Sign changes in $\pi_{q,a}(x) - \pi_{q,b}(x)$

by

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1. Introduction and summary. Let

$$\text{li}(x) = \lim_{\varepsilon \to 0^+} \int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^x \frac{dt}{\log t}$$

and let $\pi(x)$ denote the number of primes $\leq x$. Also, $\pi_{q,a}(x)$ denotes the number of primes $\leq x$ lying in the progression $a \mod q$. In 1792, Gauss observed that $\pi(x) < \text{li}(x)$ for $x < 3000000$ (see e.g. [E]) and the question of whether or not there are any sign changes of $\pi(x) - \text{li}(x)$ remained open until 1914 when J. E. Littlewood [Li] showed that there exists a positive constant $k$ such that infinitely often both $\pi(x) - \text{li}(x)$ and $\text{li}(x) - \pi(x)$ exceed

$$k \frac{x^{1/2} \log \log \log x}{\log x}.$$

Sign changes are, nonetheless, quite rare and it was not until 1955 that any upper bound was obtained for the first sign change. The upper bound of

$$10^{10^{10^{34}}}$$

was obtained by Skewes [Sk1] on the assumption of the Riemann Hypothesis, and in 1955 [Sk2] he provided the first unconditional upper bound for the first sign change, namely

$$10^{10^{10^{3}}}.$$

In 1966, Lehman [Leh] developed a new method based on an explicit formula for $\text{li}(x) - \pi(x)$ averaged by a Gaussian kernel and knowledge of zeros of the Riemann zeta function $\zeta(s)$ in the region $|\Re s| \leq 12000$. Lehman’s method drastically improves the upper bound for the first sign change. In particular, he proved that it must occur before $1.5926 \cdot 10^{1165}$ and his method was used by te Riele [tR] to lower the bound to $6.6658 \cdot 10^{370}$ and by Bays and Hudson [BH5] to lower it further to $1.39822 \cdot 10^{316}$.

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In this paper, we generalize Lehman’s method, enabling one to compare the number of primes $\leq x$ in any two arithmetic progressions $qn + a$ and $qn + b$. For reasons given in, e.g., [H2], [RS], negative values of $\pi_{q,b}(x) - \pi_{q,a}(x)$ may be relatively infrequent if $b$ is a quadratic non-residue of $q$ and $a$ a quadratic residue. This phenomenon, first noted by Chebyshev in 1853 for the case $q = 4$, is known as “Chebyshev’s bias”. It is quite pronounced when $q | 24$, $1 < b < q$, $(b,q) = 1$ and $a = 1$, and these cases have been studied extensively from a numerical point of view ([BH1]–[BH4], [Lee], [Sh]) and from a theoretical point of view ([BFHR], [H2], [K1]–[K3], [KT1], [KT2], [Li], [RS]). For example, Bays and Hudson [BH2] showed in 1978 that the smallest $x$ with $\pi_{3,2}(x) < \pi_{3,1}(x)$ is $x = 608981813029$.

Section 2 is devoted to the development of the analog of Lehman’s theorem. Our bounds are considerably sharper than in [Leh], but as a consequence the bounds are a bit more complex. In Section 3 we apply the theorem for $q | 24$ and $a = 1$. Our present knowledge of the zeros of these $L$-functions is due to Rumely ([Ru1], [Ru2]) and this is insufficient to obtain bounds which are anywhere near “best possible”. The bounds, however, are in most cases adequate to localize negative values of $\pi_{q,b}(x) - \pi_{q,1}(x)$.

2. A generalization of Lehman’s theorem. For non-real numbers $z$, define

\begin{equation}
\text{li}(e^z) := e^z \int_0^\infty \frac{e^{-t}}{z - t} \, dt
\end{equation}

and let

\begin{equation}
K(s; \alpha) = \sqrt{\frac{\alpha}{2\pi}} e^{-\alpha s^2/2}.
\end{equation}

Also, for $\varrho = \beta + i\gamma$, $0 < \beta < 1$, define

\[ J(\varrho) := \int_{\varrho - \eta}^{\varrho + \eta} K(u - \varrho; \alpha)u e^{-u/2} \text{li}(e^{\varrho u}) \, du. \]

**Lemma 2.1.** If $\varrho = 1/2 + i\gamma$ with $\gamma \neq 0$, $u \geq 1$ and $J \geq 1$, then

\[ \left| \frac{\text{li}(e^{\varrho u})}{e^{\varrho u}} - \sum_{j=1}^{J} \frac{(j-1)!}{(\varrho u)^j} \right| \leq \frac{J!}{u^{J+1}} \min \left( \frac{1}{|\gamma|^{J+1}}, \frac{2^{1.5J+2}}{(1+2|\gamma|)^{J+1}} \right). \]

**Proof.** By (2.1) and repeated integration by parts, we have for non-real $z$ the identity

\begin{equation}
e^{-z} \text{li}(e^z) - \sum_{j=1}^{J} \frac{(j-1)!}{z^j} = J! \int_0^\infty \frac{e^{-t}}{(z - t)^{J+1}} \, dt.\end{equation}
Now put $z = qu$. Since $|qu - t| \geq u|\gamma|$, the last integral is $\leq (u|\gamma|)^{-J-1}$. If $|\gamma|$ is small, we can do better by deforming the contour. If $\gamma > 0$ let $C$ be the union of the straight line segments from 0 to $\frac{1}{2}(u - iw)$ to $u$ to $\infty$ and if $\gamma < 0$ let $C$ be the union of the line segments from 0 to $\frac{1}{2}(u + iw)$ to $u$ to $\infty$. For $t \in C$, we have

$$|qu - t| \geq \frac{(1 + 2|\gamma|)u}{2^{3/2}}.$$

Together with the bound

$$\int_C |e^{-t}| \, dt \leq \sqrt{2},$$

this proves the lemma. 

