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Algebraic independence of Fredholm series

by

KUMIKO NISHIOKA (Yokohama)

1. Introduction. Let K be an algebraic number field and $d \geq 2$ be an integer. We call

$$f(z) = \sum_{h=0}^{\infty} \sigma_h z^{d^h}, \qquad \sigma_h \in K^{\times},\ \log \|\sigma_h\| = o(d^h),$$

a *Fredholm series*. The convergence radius of $f(z)$ is 1. By Hadamard's gap theorem, the unit circle is the natural boundary of $f(z)$. If α is an algebraic number with $0 < |\alpha| < 1$, then $f(\alpha)$ is transcendental (cf. Theorem 2.10.1 in Nishioka [2]). Let

$$f_d(z) = \sum_{h=0}^{\infty} \sigma_{dh} z^{d^h}, \qquad \sigma_{dh} \in K^{\times},\ \log \|\sigma_{dh}\| = o(d^h),\ d = 2, 3, \ldots$$

Then we may expect that $f_d(\alpha)$, $d = 2, 3, \ldots$, are algebraically independent. When $\sigma_{dh} = 1$ for all d, h, this is proved in Nishioka [3]. Here we will prove the following.

THEOREM 1. *If for every d, the σ_{dh} $(h = 0, 1, \ldots)$ are in a finite set of nonzero algebraic numbers, then $f_d(\alpha)$, $d = 2, 3, \ldots$, are algebraically independent for any algebraic number α with $0 < |\alpha| < 1$.*

2. Mahler's method. By $\mathbb{N}$ and $\mathbb{N}_0$ we denote the set of positive integers and the set of nonnegative integers respectively. If α is an algebraic number, we denote by $\overline{|\alpha|}$ the maximum of the absolute values of the conjugates of α and by $\mathrm{den}(\alpha)$ the least positive integer such that $\mathrm{den}(\alpha)\alpha$ is an algebraic integer, and we set $\|\alpha\| = \max\{\overline{|\alpha|}, \mathrm{den}(\alpha)\}$. Then we have the inequalities

$$|\alpha| \geq \|\alpha\|^{-2[\mathbb{Q}(\alpha):\mathbb{Q}]} \quad \text{and} \quad \|\alpha^{-1}\| \leq \|\alpha\|^{2[\mathbb{Q}(\alpha):\mathbb{Q}]}$$

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(cf. Lemma 2.10.2 in [2]). If $\Omega = (\omega_{ij})$ is an $n \times n$ matrix with nonnegative integer entries and $z = (z_1, \ldots, z_n)$ is a point of $\mathbb{C}^n$, we define a transformation $\Omega : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\Omega z = \Big(\prod_{j=1}^{n} z_j^{\omega_{1j}}, \ldots, \prod_{j=1}^{n} z_j^{\omega_{nj}} \Big).$$

Let $\{\Omega^{(k)}\}_{k \geq 0}$ be a sequence of matrices with nonnegative integer entries. We put

$$\Omega^{(k)} = (\omega_{ij}^{(k)}) \quad \text{and} \quad \Omega^{(k)} z = (z_1^{(k)}, \ldots, z_n^{(k)}).$$

For $\lambda = (\lambda_1, \ldots, \lambda_n)$, we define $z^\lambda = z_1^{\lambda_1} \ldots z_n^{\lambda_n}$ and $|\lambda| = \lambda_1 + \ldots + \lambda_n$. Let $\{f_1^{(k)}(z)\}_{k \geq 0}, \ldots, \{f_m^{(k)}(z)\}_{k \geq 0}$ be sequences of power series in $K[[z_1, \ldots, z_n]]$. Let $\chi = (z_1, \ldots, z_n)$ be the ideal generated by $z_1, \ldots, z_n$ in $K[[z_1, \ldots, z_n]]$. We assume

$$f_i^{(k)} \to f_i \quad (k \to \infty), \quad i = 1, \ldots, m,$$

under the topology defined by χ. In what follows, $c_1, c_2, \ldots$ denote positive constants independent of k.

