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On Bialostocki's conjecture for zero-sum sequences

by

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1. Introduction. A finite sequence S of terms from an (additive) abelian group is said to be a *zero-sum sequence* if the sum of the terms of S is zero. In 1961 P. Erdős, A. Ginzburg and A. Ziv [3] proved that any sequence of $2n-1$ terms from an abelian group of order n contains an n-term zero-sum subsequence. This celebrated EGZ theorem is an important result in combinatorial number theory and it has many different generalizations [5–8] including Sun's recent extension involving covering systems.

The following theorem is called the weighted EGZ theorem. It was conjectured by Y. Caro [2] and proved by D. J. Grynkiewicz [4].

THEOREM 1.1 (Weighted EGZ Theorem). *Let n be a positive integer and let $w_1,\dots,w_n \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ with $\sum_{k=1}^n w_k = 0$. If $a_1,\dots,a_{2n-1}$ is a sequence of elements from $\mathbb{Z}_n$, then $\sum_{k=1}^n w_k a_{j_k} = 0$ for some distinct $j_1,\dots,j_n \in \{1,\dots,2n-1\}$.*

Recently Bialostocki raised the following challenging conjecture.

CONJECTURE 1.1 (Bialostocki [1, Conjecture 14]). *Let n be a positive even integer. Suppose that $a_1,\dots,a_n$ and $w_1,\dots,w_n$ are zero-sum sequences with terms from $\mathbb{Z}_n$. Then there exists a permutation $\sigma \in S_n$ such that $\sum_{k=1}^n w_k a_{\sigma(k)} = 0$, where S_n denotes the symmetric group of all permutations on $\{1,\dots,n\}$.*

The conjecture has been verified for $n = 2,4,6,8$. It fails for $n = 3,5,7,\dots$. For example, $\{a_1,a_2,a_3\} = \{w_1,w_2,w_3\} = \mathbb{Z}_3$ gives a counterexample for $n=3$.

In this paper we mainly establish the following result.

THEOREM 1.2. *Let n be a positive even integer, and let $a_1,\dots,a_n \in \mathbb{Z}$ with $\sum_{k=1}^n a_k \equiv 0 \pmod{n}$. Then there exists a permutation $\sigma \in S_n$ such*

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that $\sum_{k=1}^{n} ka_{\sigma(k)} \equiv 0 \pmod{n/2}$. *Consequently, if* $w_1, \ldots, w_n \in \mathbb{Z}$ *form an arithmetic progression with even common difference, then* $\sum_{k=1}^{n} w_k a_{\sigma(k)} \equiv 0 \pmod{n}$ *for some* $\sigma \in S_n$.

We are going to present two lemmas in the next section and then give our proof of Theorem 1.2 in Section 3.

2. Two lemmas

LEMMA 2.1. *Let* $n = mq$ *with* $m, q \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ *and* $m \geq 2$. *Let* $d \in \mathbb{Z}^+$ *be a divisor of* q, *and let* $a_1, \ldots, a_n \in \mathbb{Z}$. *Then there is a partition* $I_1, \ldots, I_m$ *of* $[1, n] = \{1, \ldots, n\}$ *such that for each* $s = 1, \ldots, m$ *we have* $|I_s| = q$ *and*

$$d \Big| \sum_{i \in I_s} a_i \;\Rightarrow\; |\{a_i \bmod d : i \in I_s\}| = 1.$$

Proof. By induction on m, it suffices to show that there exists an $I \subseteq [1, n]$ with $|I| = q$ such that for each $J \in \{I, [1, n] \setminus I\}$ we have $|\{a_j \bmod d : j \in J\}| = 1$ or $\sum_{j \in J} a_j \not\equiv 0 \pmod{d}$. To achieve this we distinguish three cases.

CASE 1: $|\{a_i \bmod d : i \in [1, n]\}| = 1$. In this case, $I = [1, q]$ works.