**Lemma 2.2 (McCurley).** Let $\chi$ be a Dirichlet character of conductor $k$ and denote by $N(T, \chi)$ the number of zeros of $L(s, \chi)$ lying in the region $s = \sigma + it$, $0 < \sigma < 1$, $|\gamma| \leq T$. Then

$$\left| N(T, \chi) - \frac{T}{\pi} \log \left( \frac{kT}{2\pi e} \right) \right| \leq C_2 \log(kT) + C_3,$$

where

$$C_2 = 0.9185, \quad C_3 = 5.512.$$

**Proof.** This is Theorem 2.1 of [M] with $\eta = 1/2$. 

**Corollary 2.3.** Suppose $g$ is a continuous, positive, decreasing function for $t \geq T = 2\pi e/k$, and suppose $T_2 \geq T_1 \geq T$. Let $\chi$ be a Dirichlet character of conductor $k$ and denote by $\gamma$ the imaginary part of a generic non-trivial zero of $L(s, \chi)$. Then

$$\left| \sum_{T_1 < |\gamma| \leq T_2} g(|\gamma|) - \frac{1}{\pi} \int_{T_1}^{T_2} g(t) \log \left( \frac{kt}{2\pi} \right) \, dt \right|$$

$$\leq 2g(T_1)(C_2 \log(kT_1) + C_3) + C_2 \int_{T_1}^{T_2} \frac{g(t)}{t} \, dt.$$ 

**Proof.** Lemma 2.2 and partial summation. 

**Corollary 2.4.** If $T \geq 150$, $n \geq 2$ and $\chi$ is a Dirichlet character of conductor $k \geq 3$, then

$$\sum_{|\gamma| > T} \gamma^{-n} < \frac{T^{1-n} \log(kT)}{3}.$$
Proof. Letting \( g(\gamma) = \gamma^{-n} \) in Corollary 2.3, we obtain
\[
\sum_{|\gamma| > T} \gamma^{-n} \leq T^{1-n} \left( \frac{\log \left( \frac{kT}{2\pi} \right)}{\pi(n-1)} + \frac{1}{\pi(n-1)^2} + \frac{2C_2 \log(kT) + 2C_3 + C_2/n}{T} \right)
\]
\[
\leq T^{1-n} \log(kT) \left( \frac{1}{\pi} + \frac{2C_2}{T} \right) + T^{1-n} \left( \frac{2C_3 + C_2/2}{T} - \frac{\log(2\pi)}{\pi} \right)
\]
\[
< \frac{1}{3} T^{1-n} \log(kT) .
\]

We also use the simple bound
\[
\int_{y}^{\infty} K(u; \alpha) \, du < \sqrt{\frac{\alpha}{2\pi}} \int_{y}^{\infty} \left( \frac{u}{y} \right) e^{-\alpha u^2/2} \, du = \frac{K(y; \alpha)}{\alpha y} \quad (y > 0).
\]

We now adopt a notational convention from [Leh]: The notation \( f = \vartheta(g) \) means \( |f| \leq |g| \).

**Lemma 2.5.** Suppose\n\[
\omega \geq 30, \quad 0 < \eta \leq \omega/30, \quad |\gamma| \leq \alpha \eta/2.
\]
If \( \varrho = 1/2 + i\gamma \), then
\[
J(\varrho) = e^{i\gamma \omega - \gamma^2/(2\alpha)} \left( \frac{1}{\varrho} + \frac{1}{\omega \varrho^2} + \frac{2}{\omega^2 \varrho^3} \right) + Q_1(\gamma) + Q_2(\gamma),
\]
where
\[
|Q_1(\gamma)| \leq \frac{6}{(\omega - \eta)^2} \min \left( \frac{1}{\gamma^4}, \frac{64\sqrt{2}}{(1 + 2|\gamma|)^4} \right),
\]
\[
|Q_2(\gamma)| \leq \frac{2.2K(\eta; \alpha)}{|\varrho| \alpha \eta} + \frac{1.25}{\alpha \omega^3 |\varrho|^2} + \frac{1.27e^{-\gamma^2/(2\alpha)}}{\omega^2 \alpha |\varrho|}.
\]

Proof. Without loss of generality suppose \( \gamma > 0 \). By Lemma 2.1 and the fact that \( \int_{-\infty}^{\infty} K(u; \alpha) \, du = 1 \),
\[
\int_{-\infty}^{\infty} K(u - \omega; \alpha) e^{-u/2} \text{li}(e^{\varrho u}) \, du = I + E,
\]
where
\[
I = \int_{-\infty}^{\infty} K(u - \omega; \alpha) e^{i\gamma u} \sum_{j=1}^{J} \frac{(j-1)!}{(\varrho u)^j} \, du,
\]
\[
|E| \leq \frac{J!}{(\omega - \eta)^j} \min \left( \frac{1}{\gamma^{j+1}}, \frac{2^{1.5}J^2}{(1 + 2\gamma)^{j+1}} \right).
\]
Now make the change of variables \( u = \omega - s \) and take \( J = 3 \). By (2.5),
\[ |s/\omega| \leq 1/30 \text{ and } |\varrho\omega| \geq 15, \text{ thus } \]

\[
\frac{I}{e^{i\gamma_0}} = \int_{-\eta}^{\eta} K(s; \alpha) e^{-i\gamma s} \left( \frac{1}{q} + \frac{1}{\omega \varrho^2(1-s/\omega)} + \frac{2}{\omega^2 \varrho^3(1-s/\omega)^2} \right) \, ds
\]

\[
= \int_{-\eta}^{\eta} K(s; \alpha) e^{-i\gamma s} \times \left( \frac{1}{q} + \frac{1}{\omega \varrho^2} + \frac{2}{\omega^2 \varrho^3} + \frac{s}{\omega^2 \varrho^2} + \frac{4s}{\omega^3 \varrho^3} + \vartheta \left( \frac{1.25s^2}{\omega^3 \varrho^2} \right) \right) \, ds
\]

\[
= \left( \frac{1}{q} + \frac{1}{\omega \varrho^2} + \frac{2}{\omega^2 \varrho^3} \right) I_0 + \frac{I_1}{\omega^2 \varrho^2} \left( 1 + \frac{4}{\omega \varrho} \right) + \vartheta \left( \frac{I_2^1 \cdot 1.25}{\omega^3 \varrho^2} \right)
\]

where

\[
I_n = \int_{-\eta}^{\eta} K(s; \alpha) s^n e^{-i\gamma s} \, ds \quad (n = 0, 1)
\]

and

\[
I_2 = \int_{-\infty}^{\infty} K(s; \alpha) s^2 \, ds = 1/\alpha.
\]