Theorem 2. *Suppose that the coefficients of $f_i^{(k)}$ are in a finite set $S \subset K$ for all i and k. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in K^n$, $0 < |\alpha_i| < 1$, $i = 1, \ldots, n$, and the following three properties are satisfied, then $f_1^{(0)}(\alpha), \ldots, f_m^{(0)}(\alpha)$ are algebraically independent.*

(I) *There exists a sequence $\{r_k\}_{k \geq 0}$ of positive numbers such that*

$$\lim_{k \to \infty} r_k = \infty, \quad \omega_{ij}^{(k)} \leq c_1 r_k, \quad \log |\alpha_i^{(k)}| \leq -c_2 r_k.$$

(II) *If we put*

$$f_i^{(0)}(\alpha) = f_i^{(k)}(\Omega^{(k)} \alpha) + b_i^{(k)},$$

then $b_i^{(k)} \in K$ and

$$\log \|b_i^{(k)}\| \leq c_3 r_k.$$

(III) *For any power series $F(z)$ represented as a polynomial in $z_1, \ldots, z_n$, $f_1, \ldots, f_m$ with complex coefficients,*

$$F(z) = \sum_{\lambda,\, \mu = (\mu_1, \ldots, \mu_m)} a_{\lambda\mu} z^\lambda f_1(z)^{\mu_1} \ldots f_m(z)^{\mu_m},$$

where $a_{\lambda\mu}$ are not all zero, there exists $\lambda_0 \in (\mathbb{N}_0)^n$ such that if k is sufficiently large, then

$$|F(\Omega^{(k)} \alpha)| \geq c_4 |(\Omega^{(k)} \alpha)^{\lambda_0}|.$$

Proof of Theorem 2. The following lemma is easy to prove.

LEMMA 1. *Let* $f(z)=\sum_{\lambda_1,\ldots,\lambda_n} c_{\lambda_1\ldots\lambda_n} z_1^{\lambda_1}\ldots z_n^{\lambda_n}\in\mathbb{C}[[z_1,\ldots,z_n]]$ *converge around the origin. If* z *is sufficiently close to the origin, then*

$$\sum_{|\lambda|\ge H}|c_{\lambda_1\ldots\lambda_n}|\cdot|z_1|^{\lambda_1}\ldots|z_n|^{\lambda_n}\le\gamma^{H+1}\max_i|z_i|^H,$$

where γ *is a positive constant depending on* $f(z)$.

LEMMA 2. (i) *If* $f_i^{(k)}-f_i\in\chi^H$, *then*

$$|f_i^{(k)}(\Omega^{(k)}\alpha)-f_i(\Omega^{(k)}\alpha)|\le c_5^{H+1}e^{-c_2 r_k H}.$$

(ii) *For* $F(z)$ *in* (III) *we put*

$$F^{(k)}(z)=\sum_{\lambda,\,\mu=(\mu_1,\ldots,\mu_m)}a_{\lambda\mu}z^\lambda f_1^{(k)}(z)^{\mu_1}\ldots f_m^{(k)}(z)^{\mu_m}.$$

Then $F^{(k)}(\Omega^{(k)}\alpha)\ne 0$ *if* k *is sufficiently large.*

Proof. The assumption (I) and Lemma 1 imply (i). We choose a large H satisfying

$$e^{-c_2H}<\Big(\prod_{i=1}^n|\alpha_i|\Big)^{c_1|\lambda_0|}.$$

Using (i) we have

$$|F^{(k)}(\Omega^{(k)}\alpha)-F(\Omega^{(k)}\alpha)|\le c_6e^{-c_2Hr_k}\le\frac{1}{2}c_4\Big(\prod_{i=1}^n|\alpha_i|\Big)^{c_1|\lambda_0|r_k}$$

if k is sufficiently large. On the other hand, by (I) and (III),

$$|F(\Omega^{(k)}\alpha)|\ge c_4|(\Omega^{(k)}\alpha)^{\lambda_0}|\ge c_4\Big(\prod_{i=1}^n|\alpha_i|\Big)^{c_1|\lambda_0|r_k}.$$

This implies the lemma.

