Number of solutions of certain congruences

by

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1. Introduction and main results. Let $f(x) = x^d + a_1x^{d-1} + \cdots + a_d$, $a_1, \ldots, a_d \in \mathbb{Z}$, $d \geq 2$, be an irreducible polynomial. Let $N_f(n)$ be the number of solutions $x$ of $f(x) \equiv 0 \pmod{n}$ satisfying $0 \leq x < n$. It is an important problem to study the function $N_f(n)$.

In 1952, Erdős [2] proved the asymptotic formulae

$$\sum_{p \leq x} N_f(p) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right),$$

$$\sum_{p \leq x} \frac{N_f(p)}{p} = \log \log x + c(f) + o(1),$$

and the lower estimate

$$\sum_{n \leq x} N_f(n) \gg x,$$

where $p$ runs over primes, and $n$ runs over integers.

In 2001, Fomenko showed (see formula (4) in [3]) that

$$\sum_{n \leq x} N_f(n) = C(f)x + O\left(\frac{x}{(\log x)^{1/2-\varepsilon}}\right),$$

where

$$C(f) = e^{-\gamma + c(f)} P > 0. \tag{1.1}$$

Here $\gamma$ is the Euler constant and

$$P = \prod_p e^{-N_f(p)/p} \left(1 + \frac{N_f(p)}{p} + \frac{N_f(p^2)}{p^2} + \cdots\right).$$

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Let $L$ be the splitting field of $f$ over $\mathbb{Q}$ with Galois group $G = \text{Gal}(L/\mathbb{Q})$. If $G$ is Abelian, the field $L$ is called Abelian. In this case we also call $f(x)$ an *Abelian polynomial*. Otherwise we call $f(x)$ a *non-Abelian polynomial*.

In [3] Fomenko proved that for any Abelian polynomial $f(x)$,

$$
\sum_{n \leq x} N_f(n) = C(f)x + O(x \exp(-B \log x)^{\beta})
$$

for a certain positive constant $B$ and any fixed $\beta < 3/5$. In addition, Fomenko mentioned in Remark 1 of [3] that for any Abelian polynomial $f(x)$, under the Riemann Hypothesis on Dirichlet $L$-functions,

$$\sum_{n \leq x} N_f(n) = C(f)x + O(x^{1/2+\varepsilon}). \tag{1.2}
$$

Recently Kim [8] introduced the Langlands functoriality to this problem and proved the following two results.

(i) For any non-Abelian polynomial $f(x)$ of degree $d$, unconditionally we have

$$
\sum_{n \leq x} N_f(n) = C(f)x + O(x^{1-2/(d+4)+\varepsilon}).
$$

(ii) For any Abelian polynomial $f(x)$ of degree $d$, we have

$$
\sum_{n \leq x} N_f(n) = C(f)x + O(x^{1-3/(d+6)+\varepsilon}).
$$

Based on Kim’s method, we shall show the following results.

**Theorem 1.1.** For any Abelian polynomial $f(x)$ of degree $d$, we have

$$
\sum_{n \leq x} N_f(n) = \begin{cases} 
C(f)x + O(x^{1/2+\varepsilon}) & \text{for } d = 2, 3, \\
C(f)x + O(x^{1-3/(d+2)+\varepsilon}) & \text{for } 4 \leq d \leq 11, \\
C(f)x + O(x^{1-3/d+\varepsilon}) & \text{for } d \geq 12,
\end{cases}
$$

where $C(f)$ is defined in (3.4).

**Theorem 1.2.** For any non-Abelian polynomial $f(x)$ of degree $d$, unconditionally we have

$$
\sum_{n \leq x} N_f(n) = C(f)x + O(x^{1-3/(d+6)+\varepsilon}),
$$

where $C(f)$ is defined in (4.2).