By (2.2) and (2.4), we have

\[
I_0 = e^{-\gamma^2/(2\alpha)} + \vartheta \left( 2 \int_{\eta}^{\infty} K(s; \alpha) \, ds \right) = e^{-\gamma^2/(2\alpha)} + \vartheta \left( \frac{2K(\eta; \alpha)}{\alpha \eta} \right).
\]

In addition, by (2.5) we have

\[
|I_1| = \left| \frac{2i \sin \gamma \eta}{\alpha} K(\eta; \alpha) - \frac{i\gamma}{\alpha} I_0 \right|
\]

\[
\leq \left( \frac{2}{\alpha} + \frac{2\gamma}{\alpha^2 \eta} \right) K(\eta; \alpha) + \frac{\gamma e^{-\gamma^2/(2\alpha)}}{\alpha} \leq \frac{3K(\eta; \alpha) + \gamma e^{-\gamma^2/(2\alpha)}}{\alpha}.
\]

We thus obtain

\[
\left| I - e^{i\gamma_0 - \gamma^2/(2\alpha)} \left( \frac{1}{q} + \frac{1}{\omega \varrho^2} + \frac{2}{\omega^2 \varrho^3} \right) \right|
\]

\[
\leq \frac{1.27 \gamma e^{-\gamma^2/(2\alpha)}}{\omega^2 |\varrho|^2 \alpha} + \frac{1.25}{\omega^3 |\varrho|^2 2\alpha} + \left( \frac{3.8}{\omega^2 |\varrho|^2 \alpha} + \frac{2.16}{|\varrho| \alpha \eta} \right) K(\eta; \alpha).
\]

By (2.5), \( \omega^2 |\varrho| \geq 450 \eta \), and the lemma follows.  

The next lemma, essentially due to Lehman ([Leh], §5), shows how to deal with the contribution from large \( \gamma \) without needing to assume the truth of the Riemann Hypothesis.
Lemma 2.6. Suppose that

\begin{equation}
|\gamma| \geq 100, \quad \omega \geq 30, \quad \eta \leq \omega/15, \quad 1 \leq N \leq \min(|\gamma|\eta/2, \alpha\omega^2/100).
\end{equation}

Writing \( \varrho = \beta + i\gamma \), with \( 0 < \beta < 1 \), we have

\[ |J(\varrho)| \leq e^{(\beta-1/2)(\omega+\eta)} \left( \frac{2.4\sqrt{\alpha} e^{-\alpha\eta^2/8}}{\gamma^2} + \frac{2.8\sqrt{N}}{|\gamma|^{1+N+1}} \left( \frac{N\alpha}{e} \right)^{N/2} \right). \]

**Proof.** By Lemma 2.5, we expect \( |J(\varrho)| \) is about \( |\varrho|^{-1} e^{(\beta-1/2)\omega-\gamma^2/(2\alpha)} \).

Suppose without loss of generality that \( \varrho > 100 \). As in [Leh], we begin by considering the function

\[ f(s) := \varrho s e^{-\varrho s} \ln(e^{\varrho s}) e^{-\alpha(s-\omega)^2/2} \]

in the region \( -\pi/4 \leq \arg s \leq \pi/4 \), \( |s| > 1 \). This function is analytic in this sector because \( \gamma > 100 \). Then

\[ J(\varrho) = \frac{1}{\varrho} \sqrt{\frac{\alpha}{2\pi}} I_1, \quad I_1 = \int_{\omega-\eta}^{\omega+\eta} e^{(\varrho-1/2)u} f(u) \, du. \]

By repeated integration by parts,

\[ I_1 = \sum_{n=0}^{N} \frac{(-1)^n e^{(\varrho-1/2)\omega}}{(\varrho-1/2)^{n+1}} (e^{(\varrho-1/2)\eta} f^{(n)}(\omega + \eta) - e^{-(\varrho-1/2)\eta} f^{(n)}(\omega - \eta)) \]

\[ + \frac{(-1)^N}{(\varrho-1/2)^N} \int_{\omega-\eta}^{\omega+\eta} e^{(\varrho-1/2)u} f^{(N)}(u) \, du. \]

Choose \( r \leq \omega/10 \). Then

\begin{equation}
\int_{|s-u|=r} \frac{f(s)}{(s-u)^{n+1}} \, ds.
\end{equation}

By (2.3) we have

\[ f(s) = e^{-\alpha(s-\omega)^2/2} \left( 1 + \frac{1}{\varrho s} + \vartheta \left( \frac{2|\varrho s|}{|\Im \varrho s|^3} \right) \right). \]

Since \( |\varrho s| \geq 2000 \) and \( |\Im \varrho s| \geq \frac{1}{2} |\varrho s| \), it follows that

\[ |f(s)| \leq 1.001 e^{-(\alpha/2)\Re(s-\omega)^2}. \]

Writing \( s = u + re^{i\phi} \) and using (2.7), we deduce

\begin{equation}
|f^{(n)}(u)| \leq \frac{1.001n!}{2\pi r^n} \int_{-\pi}^{\pi} e^{(\alpha/2)(r^2-r^2\cos^2\phi-(r\cos\phi+u-\omega)^2)} d\phi.
\end{equation}
When \( u = \omega \pm \eta \), we take \( r = \eta/2 \) and get

\[
|f^{(n)}(u)| \leq \frac{1,001n!}{2\pi(\eta/2)^n} e^{-\alpha \eta^2/8} \int_{-\pi}^{\pi} e^{-(\alpha \eta^2/4)(1-\cos \phi)^2} d\phi \\
\leq 1,001n!(2/\eta)^n e^{-\alpha \eta^2/8},
\]

since the integrand above is \( \leq 1 \). We then obtain

\[
|I_1| \leq e^{(\beta-1/2)(\omega+\eta)} \left( \frac{2,002 e^{-\alpha \eta^2/8}}{\gamma} \sum_{n=0}^{N} n! \left( \frac{2}{\gamma \eta} \right)^n + \gamma^{-N} \int_{\omega-\eta}^{\omega+\eta} |f^{(N)}(u)| \, du \right).
\]