CASE 2: $|\{a_i \bmod d : i \in [1, n]\}| = 2$. Suppose that

$$\{a_i \bmod d : i \in [1, n]\} = \{r \bmod d,\ r' \bmod d\},$$

where $r, r' \in [0, d-1]$, $r \not\equiv r' \pmod{d}$, and $a_i \equiv r \pmod{d}$ for at least $n/2$ values of $i \in [1, n]$. Choose $I_0 \subseteq \{i \in [1, n] : a_i \equiv r \pmod{d}\}$ with $|I_0| = q \leq n/2$. Let $i_0 \in I_0$ and $j_0 \in \bar{I}_0 = [1, n] \setminus I_0$ with $a_{j_0} \equiv r' \pmod{d}$. When $\sum_{j \in \bar{I}_0} a_j \equiv 0 \pmod{d}$, we have both

$$\sum_{i \in (I_0 \setminus \{i_0\}) \cup \{j_0\}} a_i \equiv 0 - r + r' \not\equiv 0 \pmod{d}$$

and

$$\sum_{j \in (\bar{I}_0 \setminus \{j_0\}) \cup \{i_0\}} a_j \equiv 0 - r' + r \not\equiv 0 \pmod{d}.$$

Thus, there always exists an $I \subseteq [1, n]$ with $|I| = q$ such that

$$|\{a_i \bmod d : i \in I\}| = 1 \quad \text{or} \quad \sum_{i \in I} a_i \not\equiv 0 \pmod{d},$$

and also $\sum_{j \in \bar{I}} a_j \not\equiv 0 \pmod{d}$.

CASE 3: $|\{a_i \bmod d : i \in [1, n]\}| > 2$. Note that $q \geq d > 2$ in this case. As $n \geq 2q \geq 2q - 1$, by the EGZ theorem there is an $I_0 \subseteq [1, n]$ with $|I_0| = q$ such that $\sum_{i \in I_0} a_i \equiv 0 \pmod{q}$. For $\bar{I}_0 = [1, n] \setminus I_0$, we clearly have $|\bar{I}_0| = (m-1)q$. Set $b = a_1 + \cdots + a_n \equiv \sum_{j \in \bar{I}_0} a_j \pmod{q}$.

Suppose that $a_j - a_i \equiv 0$ or $b \pmod d$ for any $i \in I_0$ and $j \in \bar{I}_0$. Then

$$|\{a_i \bmod d : i \in I_0\}| \le 2 \quad \text{and} \quad |\{a_j \bmod d : j \in \bar{I}_0\}| \le 2.$$

If $i_1, i_2 \in I_0$, $j \in \bar{I}_0$ and $a_j \not\equiv a_{i_1}, a_{i_2} \pmod d$, then $a_j - a_{i_1} \equiv b \equiv a_j - a_{i_2} \pmod d$ and hence $a_{i_1} \equiv a_{i_2} \pmod d$. So, if $|\{a_i \bmod d : i \in I_0\}| = 2$ then $\{a_j \bmod d : j \in \bar{I}_0\} \subseteq \{a_i \bmod d : i \in I_0\}$, which contradicts $|\{a_i \bmod d : i \in I_0\}| > 2$. Similarly, if $|\{a_j \bmod d : j \in \bar{I}_0\}| = 2$ then we also have a contradiction. When

$$|\{a_i \bmod d : i \in I_0\}| = |\{a_j \bmod d : j \in \bar{I}_0\}| = 1,$$

we cannot have $|\{a_i \bmod d : i \in [1, n]\}| > 2$.

By the above, there are $i_0 \in I_0$ and $j_0 \in \bar{I}_0$ such that

$$a_{j_0} - a_{i_0} \not\equiv 0, b \pmod d.$$

Set

$$I = (I_0 \setminus \{i_0\}) \cup \{j_0\} \quad \text{and} \quad \bar{I} = [1, n] \setminus I = (\bar{I}_0 \setminus \{j_0\}) \cup \{i_0\}.$$

Then

$$\sum_{i \in I} a_i = \sum_{i \in I_0} a_i - a_{i_0} + a_{j_0} = 0 - a_{i_0} + a_{j_0} \not\equiv 0 \pmod d$$

and

$$\sum_{j \in \bar{I}} a_j = \sum_{j \in \bar{I}_0} a_j - a_{j_0} + a_{i_0} \equiv b + a_{i_0} - a_{j_0} \not\equiv 0 \pmod d.$$

Note that $|I| = q$ and $|\bar{I}| = (m-1)q$.