2. **Preliminaries.** Let $D$ denote the discriminant of the polynomial $f(x)$. By Lemma 3 in Erdős [2], $N_f(n)$ is a multiplicative function, and its
value at the power of a prime $p$ satisfies
\[ N_f(p^\alpha) \leq \begin{cases} d & \text{if } p \nmid D, \\
D^2 & \text{if } p \mid D, \end{cases} \]
where $d$ is the degree of the polynomial $f$. Then we have

\begin{equation}
N_f(n) \ll d^{\omega(n)} \ll \tau(n)^{\log d \log 2},
\end{equation}

where $\omega(n)$ is the number of distinct prime divisors of $n$, and $\tau(n)$ is the divisor function. Therefore in the half-plane $\Re s > 1$, we can define the $L$-function associated to $N_f(n)$,

\begin{equation}
L(s) = \sum_{n=1}^{\infty} \frac{N_f(n)}{n^s},
\end{equation}

where the series is absolutely convergent in this region. Since $N_f(n)$ is multiplicative, for $\Re s > 1$ we can write

\begin{equation}
L(s) = \prod_p \left(1 + \frac{N_f(p)}{p^s} + \frac{N_f(p^2)}{p^{2s}} + \cdots \right),
\end{equation}

where the product is over all primes.

Recall that $L$ is the splitting field of $f$ over $\mathbb{Q}$. Let $E = \mathbb{Q}(\alpha)$, where $\alpha$ is a root of $f$. We have $[E : \mathbb{Q}] = d$. Let $\zeta_E(s)$ be the Dedekind zeta-function of the field $E$. Then for $\Re s > 1$, we have

\[ \zeta_E(s) = \sum_{\mathfrak{a}} (N\mathfrak{a})^{-s}, \]

where the sum is extended over all integral ideals $\mathfrak{a}$ of the field $E$, and $N\mathfrak{a}$ is the norm of $\mathfrak{a}$. We can rewrite it as

\[ \zeta_E(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \left(1 + \frac{a_p}{p^s} + \frac{a_{p^2}}{p^{2s}} + \cdots \right), \]

where $a_n$ denotes the number of integral ideals in $E$ with norm $n$. From Lemma 9 in [1], it is known that $a_n$ is a multiplicative function and satisfies

\begin{equation}
a_n \ll (\tau(n))^{d-1},
\end{equation}

where $\tau(n)$ is the divisor function, and $d$ is the degree of the polynomial $f$. In addition, from page 57 in [1] we learn that

\begin{equation}
\zeta_E(s) U(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-a_p},
\end{equation}

where $U(s)$ is an infinite product over primes, which is absolutely and uniformly convergent for $\Re s > 1/2$. From (2.1), (2.3), (2.4), and (2.5), we
conclude that for \( \text{Re} \, s > 1 \),
\[
(2.6) \quad L(s) = \zeta_E(s) U(s) \prod_p \left( 1 + \frac{N_f(p)}{p^s} + \frac{N_f(p^2)}{p^{2s}} + \cdots \right) \left( 1 - \frac{1}{p^s} \right)^{a_p}.
\]

By Kummer’s Theorem on the decomposition of prime ideals in algebraic extensions (see e.g. Lemma 22 in Swinnerton-Dyer [11]), we learn that except for finitely many primes (in fact, if \( p \) does not divide the discriminant \( D \) of \( f(x) \) or the index \( [O_E : \mathbb{Z}[\alpha]] \)),
\[
(2.7) \quad a_p = N_f(p).
\]

In fact, the factorization of a prime \( p \) in the field \( E \) as
\[
(p) = pO_E = p_1 \cdots p_g,
\]
where \( N_{p_j} = p_j^{f_j} \) (\( 1 \leq j \leq g \)) corresponds to the factorization
\[
f(x) \equiv f_1(x) \cdots f_g(x) \pmod{p},
\]
where \( f_j(x) \) (\( 1 \leq j \leq g \)) are irreducible polynomials over \( \mathbb{Z}_p \), of degree \( f_j \).

Therefore the number of integral ideals with norm \( p \) corresponds to the number of linear polynomials among \( f_j(x) \). Obviously the latter number equals \( N_f(p) \). Therefore we have the identity (2.7).