Since \( n! \leq 2(N/2)^n \) for \( n \leq N \) and \( N/(\gamma \eta) \leq 1/2 \), the sum on \( n \) is \( \leq 3 \). By (2.8),

\[
\int_{\omega-\eta}^{\omega+\eta} |f^{(N)}(u)| \, du \leq \frac{1,001N!}{2\pi r^N} e^{\alpha r^2/2} \int_{-\pi}^{\pi} e^{-\alpha \eta^2/8} \sum_{n=0}^{N} n! \left( \frac{2}{\gamma \eta} \right)^n + \gamma^{-N} \int_{-\pi}^{\pi} e^{-\alpha t^2/2} \, dt \, d\phi \\
\leq \frac{1,001N!}{2\pi r^N} e^{\alpha r^2/2} \int_{-\infty}^{\infty} e^{-\alpha \eta^2/8} \, dt \, d\phi \\
= \frac{1,001N!}{r^N} e^{\alpha r^2/2} \sqrt{2\pi}/\alpha.
\]

Taking \( r = \sqrt{N/\alpha} \) and using the inequality \( N! \leq e^{1-N} N^{N+1/2} \) gives

\[
\int_{\omega-\eta}^{\omega+\eta} |f^{(N)}(u)| \, du \leq 1,001 e^{\sqrt{2\pi N/\alpha}} (\alpha e/N)^{-N/2}.
\]

The lemma now follows.

**Theorem 1.** Suppose \( \chi \) is a primitive Dirichlet character of conductor \( k \), and all the non-trivial zeros \( \rho = \beta + i\gamma \) of \( L(s, \chi) \) with \( |\gamma| \leq A \) have real part \( \beta = 1/2 \). Suppose that

(2.9) \( 150 \leq T \leq A, \quad \omega \geq 30, \quad \eta \leq \omega/30, \quad 2A/\eta \leq \alpha \leq A^2. \)

Then

\[
\sum_{\rho} J(\rho) = \sum_{|\gamma| \leq T} e^{i\gamma \omega - \gamma^2/(2\alpha)} \left( \frac{1}{\rho} + \frac{1}{\omega \rho^2} + \frac{2}{\omega^2 \rho^3} \right) + 4 \sum_{i=1}^{R} R_i(\chi, T),
\]

where

\[
|R_1(\chi, T)| \leq \frac{6}{(\omega - \eta)^3} \sum_{\rho} \min \left( \frac{1}{\gamma^4}, \frac{64 \sqrt{2}}{(1 + 2|\gamma|)^4} \right),
\]

\[
|R_2(\chi, T)| \leq \left( \frac{2.2K(\eta; \alpha)}{\alpha \eta} + \frac{1.27}{\alpha \omega^2} \right) \sum_{|\gamma| \leq A} \frac{1}{|\rho|} + \frac{1.25}{\alpha \omega^3} \sum_{\rho} \frac{1}{|\rho|^2},
\]

\[
\text{with } K(\eta; \alpha) = \frac{1}{\alpha} \left( \frac{4}{\pi \alpha} \right)^{1/2} \frac{305}{\sqrt{12}}(\cos \frac{\eta}{\alpha} - i \sin \frac{\eta}{\alpha}).
\]
\[ |R_3(\chi, T)| \leq e^{-T^2/(2\alpha)} \log(kT) \left( \frac{\alpha}{\pi T^2} + \frac{4.3}{T} \right), \]
\[ |R_4(\chi, T)| \leq e^{(\omega+\gamma)/2} \log(kA) \left( \frac{0.8\sqrt{\alpha}e^{-\alpha\gamma^2/8}}{A} + 2.56A\alpha^{-1/2}e^{-A^2/(2\alpha)} \right). \]

If the Riemann Hypothesis is true for \( L(s, \chi) \) (i.e. all the non-trivial zeros have real part 1/2), then the term \( R_4 \) may be omitted, as may the condition \( \alpha \leq A^2 \). Also, if \( A = T \), then \( R_3(\chi, T) = 0 \).

**Proof.** The main terms in the theorem come from the main terms of Lemma 2.5 for \( |\gamma| \leq T \). The first part of the theorem follows by taking
\[
R_i = R_i(\chi, T) = \sum_{|\gamma| \leq A} Q_i(\gamma) \quad (i = 1, 2),
\]
\[
R_3 = R_3(\chi, T) = \sum_{T < |\gamma| \leq A} e^{i\gamma\omega-\gamma^2/(2\alpha)} \left( \frac{1}{\varrho} + \frac{1}{\omega \varrho^2} + \frac{2}{\omega^2 \varrho^3} \right),
\]
\[
R_4 = R_4(\chi, T) = \sum_{|\gamma| > A} J(\varrho).
\]
The upper bounds for \( R_1 \) and \( R_2 \) follow from Lemma 2.5. Since \( \omega \geq 30 \), we have
\[
\left| \frac{1}{\varrho} + \frac{1}{\omega \varrho^2} + \frac{2}{\omega^2 \varrho^3} \right| \leq \frac{1}{\gamma}.
\]
Thus, by Corollary 2.3, we find that
\[
|R_3| \leq \sum_{|\gamma| > T} e^{-\gamma^2/(2\alpha)} \gamma
\leq \int_T^\infty \frac{e^{-t^2/(2\alpha)}}{\pi t} \log \left( \frac{kt}{2\pi} \right) dt + \frac{2e^{-T^2/(2\alpha)}}{T} \left( C_2 \log(kT) + C_3 \right)
+ C_2 \int_T^\infty \frac{e^{-t^2/(2\alpha)}}{t^2} dt.
\]
If \( g(t) \) is positive and decreasing for \( t \geq T \) we have
\[
\int_T^\infty g(t)e^{-bt^2} dt < \frac{g(T)}{T} \int_T^\infty te^{-bt^2} dt = \frac{g(T)e^{-bT^2}}{2bT}.
\]
Therefore,
\[
|R_3| \leq e^{-T^2/(2\alpha)} \left( \frac{\alpha \log(kT/(2\pi))}{\pi T^2} + \frac{2C_2 \log(kT) + 2C_3}{T} + \frac{\alpha C_2}{T^3} \right).
\]
The desired bound for $R_3$ now follows from the bounds $kT \geq 100$ and
\[
\frac{\alpha C_2}{T^3} \leq \frac{\alpha \log(2\pi)}{\pi T^2}.
\]
Lastly, Corollary 2.4 and Lemma 2.6 give
\[
|R_4| \leq \sum_{|\gamma| > A} |J(\gamma)| 
\leq e^{(\omega + n)/2} \log(kA) \left( \frac{0.8 \sqrt{\alpha} e^{-\alpha n^2/8}}{A} + 0.94 \sqrt{N \left( \frac{N \alpha}{eA^2} \right)^{N/2}} \right).
\]
We take $N = [A^2/\alpha]$ and note that (2.9) implies (2.6).