Combining the above and using an induction argument, we see that the desired result holds for any $m = 2, 3, 4, \ldots$. ■

LEMMA 2.2. *Let $a_1, \ldots, a_n \in \mathbb{Z}$ with $n = p^\alpha$, where p is an odd prime and α is a positive integer. If $\sum_{k=1}^n a_k \not\equiv 0 \pmod p$ or $|\{a_k \bmod p : k \in [1, n]\}| = 1$, then there exists a permutation $\sigma \in S_n$ such that $\sum_{k=1}^n k a_{\sigma(k)} \equiv 0 \pmod n$.*

Proof. If $a := \sum_{k=1}^n a_k \not\equiv 0 \pmod p$, then there is an $l \in [1, n]$ such that $al + \sum_{k=1}^n k a_k \equiv 0 \pmod{p^\alpha}$ and hence

$$\sum_{k=1}^n k a_{\sigma(k)} \equiv \sum_{k=1}^n (k+l) a_k \equiv \sum_{k=1}^n k a_k + la \equiv 0 \pmod{p^\alpha},$$

where $\sigma(k)$ is the least positive residue of $k - l$ modulo n.

In the case $a_1 \equiv \cdots \equiv a_n \pmod p$, it is clear that

$$\sum_{k=1}^p k a_k \equiv a_1 \sum_{k=1}^p k = a_1 p \frac{p+1}{2} \equiv 0 \pmod p.$$

Thus we have the desired result for $\alpha = 1$.

Now let $\alpha > 1$ and assume the desired result with α replaced by $\alpha - 1$. As mentioned above, the desired result holds if $\sum_{k=1}^n a_k \not\equiv 0 \pmod p$. Suppose that $a_1 \equiv \cdots \equiv a_n \pmod p$ and set $b_k = (a_k - a_1)/p$ for $k = 1, \ldots, n$. In light of Lemma 2.1, there exists a partition $I_1 \cup \cdots \cup I_p$ of $[1, n]$ with $|I_1| = \cdots = |I_p| = p^{\alpha-1}$ such that for any $s = 1, \ldots, p$ either $|\{b_k \bmod p : k \in I_s\}| = 1$ or $\sum_{k\in I_s} b_k \not\equiv 0 \pmod p$. By the induction hypothesis, there are one-to-one mappings $\sigma_s : [1, p^{\alpha-1}] \to I_s$ $(s = 1, \ldots, p)$ such that

$$\sum_{k=1}^{p^{\alpha-1}} kb_{\sigma_s(k)} \equiv 0 \pmod{p^{\alpha-1}} \quad \text{for all } s = 1, \ldots, p.$$

For $s \in [1, p]$ and $t \in [1, p^{\alpha-1}]$ define $\sigma(p^{\alpha-1}(s-1)+t) = \sigma_s(t)$. Then $\sigma \in S_n$ and

$$\begin{aligned}\sum_{k=1}^{n} ka_{\sigma(k)} &= \sum_{k=1}^{n} ka_1 + p\sum_{k=1}^{n} kb_{\sigma(k)}\\ &= \frac{p^\alpha(p^\alpha+1)}{2}a_1 + p\sum_{s=1}^{p}\sum_{t=1}^{p^{\alpha-1}}(p^{\alpha-1}(s-1)+t)b_{\sigma_s(t)}\\ &\equiv p\sum_{s=1}^{p}\sum_{t=1}^{p^{\alpha-1}} tb_{\sigma_s(t)} \equiv 0 \pmod{p^\alpha}.\end{aligned}$$

This concludes the induction step and we are done. ■

3. Proof of Theorem 1.2. We use induction on $\nu(n)$, the total number of prime divisors of n.

In the case $\nu(n) = 1$, clearly $n = 2$ and the desired result holds trivially.

Now let $\nu(n) > 1$ and assume the desired result for those even positive integers with fewer than $\nu(n)$ prime divisors.