We define
\[
S = \{ p : p \mid D \text{ or } p \mid [O_E : \mathbb{Z}[\alpha]] \}.
\]

Then from (2.6) and (2.7), we conclude that for \( \text{Re} \, s > 1 \),
\[
(2.8) \quad L(s) = \zeta_E(s) U(s) \prod_{p \in S} \left( 1 + \frac{N_f(p)}{p^s} + \frac{N_f(p^2)}{p^{2s}} + \cdots \right) \left( 1 - \frac{1}{p^s} \right)^{a_p}
\]
\[
\times \prod_{p \notin S} \left( 1 + \frac{N_f(p^2) - a_p^2/2 - a_p/2}{p^{2s}} + \cdots \right)
\]
\[
:= \zeta_E(s) U(s) \prod_{p \in S} \times \prod_{p \notin S}
\]
\[
:= \zeta_E(s) A(s).
\]

From (2.1), (2.4), and the finiteness of the set \( S \), we learn that the product \( \prod_{p \in S} \) is absolutely convergent for \( \text{Re} \, s > 0 \), and the product \( \prod_{p \notin S} \) is absolutely convergent for \( \text{Re} \, s > 1/2 \). Then \( A(s) = U(s) \prod_{p \in S} \times \prod_{p \notin S} \) is absolutely convergent for \( \text{Re} \, s > 1/2 \), and uniformly convergent for \( \text{Re} \, s \geq 1/2 + \varepsilon \) with any \( \varepsilon > 0 \), and hence holomorphic for \( \text{Re} \, s > 1/2 \). Therefore \( L(s) = \zeta_E(s) A(s) \) has a meromorphic continuation to the half-plane \( \text{Re} \, s > 1/2 \). Since \( \zeta_E(s) \) only has a simple pole at \( s = 1 \) in this region, so does \( L(s) \).
3. Proof of Theorem 1.1. In this section $L$ is the splitting field of $f$ over $\mathbb{Q}$ with the Abelian Galois group $G = \text{Gal}(L/\mathbb{Q})$. Then the splitting field $L$ coincides with the field $E = \mathbb{Q}(\alpha)$.

The Kronecker–Weber Theorem asserts that every Abelian extension of $\mathbb{Q}$ is cyclotomic (see e.g. Theorem 44 in Swinnerton-Dyer [11]). We let $\mathbb{Q}(\zeta_m)$ with $\zeta_m = e^{2\pi i/m}$ be the least cyclotomic field which contains the Abelian field $L$. Then we call $m$ the conductor of the Abelian field $L$. We have $\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^*$, and so $H = \text{Gal}(\mathbb{Q}(\zeta_m)/L)$ can be regarded as a subgroup of $(\mathbb{Z}/m\mathbb{Z})^*$. The characters of the finite Abelian group $\text{Gal}(L/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^*$ are also called the characters of the field $L$. We denote the character group of $L$ by $\hat{L}$. Therefore $\hat{L}$ consists of the Dirichlet characters modulo $m$ that are trivial on $H$.

As a simple corollary of Abelian class field theory we can write $\zeta_L(s)$ as a product of the Riemann zeta-function and Dirichlet $L$-functions. More precisely, we have

$$\zeta_L(s) = \prod_{\chi \in \hat{L}} L(s, \chi^*) = \zeta(s) \prod_{\chi \in \hat{L} \setminus \chi = \chi_0} L(s, \chi^*),$$

where $\chi^*$ is a primitive character modulo $m'$ with $m' | m$, which induces $\chi \mod m$. For simplicity, we shall write

$$\zeta_L(s) = \zeta(s) \prod_{j=1}^{d-1} L(s, \chi_j),$$

where $L(s, \chi_j)$ are primitive Dirichlet $L$-functions.

From (2.8) and (3.1), we have

$$L(s) = \zeta_L(s) A(s) = \zeta(s) \prod_{j=1}^{d-1} L(s, \chi_j) A(s),$$

which admits a meromorphic continuation to the half-plane $\text{Re } s > 1/2$, and only has a simple pole at $s = 1$ in this region. Here $A(s)$ is absolutely and uniformly convergent for $\text{Re } s \geq 1/2 + \varepsilon$ with any $\varepsilon > 0$.