Finally, we need explicit formulas for the number of primes in an arithmetic progression. For a primitive Dirichlet character $\chi$ modulo $k \geq 3$, let $a = 0$ if $\chi(-1) = 1$ and $a = 1$ if $\chi(-1) = -1$. By an analog of the Riemann–von Mangoldt formula ([La, p. 532]), if $L(s, \chi)$ has no positive real zeros then
\begin{equation}
S(\chi; x) := \sum_{p, m} \frac{\chi(p)^m}{m} \sum_{p^m \leq x} = -\sum_{q} \operatorname{li}(x^q) + \int_{x}^{\infty} \frac{dy}{y^{1-a}(y^2-1) \log y} + (1-a) \log \log x + K_a,
\end{equation}
where
\[
K_0 = C - \log \left( \frac{\tau(\chi) \pi}{2k} L(1, \chi) \right),
\]
\[
K_1 = \log \left( \frac{\tau(\chi)}{i\pi} L(1, \chi) \right),
\]
and
\[
\tau(\chi) = \sum_{m=1}^{k} \chi(m) e^{2\pi im/k}.
\]
Here $C = 0.5772\ldots$ is the Euler–Mascheroni constant and \log \ refers to the principal branch of the logarithm. The values of $L(1, \chi)$ are computed easily by means of the formula
\[
\tau(\chi)L(1, \chi) = -\sum_{j=1}^{k-1} \chi(j) \log(1 - e^{2\pi ij/k}).
\]
Also, the integral in (2.10) is less than $1/x$ for $x > 10$. The last formula we
need is
\begin{equation}
\pi_{q,a}(x) = \frac{1}{\phi(q)} \sum_{\chi \mod q} \overline{\chi(a)} S(\chi; x) - \sum_{p, m \leq x, m \geq 2 \atop p^m \equiv a \mod q} \frac{1}{m}.
\end{equation}

In practice the $m = 2$ terms will be very significant, while the terms with $m \geq 3$ will be negligible. In fact, we have
\begin{equation}
\sum_{p^m \leq x, m \geq 3} \frac{1}{m} \leq \frac{1.3x^{1/3}}{\log x} \quad (x \geq e^{30})
\end{equation}
which follows easily from the inequality
\[
\pi(x) \leq \frac{x}{\log x} + \frac{1.5x}{\log^2 x} \quad (x > 1)
\]
given by Theorem 1 of Rosser and Schoenfeld [RoS]. Lastly, if $\chi_0$ is the primitive character (of order $k_0$) which induces $\chi$, then
\begin{equation}
|S(\chi_0; x) - S(\chi; x)| \leq \sum_{p^m \leq x \atop p \mid k, p \mid k_0} \frac{1}{m} \leq \sum_{p \mid k, p \mid k_0} \left(1 + \log \frac{\log x}{\log p}\right)
\end{equation}
\[
\leq \left|\{p : p \mid k, p \mid k_0\}\right| \log \log x + 1 - \log 2.
\]
Here we have used the inequality $\sum_{n \leq x} 1/n \leq 1 + \log x$.

3. Primes in progressions modulo 3, 4, 8, 12 and 24. For brevity, write
\[
\Delta_{q,b,1}(x) := \pi_{q,b}(x) - \pi_{q,1}(x).
\]
In this section we give new results on the location of negative values of $\Delta_{q,b,1}(x)$. Throughout we assume $q \mid 24$, $1 < b < q$ and $(b, q) = 1$. As noted previously, such negative values are quite rare. The smallest $x$ giving $\Delta_{4,3,1}(x) < 0$ is $x = 26861$, discovered by Leech [Lee] in 1957. Shanks [Sh] computed $\Delta_{8,b,1}(x)$ for $b = 3, 5, 7$ and $x \leq 10^6$ and found that none of the functions takes negative values. Extensive computations by Bays and Hudson in the 1970s ([BH1]–[BH4]) for $x \leq 10^{12}$ led to the discovery of several more “negative regions” for $\Delta_{4,3,1}(x)$, as well as a single region for $\Delta_{3,2,1}(x)$, a single region for $\Delta_{24,13,1}(x)$ and two regions for $\Delta_{8,5,1}(x)$. By “negative region” we mean an interval $[x_1, x_2]$ where the corresponding function is negative a large percentage of time. It is not well defined, but reflects the observation that negative values of the functions $\Delta_{q,b,1}(x)$ occur in “clumps”. For example, $\Delta_{3,2,1}(x) < 0$ for about 15.9% of the integers in the interval [608981813029, 610968213796]. On the other hand, the computations show that
\[
\Delta_{q,b,1}(x) \geq 0 \quad (x \leq 10^{12})
\]
Sign changes in $\pi_{q,a}(x) - \pi_{q,b}(x)$

for

$$q = 8, \ b \in \{3, 7\} \quad \text{and} \quad q = 24, \ b \in \{5, 7, 11, 17, 19, 23\}.$$  

With modern computers, the search could easily be extended to $10^{14}$ or even $10^{15}$, and we will show that in fact there are regions in this range where $\Delta_{q,b,1}(x) < 0$ for some of the pairs $q, b$ given in (3.1). Our method, though, takes only seconds versus weeks for an exhaustive search.

From a theoretical standpoint, Littlewood [Li] proved in 1914 that $\Delta_{4,3,1}(x)$ and $\Delta_{3,2,1}(x)$ change sign infinitely often. Knapowski and Turán (Part II of [KT1]) generalized this substantially, showing that $\Delta_{q,b,1}(x)$ changes sign infinitely often whenever $q | 24, 1 < b < q$ and $(b, q) = 1$ (in addition to other $q, b$). Later papers ([KT1], [KT2]) deal with the frequency of sign changes, but the bounds for the first sign change are of the “towering exponentials” type, similar to Skewes’ results.

