + \cdots\right), \tag{2.3}$$

where the product is over all primes.

Recall that L is the splitting field of f over $\mathbb{Q}$. Let $E = \mathbb{Q}(\alpha)$, where α is a root of f. We have $[E : \mathbb{Q}] = d$. Let $\zeta_E(s)$ be the Dedekind zeta-function of the field E. Then for $\operatorname{Re} s > 1$, we have

$$\zeta_E(s) = \sum_{\mathfrak{a}} (N\mathfrak{a})^{-s},$$

where the sum is extended over all integral ideals $\mathfrak{a}$ of the field E, and $N\mathfrak{a}$ is the norm of $\mathfrak{a}$. We can rewrite it as

$$\zeta_E(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \left(1 + \frac{a_p}{p^s} + \frac{a_{p^2}}{p^{2s}} + \cdots\right),$$

where a_n denotes the number of integral ideals in E with norm n. From Lemma 9 in [1], it is known that a_n is a multiplicative function and satisfies

$$a_n \ll (\tau(n))^{d-1}, \tag{2.4}$$

where $\tau(n)$ is the divisor function, and d is the degree of the polynomial f. In addition, from page 57 in [1] we learn that

$$\zeta_E(s)U(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-a_p}, \tag{2.5}$$

where $U(s)$ is an infinite product over primes, which is absolutely and uniformly convergent for $\operatorname{Re} s > 1/2$. From (2.1), (2.3), (2.4), and (2.5), we conclude that for $\operatorname{Re} s > 1$,

$$L(s) = \zeta_E(s)U(s)\prod_p \left(1 + \frac{N_f(p)}{p^s} + \frac{N_f(p^2)}{p^{2s}} + \cdots\right)\left(1 - \frac{1}{p^s}\right)^{a_p}. \tag{2.6}$$

By Kummer's Theorem on the decomposition of prime ideals in algebraic extensions (see e.g. Lemma 22 in Swinnerton-Dyer [11]), we learn that except for finitely many primes (in fact, if p does not divide the discriminant D of $f(x)$ or the index $[O_E : \mathbb{Z}[\alpha]]$),

$$a_p = N_f(p). \tag{2.7}$$

In fact, the factorization of a prime p in the field E as

$$(p) = pO_E = \mathfrak{p}_1 \cdots \mathfrak{p}_g,$$

where $N\mathfrak{p}_j = p^{f_j}$ $(1 \le j \le g)$ corresponds to the factorization

$$f(x) \equiv f_1(x) \cdots f_g(x) \pmod p,$$

where $f_j(x)$ $(1 \le j \le g)$ are irreducible polynomials over $\mathbb{Z}_p$, of degree f_j. Therefore the number of integral ideals with norm p corresponds to the number of linear polynomials among $f_j(x)$. Obviously the latter number equals $N_f(p)$. Therefore we have the identity (2.7).

We define

$$S = \{p : p \mid D \text{ or } p \mid [O_E : \mathbb{Z}[\alpha]]\}.$$

Then from (2.6) and (2.7), we conclude that for $\operatorname{Re} s > 1$,

$$\begin{aligned} L(s) &= \zeta_E(s)U(s)\prod_{p\in S} \left(1 + \frac{N_f(p)}{p^s} + \frac{N_f(p^2)}{p^{2s}} + \cdots\right)\left(1 - \frac{1}{p^s}\right)^{a_p} \\ &\quad\times \prod_{p\notin S}\left(1 + \frac{N_f(p^2) - a_{p^2}/2 - a_p/2}{p^{2s}} + \cdots\right) \\ &:= \zeta_E(s)U(s)\prod_{p\in S} \times \prod_{p\notin S} \\ &:= \zeta_E(s)A(s). \end{aligned} \tag{2.8}$$