Now we begin the proof. First we assume that $4 \leq d \leq 11$. By (2.1), (2.2) and Perron's formula (see Proposition 5.54 in [7]), we have

$$\sum_{n \leq x} N_f(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} L(s) \frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right),$$

where $b = 1 + \varepsilon$ and $1 \leq T \leq x$ is a parameter to be chosen later.

Next we move the integration to the parallel segment with $\text{Re } s = 1/2 + \varepsilon$. By Cauchy's residue theorem, we have
\[
\sum_{n \leq x} N_f(n) = \frac{1}{2\pi i} \left\{ \int_{1/2+\varepsilon-iT}^{1/2+\varepsilon+iT} + \int_{1/2+\varepsilon+iT}^{1/2+\varepsilon-iT} + \int_{b-iT}^{b+iT} \right\} L(s) \frac{x^s}{s} \, ds
\]

\[+ \text{Res}_{s=1} L(s) x + O(x^{1+\varepsilon}/T)\]

\[=: I_1 + I_2 + I_3 + C(f) x + O(x^{1+\varepsilon}/T),\]

where \(C(f) = \text{Res}_{s=1} L(s)\).

It is well known that
\[\zeta(1/2 + it) \ll (1 + |t|)^{1/6} \log(|t| + 1)\]

and
\[L(1/2 + it, \chi) \ll (1 + |t|)^{1/6} \log(|t| + 1)\]

(see e.g. Theorems 24.1.1 and 24.2.1 in Pan and Pan [10]). Then by the Phragmén–Lindelöf principle for a strip (see e.g. Theorem 5.53 in Iwaniec and Kowalski [7]), we deduce that for \(1/2 \leq \sigma \leq 1 + \varepsilon\),

\[\zeta(\sigma + it) \ll (1 + |t|)^{(1-\sigma)/3+\varepsilon}\]

and
\[L(\sigma + it, \chi) \ll (1 + |t|)^{(1-\sigma)/3+\varepsilon},\]

where we have used
\[\zeta(1 + \varepsilon + it) \ll 1 \quad \text{and} \quad L(1 + \varepsilon + it, \chi) \ll 1.\]

Hence we have
\[\zeta(1/2 + \varepsilon + it) \ll (1 + |t|)^{1/6+\varepsilon}, \quad L(1/2 + \varepsilon + it, \chi) \ll (1 + |t|)^{1/6+\varepsilon}.\]

For \(I_1\), by (2.8) or (3.2) we have
\[I_1 \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \int_{1/2+\varepsilon}^{T} |L(1/2 + \varepsilon + it)| t^{-1} \, dt\]

\[\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \int_{1/2+\varepsilon}^{T} |\zeta_L(1/2 + \varepsilon + it) A(1/2 + \varepsilon + it)| t^{-1} \, dt\]

\[\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \int_{1/2+\varepsilon}^{T} |\zeta_L(1/2 + \varepsilon + it)| t^{-1} \, dt,\]

where we have used that \(A(s)\) is absolutely convergent in the region \(\text{Re} \, s \geq 1/2 + \varepsilon\) and is \(O(1)\) there.

By (3.1) and (3.6), we have
\[I_1 \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \int_{1/2+\varepsilon}^{T} |\zeta(1/2 + \varepsilon + it) \prod_{j=1}^{3} L(1/2 + \varepsilon + it, \chi_j)\]

\[\times \prod_{j=4}^{d-1} L(1/2 + \varepsilon + it, \chi_j)| t^{-1} \, dt\]

\[\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \int_{1/2+\varepsilon}^{T} |\zeta(1/2 + \varepsilon + it) \prod_{j=1}^{3} L(1/2 + \varepsilon + it, \chi_j)| t^{(d-4)/6-1} \, dt.\]
Then by Hölder’s inequality, we have

\begin{equation}
I_1 \ll x^{1/2+\varepsilon} \log T \max_{T_1 \leq T} \left\{ T_1^{(d-4)/6-1} \left( \int_{T_1/2}^{T_1} |\zeta(1/2 + \varepsilon + it)|^4 \, dt \right)^{1/4} \times \prod_{j=1}^{3} \left( \int_{T_1/2}^{T_1} |L(1/2 + \varepsilon + it, \chi_j)|^4 \, dt \right)^{1/4} \right\} + x^{1/2+\varepsilon} \ll x^{1/2+\varepsilon} T^{(d-4)/6+\varepsilon} + x^{1/2+\varepsilon},
\end{equation}