In what follows, $\chi_k$ denotes the unique primitive character modulo $k$ and $\chi_{k,i}$ ($i = 1, \ldots, h$) denote the primitive characters modulo $k$ if there are more than one. In particular, $\chi_{8,1}(-1) = -1$ and $\chi_{24,1}(-1) = -1$. Table 1 below lists some parameters which we will need. Here

$$\Sigma_1 = \sum_{\gamma} \frac{1}{|\gamma|^2}, \quad \Sigma_2 = \sum_{\gamma} \min \left( \frac{1}{\gamma^4}, \frac{64\sqrt{2}}{(1 + 2|\gamma|)^4} \right), \quad \Sigma_3 = \sum_{|\gamma| \leq 10000} \frac{1}{|\gamma|}.$$

The entries in the second, third, and fourth columns are rigorous upper bounds, obtained from Rumely’s lists of zeros [Ru2] and Corollary 2.4. The number $N$ denotes the number of zeros with $0 < \gamma < 10000$. It is desirable in applications to know the zeros of all the required $L$-functions to the same height. Rumely [Ru1] originally computed zeros to height 10000 for characters with conductor $\leq 13$ and to height 2600 for other characters. For the two primitive characters modulo 24, Rumely’s original programs were run to compute the zeros to height $T = 10000$, and the output was checked against his original list of zeros to height 2600. In all of our computations, we take $T = 10000$ for every character. Recently Rumely [Ru2] has extended the computations to height 100000 for characters of conductor $< 10$. So for such characters we may take $A = 100000$.

When $q | 24$, all the characters modulo $q$ are real, and furthermore the only quadratic residue modulo $q$ is 1. When $x \geq e^{32.3}$, for each character in Table 1,

$$|(1 - a) \log \log x + K_a| \leq |\log \log x + \log 3| \leq 0.00312 \frac{x^{1/3}}{\log x}.$$

Further, if $\chi_0$ is the primitive character (modulo $k_0$) which induces $\chi$ (for
one of the seven characters in Table 1), then

$$(\log \log x + 0.31)|\{p : p | k, \ p \nmid k^0\}| \leq \log \log x + 0.31 \leq 0.0026 \frac{x^{1/3}}{\log x}.$$  

Together with (2.10)–(2.13), we obtain the formula

$$(3.2) \quad \pi_{q,b}(x) - \pi_{q,1}(x) = \frac{2}{\phi(q)} \sum_{\chi \bmod q} \sum_{\rho \in \chi(b) = -1} \log(x^\rho) + \frac{\pi(\sqrt{x})}{2} + \vartheta\left(\frac{1.31x^{1/3}}{\log x}\right).$$

We need a tight upper bound on $\pi(\sqrt{x})$, given by the next lemma.

**Lemma 3.1.** For $x \geq 10^{14}$, we have $\pi(x) \leq 1.000011 \ln(x)$.

**Proof.** From Table 3 of [Ri], we have $\pi(10^{14}) < \ln(10^{14})$. Defining $\theta(x) = \sum_{p \leq x} \log p$, we have

$$|\theta(x) - x| \leq 0.0000055x \quad (x \geq e^{32}),$$

which follows from Theorem 5.1.1 of [RR], upon taking $x = e^{32}$, $m = 18$, $H = 70000000$, and $\delta = 6.59668 \cdot 10^{-8}$. By partial summation, for $x \geq 10^{14}$ we obtain

$$\pi(x) \leq \ln(10^{14}) + \int_{10^{14}}^{x} \frac{d\theta(t)}{\log t} \leq (1 + 2(0.0000055)) \ln(x).$$

Define

$$W(\chi; x) = \sum_{\rho} \log(x^\rho),$$

where the sum is over zeros $\rho$ of $L(s, \chi)$ lying in the critical strip. Since we are primarily interested in locations where $\pi_{q,b}(x) - \pi_{q,1}(x)$ is negative, we apply Lemma 3.1 to obtain from (3.2) the inequality

$$\pi_{q,b}(x) - \pi_{q,1}(x) \leq \frac{2}{\phi(q)} \sum_{\chi \bmod q \chi(b) = -1} W(\chi; x) + \frac{1}{2}(1.000011) \ln(\sqrt{x}) + \frac{1.31x^{1/3}}{\log x}.$$  

It is easy to show that
\[ \text{li}(x) \leq \frac{x}{\log x} \left( 1 + \frac{1}{\log x} + \frac{2}{\log^2 x} + \frac{h(x)}{\log^3 x} \right), \]

where

\[ h(x) = \begin{cases} 
8.326, & e^{16} \leq x < e^{21}, \\
7.538, & e^{21} \leq x \leq e^{29.3}, \\
7, & x \geq e^{29.3}.
\end{cases} \]

By Theorem 1, we therefore have

**Theorem 2.** Suppose that \( \omega - \eta \geq 32.3 \) and \( 0 < \eta \leq \omega/30 \). Suppose \( q \mid 24 \), \( (b, q) = 1 \) and \( 1 < b < q \). For each Dirichlet character \( \chi \mod q \) with \( \chi(b) = -1 \), suppose that all the zeros of \( L(s, \chi) \) which lie in the rectangle \( 0 < \Re s < 1 \), \( -A_{\chi} \leq \Im s \leq A_{\chi} \), actually lie on the critical line \( \Re s = 1/2 \). Further suppose that

\[ 150 \leq T_{\chi} \leq A_{\chi}, \quad 2A_{\chi}/\eta \leq \alpha \leq A_{\chi}^2 \]

for every \( \chi \). Then

\[
\begin{align*}
&\int_{\omega - \eta}^{\omega + \eta} K(u - \omega; \alpha)ue^{-u/2}(\pi_{q,b}(e^u) - \pi_{q,1}(e^u)) \, du \\
&\quad \leq (1.000011) \left( 1 + \frac{2}{\omega - \eta} + \frac{8}{(\omega - \eta)^2} + \frac{8h(e^{(\omega - \eta)/2})}{(\omega - \eta)^3} \right) + 1.31e^{-(\omega - \eta)/6} \\
&\quad + \frac{2}{\phi(q)} \sum_{\chi \mod q, \chi(b) = -1} \left( \sum_{|\gamma| \leq T_{\chi}} e^{i\gamma\omega - \gamma^2/(2\alpha)} \left( \frac{1}{\varrho} + \frac{1}{\omega \varrho^2} + \frac{2}{\omega^2 \varrho^3} \right) \right) \\
&\quad + \sum_{i=1}^{4} |R_i(\chi, T_{\chi})|.
\end{align*}
\]

The error terms \( R_i(\chi, T_{\chi}) \) are as given in Theorem 1, with \( T = T_{\chi} \) and \( A = A_{\chi} \). Furthermore, if \( A_{\chi} = T_{\chi} \) then the corresponding \( R_3(\chi, T) \) is 0, and if the Riemann Hypothesis holds for \( L(s, \chi) \), then we have \( R_4(\chi, T) = 0 \) and the condition \( \alpha \leq A_{\chi}^2 \) may be omitted.