From (2.1), (2.4), and the finiteness of the set S, we learn that the product $\prod_{p\in S}$ is absolutely convergent for $\operatorname{Re} s > 0$, and the product $\prod_{p\notin S}$ is absolutely convergent for $\operatorname{Re} s > 1/2$. Then $A(s) = U(s)\prod_{p\in S} \times \prod_{p\notin S}$ is absolutely convergent for $\operatorname{Re} s > 1/2$, and uniformly convergent for $\operatorname{Re} s \ge 1/2 + \varepsilon$ with any $\varepsilon > 0$, and hence holomorphic for $\operatorname{Re} s > 1/2$. Therefore $L(s) = \zeta_E(s)A(s)$ has a meromorphic continuation to the half-plane $\operatorname{Re} s > 1/2$. Since $\zeta_E(s)$ only has a simple pole at $s = 1$ in this region, so does $L(s)$.

3. Proof of Theorem 1.1. In this section L is the splitting field of f over $\mathbb{Q}$ with the Abelian Galois group $G = \mathrm{Gal}(L/\mathbb{Q})$. Then the splitting field L coincides with the field $E = \mathbb{Q}(\alpha)$.

The Kronecker–Weber Theorem asserts that every Abelian extension of $\mathbb{Q}$ is cyclotomic (see e.g. Theorem 44 in Swinnerton-Dyer [11]). We let $\mathbb{Q}(\zeta_m)$ with $\zeta_m = e^{2\pi i/m}$ be the least cyclotomic field which contains the Abelian field L. Then we call m the *conductor* of the Abelian field L. We have $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^*$, and so $H = \mathrm{Gal}(\mathbb{Q}(\zeta_m)/L)$ can be regarded as a subgroup of $(\mathbb{Z}/m\mathbb{Z})^*$. The characters of the finite Abelian group $\mathrm{Gal}(L/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^*/H$ are also called the characters of the field L. We denote the character group of L by $\widehat{L}$. Therefore $\widehat{L}$ consists of the Dirichlet characters modulo m that are trivial on H.

As a simple corollary of Abelian class field theory we can write $\zeta_L(s)$ as a product of the Riemann zeta-function and Dirichlet L-functions. More precisely, we have

$$\zeta_L(s) = \prod_{\chi\in\widehat{L}} L(s,\chi^*) = \zeta(s) \prod_{\substack{\chi\in\widehat{L}\\ \chi\neq\chi_0}} L(s,\chi^*),$$

where χ^* is a primitive character modulo m' with $m' \mid m$, which induces $\chi \bmod m$. For simplicity, we shall write

$$\zeta_L(s) = \zeta(s) \prod_{j=1}^{d-1} L(s,\chi_j), \tag{3.1}$$

where $L(s,\chi_j)$ are primitive Dirichlet L-functions.

From (2.8) and (3.1), we have

$$L(s) = \zeta_L(s)A(s) = \zeta(s) \prod_{j=1}^{d-1} L(s,\chi_j)A(s), \tag{3.2}$$

which admits a meromorphic continuation to the half-plane $\mathrm{Re}\, s > 1/2$, and only has a simple pole at $s = 1$ in this region. Here $A(s)$ is absolutely and uniformly convergent for $\mathrm{Re}\, s \geq 1/2 + \varepsilon$ with any $\varepsilon > 0$.

Now we begin the proof. First we assume that $4 \leq d \leq 11$. By (2.1), (2.2) and Perron's formula (see Proposition 5.54 in [7]), we have

$$\sum_{n\leq x} N_f(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} L(s)\, \frac{x^s}{s}\, ds + O\biggl(\frac{x^{1+\varepsilon}}{T}\biggr), \tag{3.3}$$

where $b = 1 + \varepsilon$ and $1 \leq T \leq x$ is a parameter to be chosen later.