where we have used

\begin{equation}
\int_{T_1/2}^{T_1} |\zeta(1/2 + \varepsilon + it)|^4 \, dt \ll T_1^{1+\varepsilon}
\end{equation}

and

\begin{equation}
\int_{T_1/2}^{T_1} |L(1/2 + \varepsilon + it, \chi_j)|^4 \, dt \ll T_1^{1+\varepsilon}.
\end{equation}

These results can be established by using Gabriel’s convexity theorem (see e.g. Lemma 8.3 in Ivić [6]), and the following two classical results (see e.g. Theorems 29.3.1 and 29.3.4 in Pan and Pan [10]):

\begin{equation}
\int_{T_1/2}^{T_1} |\zeta(1/2 + it)|^4 \, dt \ll T_1 (\log T_1)^4
\end{equation}

and

\begin{equation}
\int_{T_1/2}^{T_1} |L(1/2 + it, \chi_j)|^4 \, dt \ll T_1 (\log T_1)^4.
\end{equation}

By (3.1) and (3.5), we conclude that for $1/2 \leq \sigma \leq 1 + \varepsilon$,

\[ \zeta_L(\sigma + it) \ll (1 + |t|)^{\frac{d}{3}(1-\sigma)+\varepsilon}. \]

Therefore for the integrals over the horizontal segments we have

\begin{equation}
I_2 + I_3 \ll \int_{1/2+\varepsilon}^{b} x^\sigma |\zeta_L(\sigma + iT)| T^{-1} \, d\sigma
\end{equation}

\begin{equation}
\ll \max_{1/2+\varepsilon \leq \sigma \leq b} x^\sigma T^{\frac{d}{3}(1-\sigma)+\varepsilon} T^{-1} = \max_{1/2+\varepsilon \leq \sigma \leq b} \left( \frac{x}{T^{d/3}} \right)^\sigma T^{-d/3+1+\varepsilon}
\end{equation}

\begin{equation}
\ll x^{1+\varepsilon}/T + x^{1/2+\varepsilon} T^{d/3+1+\varepsilon}.
\end{equation}
From (3.4), (3.7) and (3.8), we have

\[ \sum_{n \leq x} N_f(n) = C(f)x + O(x^{1+\varepsilon}/T) + O(x^{1/2+\varepsilon}T^{(d-4)/6+\varepsilon}). \]  

(3.9)

On taking \( T = x^{3/(d+2)} \) in (3.9), we have

\[ \sum_{n \leq x} N_f(n) = C(f)x + O(x^{1-3/(d+2)+\varepsilon}). \]

Now we consider the case \( d \geq 12 \). From the context we only need to estimate the integral \( I_1 \). We have

\[ I_1 \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \left( \int_1^T |L(1/2 + \varepsilon + it)|t^{-1} dt \right) \]

\[ \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \left( \int_1^T |\zeta(1/2 + \varepsilon + it)|t^{-1} dt \right) \]

\[ \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \left( \int_1^T |\zeta(1/2 + \varepsilon + it)| \prod_{j=1}^{11} L(1/2 + \varepsilon + it, \chi_j) \right. \]

\[ \times \left. \prod_{j=12}^{d-1} L(1/2 + \varepsilon + it, \chi_j) \right| t^{-1} dt \]

\[ \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \left( \int_1^T |\zeta(1/2 + \varepsilon + it)| \prod_{j=1}^{11} L(1/2 + \varepsilon + it, \chi_j) \right| t^{(d-12)/6-1} dt. \]