Locating likely candidates for regions where \( \Delta_{q,b,1}(x) \) takes negative values is relatively simple. We search for values of \( \omega \) for which

\[ K^* = K^*(q, b; \omega) = \frac{\text{li}(\sqrt{x}) \log x}{2\sqrt{x}} + \frac{2}{\phi(q)} \sum_{\chi \mod q, \chi(b) = -1} \sum_{|\gamma| \leq T_{\chi}} e^{i\gamma\omega} \varrho < 0. \]

Heuristically, \( K^* \) is a good predictor for the average of \( ue^{-u/2}\Delta_{q,b,1}(e^u) \) for \( u \) near \( \omega \). For example, \( K^*(24, 13; \omega) \) reaches a relative minimum of \(-0.15873\) at about \( \omega = 27.617477 \), while Bays and Hudson [BH3] computed at \( x = \)
9.866 \cdot 10^{11} \approx e^{27.61753}$ the value $\Delta_{24,13,1}(x) = -6091 \approx -0.169357 \frac{\sqrt{x}}{\log x}$ (it is possible that $\Delta_{24,13,1}(x)$ takes smaller values in this vicinity, but this is the smallest value listed in the paper). Using $K^*$ as an approximation for $ue^{-u/2}D_{q,b,1}(e^u)$ is also useful in computing a numerical value for Chebyshev’s bias (see [RS], [BFHR]).

In practice, since $\omega$ is large, $\eta$ is small, and $T$ is large ($\geq 10000$), the most critical of the error terms is $R_4(\chi, T\chi)$ because it controls the maximum practical value for $\alpha$. We want to take $\alpha$ as large as possible, so the sums over $e^{i\gamma\omega - \gamma^2/(2\alpha)}/q$, which are required to be “large” negative, are not damped out too much by the $e^{-\gamma^2/(2\alpha)}$ factor.

The computations were performed with a C program running on a Sun Ultra-10 workstation using double precision floating point arithmetic, which provides about 16 digits of precision. The zeros of the $L$-functions in Rumely’s lists are all accurate to within $10^{-12}$. Values computed for the right side of the inequality in Theorem 2 were rounded up in the 4th decimal place.

**Theorem 3.** For each row of Tables 2 and 3 for which a value of $K$ is given, we have

(3.3) $\min_{\omega-\eta \leq u \leq \omega+\eta} ue^{-u/2}(\pi_{q,b}(e^u) - \pi_{q,1}(e^u)) \leq K.$

**Proof.** Take the indicated values of the parameters in Theorem 2. Here $T\chi = 10000$ for every $\chi$, $A\chi = 100000$ in Table 2 and $A\chi = 10000$ in Table 3. In the case where a value of $K$ is not given, we could not prove that $K < 0$ with any choice of parameters.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$b$</th>
<th>$\omega$</th>
<th>$K^*$</th>
<th>$\eta$</th>
<th>$\alpha$</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>45.12686</td>
<td>-0.0798</td>
<td>0.02</td>
<td>$10^7$</td>
<td>-0.0650</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>58.36855</td>
<td>-0.1710</td>
<td>0.02</td>
<td>$10^7$</td>
<td>-0.1525</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>2179.77584</td>
<td>-0.8109</td>
<td>0.05</td>
<td>$400000$</td>
<td>-0.7761</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>78683.67818</td>
<td>-1.0480</td>
<td>2.00</td>
<td>$1200000$</td>
<td>-0.8372</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>43.36630</td>
<td>-0.0249</td>
<td>0.02</td>
<td>$10^7$</td>
<td>-0.0013</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>54.94255</td>
<td>-0.0490</td>
<td>0.02</td>
<td>$10^7$</td>
<td>-0.0280</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>32.89388</td>
<td>-0.0716</td>
<td>0.02</td>
<td>$10^7$</td>
<td>-0.0503</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>34.46826</td>
<td>-0.0051</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>57.48058</td>
<td>-0.2136</td>
<td>0.02</td>
<td>$10^7$</td>
<td>-0.1915</td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>32.89284</td>
<td>-0.0136</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>45.34991</td>
<td>-0.0868</td>
<td>0.02</td>
<td>$10^7$</td>
<td>-0.0508</td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>48.79950</td>
<td>-0.1889</td>
<td>0.02</td>
<td>$10^7$</td>
<td>-0.1724</td>
</tr>
<tr>
<td>12</td>
<td>11</td>
<td>187.53674</td>
<td>-0.0410</td>
<td>0.02</td>
<td>$10^7$</td>
<td>-0.0191</td>
</tr>
<tr>
<td>12</td>
<td>11</td>
<td>191.89007</td>
<td>-0.0415</td>
<td>0.02</td>
<td>$10^7$</td>
<td>-0.0182</td>
</tr>
</tbody>
</table>
Example. The “error terms” \( R_3 \) and \( R_4 \) force \( \alpha \) to be less than \( \min(A^2/\omega, T^2) \) for practical purposes. For row 5 of Table 2, with the indicated values of the parameters, we compute (rounded in the last place after the decimal point)

<table>
<thead>
<tr>
<th>Char</th>
<th>Sum on ( \varrho )</th>
<th>( R_1 )</th>
<th>( R_2 )</th>
<th>( R_3 )</th>
<th>( R_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_4 )</td>
<td>-0.802723684</td>
<td>0.000000137</td>
<td>0.000000002</td>
<td>0.002303420</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_{8,2} )</td>
<td>-1.308816425</td>
<td>0.000000326</td>
<td>0.000000003</td>
<td>0.002454092</td>
<td>0</td>
</tr>
</tbody>
</table>

Here the second column is the sum over \(|\gamma| \leq T\chi\) in Theorem 2. The first line of the right side of the inequality in Theorem 2 is computed as \( 1.0521043 \). All of these values are rounded in the 9th decimal place.