We assume $f_1^{(0)}(\alpha),\ldots,f_m^{(0)}(\alpha)$ are algebraically dependent and deduce a contradiction. There exist a positive integer L and integers τ_μ, not all zero, for $\mu=(\mu_1,\ldots,\mu_m)$ with $0\le\mu_i\le L$ such that

$$\sum_\mu\tau_\mu f_1^{(0)}(\alpha)^{\mu_1}\ldots f_m^{(0)}(\alpha)^{\mu_m}=0.$$

Let $w_1,\ldots,w_m,y_1,\ldots,y_m$ and t_μ $(\mu=(\mu_1,\ldots,\mu_m),\ 0\le\mu_i\le L)$ be variables and put

$$F^{(k)}(z;t)=\sum_\mu t_\mu f_1^{(k)}(z)^{\mu_1}\ldots f_m^{(k)}(z)^{\mu_m},$$

$$F(z;t)=\sum_\mu t_\mu f_1(z)^{\mu_1}\ldots f_m(z)^{\mu_m}$$

and

$$\sum_{\mu} t_\mu (w_1 + y_1)^{\mu_1} \dots (w_m + y_m)^{\mu_m} = \sum_{\mu} T_\mu(t; y) w_1^{\mu_1} \dots w_m^{\mu_m}.$$

Then we obtain

$$\begin{aligned} 0 = F^{(0)}(\alpha; \tau) &= \sum_{\mu} \tau_\mu (f_1^{(k)}(\Omega^{(k)}\alpha) + b_1^{(k)})^{\mu_1} \dots (f_m^{(k)}(\Omega^{(k)}\alpha) + b_m^{(k)})^{\mu_m} \\ &= \sum_{\mu} T_\mu(\tau; b^{(k)}) f_1^{(k)}(\Omega^{(k)}\alpha)^{\mu_1} \dots f_m^{(k)}(\Omega^{(k)}\alpha)^{\mu_m} \\ &= F^{(k)}(\Omega^{(k)}\alpha; T(\tau; b^{(k)})). \end{aligned}$$

We put $R = K[t] = K[\{t_\mu\}_{\mu=(\mu_1,\dots,\mu_m),\ 0 \le \mu_i \le L}]$ and

$$V(\tau) = \{Q(t) \in R \mid Q(T(\tau; y)) = 0\}.$$

Then $V(\tau)$ is a prime ideal of R.

Definition. For $P(z; t) = \sum_\lambda P_\lambda(t) z^\lambda \in R[[z_1, \dots, z_n]]$, we define

$$\operatorname{index} P(z; t) = \min\{|\lambda| \mid P_\lambda \notin V(\tau)\}.$$

If $P_\lambda(t) \in V(\tau)$ for any λ, then we define $\operatorname{index} P(z; t) = \infty$.

Since $R/V(\tau)$ is an integral domain, we have

$$\operatorname{index} P_1(z; t) P_2(z; t) = \operatorname{index} P_1(z; t) + \operatorname{index} P_2(z; t).$$

Lemma 3. *The following two properties are equivalent for any $P(z; t) \in R[z]$.*

(i) $P(\Omega^{(k)}\alpha; T(\tau; b^{(k)})) = 0$ *for all large* k.
(ii) $\operatorname{index} P(z; t) = \infty$.

Proof. We put

$$P(z; t) = \sum_{\lambda} Q_\lambda(t) z^\lambda, \qquad Q_\lambda(t) \in R,$$

and

$$Q_\lambda(T(\tau; f^{(0)}(\alpha) - w)) = \sum_{\mu} a_{\lambda\mu} w_1^{\mu_1} \dots w_m^{\mu_m}.$$

We assume (i). Since $b_i^{(k)} = f_i^{(0)}(\alpha) - f_i^{(k)}(\Omega^{(k)}\alpha)$, we have

$$\begin{aligned} 0 &= P(\Omega^{(k)}\alpha; T(\tau; b^{(k)})) \\ &= \sum_{\lambda} \sum_{\mu} a_{\lambda\mu} (\Omega^{(k)}\alpha)^\lambda f_1^{(k)}(\Omega^{(k)}\alpha)^{\mu_1} \dots f_m^{(k)}(\Omega^{(k)}\alpha)^{\mu_m}, \end{aligned}$$

for all large k. Lemma 2 implies $a_{\lambda\mu} = 0$ for all λ, μ. Hence

$$Q_\lambda(T(\tau; f^{(0)}(\alpha) - w)) = 0.$$

Since $w_1, \ldots, w_m$ are variables, $Q_\lambda(T(\tau; y)) = 0$, which implies (ii). The opposite is trivial.