Then by Hölder’s inequality, we have

\[ I_1 \ll x^{1/2+\varepsilon} \log T \max_{T_1 \leq T} \left\{ \int_{T_1/2}^{T_1} |\zeta(1/2 + \varepsilon + it)|^{12} dt \right\}^{1/12} \]

\[ \times \prod_{j=1}^{11} \left( \int_{T_1/2}^{T_1} |L(1/2 + \varepsilon + it, \chi_j)|^{12} dt \right)^{1/12} \}

\[ + x^{1/2+\varepsilon} \]

\[ \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} T^{d/6-1+\varepsilon}, \]

where we have used

\[ \int_{T_1/2}^{T_1} |\zeta(1/2 + \varepsilon + it)|^{12} dt \ll T_1^{2+\varepsilon} \]

and

\[ \int_{T_1/2}^{T_1} |L(1/2 + \varepsilon + it, \chi)|^{12} dt \ll T_1^{2+\varepsilon}. \]
These results can be deduced from Gabriel’s convexity theorem (see e.g. Lemma 8.3 in Ivić [6]), and the results of Heath-Brown [4] and Meurman [9] respectively, which state that
\[ \int_{T_1/2}^{T_1} |\zeta(1/2 + it)|^{12} \, dt \ll T_1^2 (\log T_1)^{17} \]
and
\[ \int_{T_1/2}^{T_1} |L(1/2 + it, \chi)|^{12} \, dt \ll T_1^{2+\varepsilon}. \]

Then on taking \( T = x^{3/d} \), we have
\[ \sum_{n \leq x} N_f(n) = C(f)x + O(x^{1-3/d+\varepsilon}). \]

Finally, we consider the cases \( d = 2, 3 \). For \( d = 2 \), we have
\[
I_1 \ll x^{1/2+\varepsilon} \log T \max_{T_1 \leq T} \left\{ T_1^{-1} \left( \int_{T_1/2}^{T_1} |\zeta(1/2 + \varepsilon + it)|^2 \, dt \right)^{1/2} \right. \\
\left. \times \left( \int_{T_1/2}^{T_1} |L(1/2 + \varepsilon + it, \chi)|^2 \, dt \right)^{1/2} \right\} \ll x^{1/2+\varepsilon},
\]
where we have used
\[ \int_{T_1/2}^{T_1} |\zeta(1/2 + \varepsilon + it)|^2 \, dt \ll T_1^{1+\varepsilon} \]
and
\[ \int_{T_1/2}^{T_1} |L(1/2 + \varepsilon + it, \chi)|^2 \, dt \ll T_1^{1+\varepsilon}. \]

These results can also be established by applying Gabriel’s convexity theorem (see e.g. Lemma 8.3 in Ivić [6]), and the following two classical results (see e.g. Theorems 25.2.1 and 25.3.1 in Pan and Pan [10]):
\[ \int_{T_1/2}^{T_1} |\zeta(1/2 + it)|^2 \, dt \ll T_1 \log T_1 \]
and
\[ \int_{T_1/2}^{T_1} |L(1/2 + it, \chi)|^2 \, dt \ll T_1 \log T_1. \]
Then on taking $T = x^{1/2}$, we have
\[
\sum_{n \leq x} N_f(n) = C(f)x + O(x^{1/2+\varepsilon}).
\]

For the case $d = 3$, we have
\[
I_1 \ll x^{1/2+\varepsilon} \log T \max_{T_1 \leq T} \left\{ T_1^{-1} \left( \int_{T_1/2}^{T_1} |\zeta(1/2 + \varepsilon + it)|^2 dt \right)^{1/2} \times \prod_{j=1}^{2} \left( \int_{T_1/2}^{T_1} |L(1/2 + \varepsilon + it, \chi_j)|^4 dt \right)^{1/4} \right\}
\ll x^{1/2+\varepsilon}.
\]

Then on taking $T = x^{1/2}$, we also have
\[
\sum_{n \leq x} N_f(n) = C(f)x + O(x^{1/2+\varepsilon}).
\]

4. Proof of Theorem 1.2. Recall that $L$ is the splitting field of $f$ over $\mathbb{Q}$ with Galois group $G = \text{Gal}(L/\mathbb{Q})$ and $E = \mathbb{Q}(\alpha)$, where $\alpha$ is a root of $f$. From our assumption, $G$ is not Abelian in this section.