Corollary 4. For each \( b \in \{3, 5, 7\} \), \( \pi_{8,b}(x) < \pi_{8,1}(x) \) for some \( x < 5 \cdot 10^{19} \). For each \( b \in \{5, 7, 11\} \), \( \pi_{12,b}(x) < \pi_{12,1}(x) \) for some \( x < 10^{34} \). For each \( b \in \{5, 7, 11, 13, 17, 19, 23\} \), \( \pi_{24,b}(x) < \pi_{24,1}(x) \) for some \( x < 10^{353} \). Finally, if the zeros of \( L(s, \chi_4) \) lying in the critical strip to height \( A = 630000 \) all have real part equal to 1/2, then for some \( x \) in the vicinity of \( e^{76683.7} \) we have

\[
\pi_{4,1}(x) - \pi_{4,3}(x) > \sqrt{x}/\log x.
\]

The significance of the last statement is that we now know (once the zeros of \( L(s, \chi_4) \) are computed to height 630000) a specific region where \( \pi_{4,1}(x) \) runs ahead of \( \pi_{4,3}(x) \) as much as it usually runs behind (this is the smallest \( x \) for which \( K^* < -1 \)). The idea is that the terms on the right side of (3.2) corresponding to the zeros \( \varrho \) are oscillatory, so that on average \( \Delta_{q,b,1}(x) \) is about \( \pi(\sqrt{x})/2 \approx \sqrt{x}/\log x \). Subject to certain unproven hypotheses, this notion can be made very precise (e.g. \([RS]\)). The two rows for \( q = 4 \) were chosen because of the large negative values of \( K^* \).

In Tables 2 and 3, we have confined our calculations to locating regions with \( x \geq e^{32.3} \approx 10^{14} \), smaller \( x \) being easily dealt with by exhaustive computer search. The listed values of \( K^* \) and \( K \) are rounded up in the last decimal place. For each pair \((q, b)\) except \((4, 3)\), the first few likely regions of negative values of \( \Delta_{q,b,1}(x) \) are listed. The lists continue until a region is found where a negative value can be proved with \( A = 10000 \). In some regions, a negative value can be proved with a larger value of \( A \) and in other regions no negative value could be proved even with \( A = \infty \). These latter rows have no \( K \) value listed. However, when \( \omega \leq 44 \) or so, it is possible to find specific values of \( x \) with \( \Delta_{q,b,1}(x) < 0 \) by computing this function exactly by means of Hudson’s extension of Meissel’s formula \([H1]\). This formula makes it practical to compute exact values of \( \pi_{q,a}(x) \) for \( x \) as large as \( 10^{20} \). The first author is currently writing a computer program for this, and one preliminary result can be announced now. At \( x = 1.9282 \cdot 10^{14} \)
we have $\Delta_{8,7,1}(x) = -105$, and this computation took 10 minutes on a Sun Ultra-10 workstation.

For all pairs $q, b$, the values of $\omega$ given in Tables 2 and 3 represent the minimum of $K$, and this does not necessarily correspond to the minimum of $K^*$. The difference $|K - K^*|$ varies substantially, and this is expected due to the factors $e^{-\gamma^2/(2\alpha)}$ in Theorem 2. To illustrate the difference, Graph 1 depicts the functions $K$ and $K^*$ for $q = 12, b = 11$ in the vicinity of $e^{187.536}$.

Also as expected, larger values of $A$, which permit larger values of $\alpha$, narrow the difference appreciably.

A shortcoming of our method is the inability to compare three or more progressions. For example, Shanks [Sh] asked if $\pi_{8,1}(x)$ will ever be greater than each of $\pi_{8,3}(x), \pi_{8,5}(x)$ and $\pi_{8,7}(x)$ simultaneously. Based on computations of the functions $K^*$, it is likely that this occurs in the vicinity of $e^{389.3712}$, but this cannot be proved by the methods of this paper. It is, however, possible to detect negative values of any linear combination of the functions $\pi_{q,b}(x)$. For example, by Theorem 2 it follows that for some $x$ with $|\log x - 158.64233| \leq 0.01$, we have

$$\pi_{8,1}(x) > \frac{1}{3}(\pi_{8,3}(x) + \pi_{8,5}(x) + \pi_{8,7}(x)).$$

We are really looking for negative values of $\frac{1}{3}(\Delta_{8,3,1}(x) + \Delta_{8,5,1}(x) + \Delta_{8,7,1}(x))$, and take $A = 100000, \alpha = 10^7$ and $\eta = 0.02$ and obtain $K < -0.0265$.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$b$</th>
<th>$\omega$</th>
<th>$K^*$</th>
<th>$\eta$</th>
<th>$\alpha$</th>
<th>$K$</th>
</tr>
</thead>
</table>
| 12 | 5 | 39.12815 | -0.0071 | 12 | 5 | 69.00554 | -0.0210 | 12 | 5 | 73.93306 | -0.0117 | 12 | 5 | 88.98310 | -0.0104 | 12 | 5 | 102.08460 | -0.0344 | 12 | 5 | 103.73736 | -0.0611 | 0.02 1550000 | 0.0445 | 12 | 7 | 39.12144 | -0.2063 | 0.02 1550000 | -0.0140 | 12 | 7 | 45.87795 | -0.1468 | 0.02 1400000 | -0.0871 | 24 | 5 | 161.18837 | -0.1176 | 0.04 525000 | -0.0920 | 24 | 7 | 92.49622 | -0.0693 | 0.03 830000 | -0.0530 | 24 | 11 | 111.54595 | -0.0023 | 24 | 11 | 812.63677 | -0.0526 | 0.20 118000 | -0.0104 | 24 | 13 | 34.14425 | -0.4810 | 0.02 1700000 | -0.3521 | 24 | 17 | 34.05708 | -0.0387 | 24 | 17 | 34.19749 | -0.0208 | 0.02 1650000 | -0.0110 | 24 | 19 | 34.20322 | -0.1473 | 0.02 1650000 | -0.1362 | 24 | 23 | 43.45318 | -0.0204 | 24 | 23 | 94.46170 | -0.0376 | 0.03 800000 | -0.0113

Table 3
Sign changes in \( \pi_{q,a}(x) - \pi_{q,b}(x) \)

### Acknowledgments
The authors would like to thank Robert Rumely for providing the lists of the zeros of \( L \)-functions and copies of his original programs. The authors also thank Peter Sarnak for valuable discussions and encouragement.

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