Next we move the integration to the parallel segment with $\mathrm{Re}\, s = 1/2+\varepsilon$. By Cauchy's residue theorem, we have

$$\sum_{n\le x} N_f(n) = \frac{1}{2\pi i}\Big\{\int_{1/2+\varepsilon-iT}^{1/2+\varepsilon+iT} + \int_{1/2+\varepsilon+iT}^{b+iT} + \int_{b-iT}^{1/2+\varepsilon-iT}\Big\} L(s)\frac{x^s}{s}\,ds \tag{3.4}$$
$$+ \operatorname{Res}_{s=1} L(s)x + O(x^{1+\varepsilon}/T)$$
$$:= I_1 + I_2 + I_3 + C(f)x + O(x^{1+\varepsilon}/T),$$

where $C(f) = \operatorname{Res}_{s=1} L(s)$.

It is well known that

$$\zeta(1/2+it) \ll (1+|t|)^{1/6}\log(|t|+1)$$

and

$$L(1/2+it,\chi) \ll (1+|t|)^{1/6}\log(|t|+1)$$

(see e.g. Theorems 24.1.1 and 24.2.1 in Pan and Pan [10]). Then by the Phragmén–Lindelöf principle for a strip (see e.g. Theorem 5.53 in Iwaniec and Kowalski [7]), we deduce that for $1/2 \le \sigma \le 1+\varepsilon$,

$$\zeta(\sigma+it) \ll (1+|t|)^{(1-\sigma)/3+\varepsilon} \quad \text{and} \quad L(\sigma+it,\chi) \ll (1+|t|)^{(1-\sigma)/3+\varepsilon}, \tag{3.5}$$

where we have used

$$\zeta(1+\varepsilon+it) \ll 1 \quad \text{and} \quad L(1+\varepsilon+it,\chi) \ll 1.$$

Hence we have

$$\zeta(1/2+\varepsilon+it) \ll (1+|t|)^{1/6+\varepsilon}, \quad L(1/2+\varepsilon+it,\chi) \ll (1+|t|)^{1/6+\varepsilon}. \tag{3.6}$$

For I_1, by (2.8) or (3.2) we have

$$\begin{aligned} I_1 &\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon}\int_1^T |L(1/2+\varepsilon+it)|t^{-1}\,dt \\ &\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon}\int_1^T |\zeta_L(1/2+\varepsilon+it)A(1/2+\varepsilon+it)|t^{-1}\,dt \\ &\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon}\int_1^T |\zeta_L(1/2+\varepsilon+it)|t^{-1}\,dt, \end{aligned}$$

where we have used that $A(s)$ is absolutely convergent in the region $\operatorname{Re} s \ge 1/2+\varepsilon$ and is $O(1)$ there.

By (3.1) and (3.6), we have

$$\begin{aligned} I_1 &\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon}\int_1^T \Big|\zeta(1/2+\varepsilon+it)\prod_{j=1}^{3} L(1/2+\varepsilon+it,\chi_j) \\ &\qquad\qquad \times \prod_{j=4}^{d-1} L(1/2+\varepsilon+it,\chi_j)\Big| t^{-1}\,dt \\ &\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon}\int_1^T \Big|\zeta(1/2+\varepsilon+it)\prod_{j=1}^{3} L(1/2+\varepsilon+it,\chi_j)\Big| t^{(d-4)/6-1}\,dt. \end{aligned}$$

Then by Hölder's inequality, we have

$$I_1 \ll x^{1/2+\varepsilon} \log T \max_{T_1 \le T} \Big\{ T_1^{(d-4)/6-1} \Big(\int_{T_1/2}^{T_1} |\zeta(1/2+\varepsilon+it)|^4 \, dt \Big)^{1/4} \times \prod_{j=1}^{3} \Big(\int_{T_1/2}^{T_1} |L(1/2+\varepsilon+it,\chi_j)|^4 \, dt \Big)^{1/4} \Big\} + x^{1/2+\varepsilon} \ll x^{1/2+\varepsilon} T^{(d-4)/6+\varepsilon} + x^{1/2+\varepsilon}, \tag{3.7}$$

where we have used

$$\int_{T_1/2}^{T_1} |\zeta(1/2+\varepsilon+it)|^4 \, dt \ll T_1^{1+\varepsilon}$$

and

$$\int_{T_1/2}^{T_1} |L(1/2+\varepsilon+it,\chi_j)|^4 \, dt \ll T_1^{1+\varepsilon}.$$