By (2.1), (2.2), and Perron’s formula (see Proposition 5.54 in [7]), we have
\[
\sum_{n \leq x} N_f(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} L(s) \frac{x^s}{s} ds + O \left( \frac{x^{1+\varepsilon}}{T} \right),
\]
where $b = 1 + \varepsilon$ and $1 \leq T \leq x$ is a parameter to be chosen later.

Next we move the integration to the parallel segment with $\text{Re } s = 1/2+\varepsilon$. By Cauchy’s residue theorem, we have
\[
\sum_{n \leq x} N_f(n) = \frac{1}{2\pi i} \left\{ \int_{1/2+\varepsilon-iT}^{1/2+\varepsilon+iT} L(s) \frac{x^s}{s} ds + \int_{1/2+\varepsilon+iT}^{1/2+\varepsilon-iT} L(s) \frac{x^s}{s} ds + \int_{b-iT}^{b+iT} L(s) \frac{x^s}{s} ds \right\} + \text{Res}_{s=1} L(s)x + O \left( \frac{x^{1+\varepsilon}}{T} \right)
\]
\[:= J_1 + J_2 + J_3 + C(f)x + O \left( \frac{x^{1+\varepsilon}}{T} \right).\]

For $J_1$, by (2.8) we have
\[
J_1 \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \int_{1}^{T} |L(1/2 + \varepsilon + it)| t^{-1} dt
\]
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\[ \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \int_1^T |\zeta_E(1/2 + \varepsilon + it)A(1/2 + \varepsilon + it)| t^{-1} \, dt \]
\[ \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \int_1^T |\zeta_E(1/2 + \varepsilon + it)| t^{-1} \, dt, \]

where we have used that \( A(s) \) is absolutely convergent in the region \( \text{Re} \, s \geq 1/2 + \varepsilon \) and is \( O(1) \) there.

To go further, we cite a result of Heath-Brown [5] about the subconvexity bound for the Dedekind zeta-function on the critical line, which states that if \( E \) is an algebraic number field of degree \( d \), then

\[ \zeta_E(1/2 + it) \ll_E t^{d/6+\varepsilon} \quad (t \geq 1) \]

for any fixed \( \varepsilon > 0 \). Then by the Phragmén–Lindelöf principle for a strip (see e.g. Theorem 5.53 in Iwaniec and Kowalski [7]), we deduce that for \( 1/2 \leq \sigma \leq 1 + \varepsilon \),

\[ \zeta_E(\sigma + it) \ll (1 + |t|)^{d/2} \ll (1 + |t|)^{d/2 + \varepsilon} \]

where we have used the estimate \( \zeta_E(1 + \varepsilon) \ll 1 \).

By (4.3), we have

\[ J_1 \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \int_1^T |\zeta_E(1/2 + \varepsilon + it)| t^{-1} \, dt \]
\[ \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \int_1^T t^{d/6-1+\varepsilon} \, dt \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} T^{d/6+\varepsilon}. \]

For the integrals over the horizontal segments, by (4.3) we have

\[ J_2 + J_3 \ll \int_{1/2+\varepsilon}^b x^\sigma |\zeta_E(\sigma + iT)| T^{-1} \, d\sigma \]
\[ \ll \max_{1/2+\varepsilon \leq \sigma \leq b} x^\sigma T^{d/3} T^{d-1+\varepsilon} \]
\[ = \max_{1/2+\varepsilon \leq \sigma \leq b} \left( \frac{x}{T^{d/3}} \right)^\sigma T^{d/3-1+\varepsilon} \]
\[ \ll x^{1+\varepsilon}/T + x^{1/2+\varepsilon} T^{d/6-1+\varepsilon}. \]

From (4.2), (4.4) and (4.5), we have

\[ \sum_{n \leq x} N_f(n) = C(f)x + O(x^{1+\varepsilon}/T) + O(x^{1/2+\varepsilon} T^{d/6+\varepsilon}). \]
On taking $T = x^{3/(d+6)}$ in (4.6), we have

$$\sum_{n \leq x} N_f(n) = C(f)x + O(x^{1-3/(d+6)+\varepsilon}).$$

This completes the proof of Theorem 1.2.

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