Case 1: $n = 2^\alpha$ *for some* $\alpha \geq 2$. By the EGZ theorem, there is an $I \subseteq [1, n]$ with $|I| = n/2 = 2^{\alpha-1}$ such that $\sum_{i\in I} a_i \equiv 0 \pmod{2^{\alpha-1}}$. Note that for $\bar{I} = [1, n] \setminus I$ we also have

$$\sum_{j\in\bar{I}} a_j = \sum_{k=1}^{n} a_k - \sum_{i\in I} a_i \equiv 0 \pmod{2^{\alpha-1}}.$$

By the induction hypothesis, for some one-to-one mappings $\sigma_0 : [1, n/2] \to I$ and $\sigma_1 : [1, n/2] \to \bar{I}$ we have

$$2\sum_{k=1}^{2^{\alpha-1}} ka_{\sigma_0(k)} \equiv 2\sum_{k=1}^{2^{\alpha-1}} ka_{\sigma_1(k)} \equiv 0 \pmod{2^{\alpha-1}}.$$

Observe that

$$\sum_{k=1}^{2^{\alpha-1}}(2k-1)a_{\sigma_1(k)} \equiv 2\sum_{k=1}^{2^{\alpha-1}} ka_{\sigma_1(k)} - \sum_{j\in\bar{I}} a_j \equiv 0 \pmod{2^{\alpha-1}}.$$

For $k \in [1, n/2]$ and $r \in [0,1]$ define $\sigma(2k-r) = \sigma_r(k)$. Then $\sigma \in S_n$ and

$$\sum_{j=1}^{n} ja_{\sigma(j)} = 2\sum_{k=1}^{2^{\alpha-1}} ka_{\sigma_0(k)} + \sum_{k=1}^{2^{\alpha-1}}(2k-1)a_{\sigma_1(k)} \equiv 0 \pmod{2^{\alpha-1}}.$$

Thus we have the desired result for $n = 2^\alpha$.

CASE 2: *n has an odd prime divisor p.* Write $n = p^\alpha m$ with $\alpha, m > 0$ and $p \nmid m$. In view of Lemma 2.1 there is a partition $I_1 \cup \cdots \cup I_m$ of $[1, n]$ with $|I_1| = \cdots = |I_m| = p^\alpha$ such that for each $s = 1, \ldots, m$ either $|\{a_i \bmod p : i \in I_s\}| = 1$ or $\sum_{i\in I_s} a_i \not\equiv 0 \pmod{p}$. Combining this with Lemma 2.2, we see that for each $s \in [1, m]$ there is a one-to-one mapping $\sigma_s : [1, p^\alpha] \to I_s$ such that $\sum_{t=1}^{p^\alpha} ta_{\sigma_s(t)} \equiv 0 \pmod{p^\alpha}$.

Set $b_s = \sum_{k\in I_s} a_k$ for $s = 1, \ldots, m$. Then

$$\sum_{s=1}^{m} b_s = \sum_{k\in I_1\cup\cdots\cup I_m} a_k = \sum_{k=1}^{n} a_k \equiv 0 \pmod{m}.$$

As $2 \mid m$ and $\nu(m) < \nu(n)$, by the induction hypothesis, for some $\tau \in S_m$ we have

$$2\sum_{s=1}^{m} sb_{\tau(s)} \equiv 0 \pmod{m}$$

and hence

$$2\sum_{s=1}^{m}\sum_{t=1}^{p^\alpha} sa_{\sigma_{\tau(s)}(t)} = 2\sum_{s=1}^{m} sb_{\tau(s)} \equiv 0 \pmod{m}.$$

Note also that

$$\sum_{s=1}^{m}\sum_{t=1}^{p^\alpha} ta_{\sigma_{\tau(s)}(t)} = \sum_{s=1}^{m}\sum_{t=1}^{p^\alpha} ta_{\sigma_s(t)} \equiv 0 \pmod{p^\alpha}.$$

Therefore

$$2\sum_{s=1}^{m}\sum_{t=1}^{p^\alpha}(p^\alpha s + mt)a_{\sigma_{\tau(s)}(t)} \equiv 0 \pmod{p^\alpha m}.$$

As p^α is relatively prime to m,

$$\{p^\alpha s + mt : s \in [1, m] \text{ and } t \in [1, p^\alpha]\}$$

is a complete system of residues modulo $n = p^\alpha m$. For any $k \in [1, n]$, there are unique $s \in [1, m]$ and $t \in [1, p^\alpha]$ such that $k \equiv p^\alpha s + mt \pmod{n}$, and we define $\sigma(k) = \sigma_{\tau(s)}(t)$. Then $\sigma \in S_n$ and also

$$2\sum_{k=1}^{n} ka_{\sigma(k)} \equiv 0 \pmod{n}.$$

This concludes the induction step.

In view of the above, we have completed the proof of Theorem 1.2. ∎

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