These results can be established by using Gabriel's convexity theorem (see e.g. Lemma 8.3 in Ivić [6]), and the following two classical results (see e.g. Theorems 29.3.1 and 29.3.4 in Pan and Pan [10]):

$$\int_{T_1/2}^{T_1} |\zeta(1/2+it)|^4 \, dt \ll T_1 (\log T_1)^4$$

and

$$\int_{T_1/2}^{T_1} |L(1/2+it,\chi_j)|^4 \, dt \ll T_1 (\log T_1)^4.$$

By (3.1) and (3.5), we conclude that for $1/2 \le \sigma \le 1+\varepsilon$,

$$\zeta_L(\sigma+it) \ll (1+|t|)^{\frac{d}{3}(1-\sigma)+\varepsilon}.$$

Therefore for the integrals over the horizontal segments we have

$$\begin{aligned} I_2 + I_3 &\ll \int_{1/2+\varepsilon}^{b} x^{\sigma} |\zeta_L(\sigma+iT)| T^{-1} \, d\sigma \\ &\ll \max_{1/2+\varepsilon \le \sigma \le b} x^{\sigma} T^{\frac{d}{3}(1-\sigma)+\varepsilon} T^{-1} \\ &= \max_{1/2+\varepsilon \le \sigma \le b} \Big(\frac{x}{T^{d/3}} \Big)^{\sigma} T^{d/3-1+\varepsilon} \\ &\ll x^{1+\varepsilon}/T + x^{1/2+\varepsilon} T^{d/6-1+\varepsilon}. \end{aligned} \tag{3.8}$$

From (3.4), (3.7) and (3.8), we have

$$\sum_{n\le x} N_f(n) = C(f)x + O(x^{1+\varepsilon}/T) + O(x^{1/2+\varepsilon}T^{(d-4)/6+\varepsilon}). \tag{3.9}$$

On taking $T = x^{3/(d+2)}$ in (3.9), we have

$$\sum_{n\le x} N_f(n) = C(f)x + O(x^{1-3/(d+2)+\varepsilon}).$$

Now we consider the case $d \ge 12$. From the context we only need to estimate the integral I_1. We have

$$\begin{aligned}
I_1 &\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon}\int_1^T |L(1/2+\varepsilon+it)|t^{-1}\,dt \\
&\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon}\int_1^T |\zeta_L(1/2+\varepsilon+it)|t^{-1}\,dt \\
&\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon}\int_1^T \Big|\zeta(1/2+\varepsilon+it)\prod_{j=1}^{11} L(1/2+\varepsilon+it,\chi_j) \\
&\qquad\qquad \times \prod_{j=12}^{d-1} L(1/2+\varepsilon+it,\chi_j)\Big| t^{-1}\,dt \\
&\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon}\int_1^T \Big|\zeta(1/2+\varepsilon+it)\prod_{j=1}^{11} L(1/2+\varepsilon+it,\chi_j)\Big| t^{(d-12)/6-1}\,dt.
\end{aligned}$$

Then by Hölder's inequality, we have

$$\begin{aligned}
I_1 &\ll x^{1/2+\varepsilon}\log T \max_{T_1\le T}\Big\{T_1^{(d-12)/6-1}\Big(\int_{T_1/2}^{T_1} |\zeta(1/2+\varepsilon+it)|^{12}\,dt\Big)^{1/12} \\
&\qquad \times \prod_{j=1}^{11}\Big(\int_{T_1/2}^{T_1} |L(1/2+\varepsilon+it,\chi_j)|^{12}\,dt\Big)^{1/12}\Big\} + x^{1/2+\varepsilon} \\
&\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon}T^{d/6-1+\varepsilon},
\end{aligned}$$

where we have used

$$\int_{T_1/2}^{T_1} |\zeta(1/2+\varepsilon+it)|^{12}\,dt \ll T_1^{2+\varepsilon}$$

and

$$\int_{T_1/2}^{T_1} |L(1/2+\varepsilon+it,\chi)|^{12}\,dt \ll T_1^{2+\varepsilon}.$$

These results can be deduced from Gabriel's convexity theorem (see e.g. Lemma 8.3 in Ivić [6]), and the results of Heath-Brown [4] and Meurman [9] respectively, which state that

$$\int_{T_1/2}^{T_1} |\zeta(1/2+it)|^{12}\,dt \ll T_1^2(\log T_1)^{17}$$

and

$$\int_{T_1/2}^{T_1} |L(1/2+it,\chi)|^{12}\,dt \ll T_1^{2+\varepsilon}.$$

Then on taking $T = x^{3/d}$, we have

$$\sum_{n\le x} N_f(n) = C(f)x + O(x^{1-3/d+\varepsilon}).$$

Finally, we consider the cases $d = 2, 3$. For $d = 2$, we have

$$\begin{aligned} I_1 &\ll x^{1/2+\varepsilon}\log T \max_{T_1\le T}\Big\{T_1^{-1}\Big(\int_{T_1/2}^{T_1} |\zeta(1/2+\varepsilon+it)|^2\,dt\Big)^{1/2} \\ &\qquad\qquad \times\Big(\int_{T_1/2}^{T_1} |L(1/2+\varepsilon+it,\chi)|^2\,dt\Big)^{1/2}\Big\} \\ &\ll x^{1/2+\varepsilon}, \end{aligned}$$

where we have used

$$\int_{T_1/2}^{T_1} |\zeta(1/2+\varepsilon+it)|^2\,dt \ll T_1^{1+\varepsilon}$$

and

$$\int_{T_1/2}^{T_1} |L(1/2+\varepsilon+it,\chi_j)|^2\,dt \ll T_1^{1+\varepsilon}.$$

These results can also be established by applying Gabriel's convexity theorem (see e.g. Lemma 8.3 in Ivić [6]), and the following two classical results (see e.g. Theorems 25.2.1 and 25.3.1 in Pan and Pan [10]):

$$\int_{T_1/2}^{T_1} |\zeta(1/2+it)|^2\,dt \ll T_1\log T_1$$

and

$$\int_{T_1/2}^{T_1} |L(1/2+it,\chi_j)|^2\,dt \ll T_1\log T_1.$$

Then on taking $T = x^{1/2}$, we have

$$\sum_{n\le x} N_f(n) = C(f)x + O(x^{1/2+\varepsilon}).$$

For the case $d = 3$, we have

$$I_1 \ll x^{1/2+\varepsilon}\log T \max_{T_1\le T}\Big\{T_1^{-1}\Big(\int_{T_1/2}^{T_1}|\zeta(1/2+\varepsilon+it)|^2\,dt\Big)^{1/2} \times \prod_{j=1}^{2}\Big(\int_{T_1/2}^{T_1}|L(1/2+\varepsilon+it,\chi_j)|^4\,dt\Big)^{1/4}\Big\} \ll x^{1/2+\varepsilon}.$$

Then on taking $T = x^{1/2}$, we also have

$$\sum_{n\le x} N_f(n) = C(f)x + O(x^{1/2+\varepsilon}).$$

4. Proof of Theorem 1.2. Recall that L is the splitting field of f over $\mathbb{Q}$ with Galois group $G = \mathrm{Gal}(L/\mathbb{Q})$ and $E = \mathbb{Q}(\alpha)$, where α is a root of f. From our assumption, G is not Abelian in this section.

By (2.1), (2.2), and Perron's formula (see Proposition 5.54 in [7]), we have

$$\sum_{n\le x} N_f(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} L(s)\,\frac{x^s}{s}\,ds + O\Big(\frac{x^{1+\varepsilon}}{T}\Big), \tag{4.1}$$

where $b = 1+\varepsilon$ and $1\le T\le x$ is a parameter to be chosen later.

Next we move the integration to the parallel segment with $\mathrm{Re}\, s = 1/2+\varepsilon$. By Cauchy's residue theorem, we have

$$\begin{aligned}\sum_{n\le x} N_f(n) &= \frac{1}{2\pi i}\Big\{\int_{1/2+\varepsilon-iT}^{1/2+\varepsilon+iT} + \int_{1/2+\varepsilon+iT}^{b+iT} + \int_{b-iT}^{1/2+\varepsilon-iT}\Big\}L(s)\,\frac{x^s}{s}\,ds \\ &\quad + \mathrm{Res}_{s=1} L(s)x + O\Big(\frac{x^{1+\varepsilon}}{T}\Big) \\ &:= J_1 + J_2 + J_3 + C(f)x + O\Big(\frac{x^{1+\varepsilon}}{T}\Big).\end{aligned} \tag{4.2}$$

For J_1, by (2.8) we have

$$J_1 \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon}\int_1^T |L(1/2+\varepsilon+it)|t^{-1}\,dt$$

$$\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \int_1^T |\zeta_E(1/2+\varepsilon+it)A(1/2+\varepsilon+it)| t^{-1}\, dt$$
$$\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \int_1^T |\zeta_E(1/2+\varepsilon+it)| t^{-1}\, dt,$$

where we have used that $A(s)$ is absolutely convergent in the region $\mathrm{Re}\, s \geq 1/2+\varepsilon$ and is $O(1)$ there.

To go further, we cite a result of Heath-Brown [5] about the subconvexity bound for the Dedekind zeta-function on the critical line, which states that if E is an algebraic number field of degree d, then

$$\zeta_E(1/2+it) \ll_E t^{d/6+\varepsilon} \qquad (t \geq 1)$$

for any fixed $\varepsilon > 0$. Then by the Phragmén–Lindelöf principle for a strip (see e.g. Theorem 5.53 in Iwaniec and Kowalski [7]), we deduce that for $1/2 \leq \sigma \leq 1+\varepsilon$,

$$\zeta_E(\sigma+it) \ll (1+|t|)^{\frac{d}{3}(1-\sigma)+\varepsilon}, \tag{4.3}$$

where we have used the estimate $\zeta_E(1+\varepsilon) \ll 1$.

By (4.3), we have

$$J_1 \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \int_1^T |\zeta_E(1/2+\varepsilon+it)| t^{-1}\, dt \tag{4.4}$$
$$\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \int_1^T t^{d/6-1+\varepsilon}\, dt \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} T^{d/6+\varepsilon}.$$

For the integrals over the horizontal segments, by (4.3) we have

$$J_2 + J_3 \ll \int_{1/2+\varepsilon}^{b} x^{\sigma} |\zeta_E(\sigma+iT)| T^{-1}\, d\sigma \tag{4.5}$$
$$\ll \max_{1/2+\varepsilon\leq\sigma\leq b} x^{\sigma} T^{\frac{d}{3}(1-\sigma)+\varepsilon} T^{-1}$$
$$= \max_{1/2+\varepsilon\leq\sigma\leq b} \left(\frac{x}{T^{d/3}}\right)^{\sigma} T^{d/3-1+\varepsilon}$$
$$\ll x^{1+\varepsilon}/T + x^{1/2+\varepsilon} T^{d/6-1+\varepsilon}.$$

From (4.2), (4.4) and (4.5), we have

$$\sum_{n\leq x} N_f(n) = C(f)x + O(x^{1+\varepsilon}/T) + O(x^{1/2+\varepsilon} T^{d/6+\varepsilon}). \tag{4.6}$$

On taking $T = x^{3/(d+6)}$ in (4.6), we have

$$\sum_{n \le x} N_f(n) = C(f)x + O(x^{1-3/(d+6)+\varepsilon}).$$

This completes the proof of Theorem 1.2.

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