LEMMA 4. index $F(z;t) < \infty$.

Proof. By the property (III), there exists k_0 such that $F(\Omega^{(k_0)}\alpha; \tau) \neq 0$. If index $F(z;t) = \infty$, then

$$F(z;t) = \sum_\lambda P_\lambda(t) z^\lambda, \qquad P_\lambda(t) \in V(\tau).$$

Noting $T_\mu(\tau; 0) = \tau_\mu$, we have

$$F(\Omega^{(k_0)}\alpha; \tau) = \sum_\lambda P_\lambda(\tau)(\Omega^{(k_0)}\alpha)^\lambda = 0,$$

which is a contradiction.

For a positive integer p, we define

$$\begin{aligned} R(p) &= \{g(t) \in R \mid \deg_{t_\mu} g(t) \leq p\}, \\ \overline{R(p)} &= R(p)/R(p) \cap V(\tau), \\ d(p) &= \dim_K \overline{R(p)}. \end{aligned}$$

LEMMA 5. $d(2p) \leq 2^{(L+1)^m} d(p)$.

Proof. If $P(t) \in R(2p)$, it can be expressed as

$$P(t) = \sum_\varepsilon Q_\varepsilon(t) \prod_\mu t_\mu^{\varepsilon(\mu)p},$$

where $Q_\varepsilon(t) \in R(p)$, ε is a mapping from the set of μ to $\{0, 1\}$ and the sum is taken over all such mappings. If $\{\overline{Q_1(t)}, \ldots, \overline{Q_{d(p)}(t)}\}$ is a base of $\overline{R(p)}$, then the set

$$\Big\{\overline{Q_i(t) \prod_\mu t_\mu^{\varepsilon(\mu)p}}\Big\}_{1 \leq i \leq d(p),\, \varepsilon}$$

generates $\overline{R(2p)}$ and the lemma is proved.

LEMMA 6. *Let p be a sufficiently large integer. Then there exist polynomials $P_0(z;t), \ldots, P_p(z;t) \in K[z;t]$ with degree at most p in each variable such that the following properties are satisfied.*

(i) index $P_0(z;t) < \infty$.
(ii) *If we put $E_p(z;t) = \sum_{h=0}^p P_h(z;t) F(z;t)^h$, then*

$$\operatorname{index} E_p(z;t) \geq c_7 (p+1)^{1+1/n}.$$

Proof. If we express

$$P_h(z;t) = \sum_\lambda P_{h\lambda}(t) z^\lambda, \qquad h = 0, \ldots, p,$$

$$F(z;t)^h = \sum_{\lambda} Q_{h\lambda}(t) z^{\lambda}, \quad h = 0, \dots, p,$$

then

$$\sum_{h=0}^{p} P_h(z;t) F(z;t)^h = \sum_{\nu} \Big(\sum_{h,\lambda,\mu,\lambda+\mu=\nu} P_{h\lambda}(t) Q_{h\mu}(t) \Big) z^{\nu}.$$

We will choose $P_{h\lambda}(t)$ satisfying

$$\sum_{h,\lambda,\mu,\lambda+\mu=\nu} \overline{P_{h\lambda}(t)}\,\overline{Q_{h\mu}(t)} = \overline{0} \quad \text{in } \overline{R(2p)},$$

for any $\nu = (\nu_1, \dots, \nu_n)$ $(\nu_i \le J-1)$, where J will be defined below. We define a linear map from $\overline{R(p)}^{(p+1)^{n+1}}$ to $\overline{R(2p)}^{J^n}$ by

$$(\overline{P_{h\lambda}(t)})_{h,\lambda} \mapsto \Big(\sum_{h,\lambda,\mu,\lambda+\mu=\nu} \overline{P_{h\lambda}(t)}\,\overline{Q_{h\mu}(t)} \Big)_{\nu}.$$

Since

$$\dim_K \overline{R(p)}^{(p+1)^{n+1}} = d(p)(p+1)^{n+1}, \quad \dim_K \overline{R(2p)}^{J^n} = d(2p)J^n,$$

if $d(p)(p+1)^{n+1} > d(2p)J^n$, then there is a nontrivial solution $(\overline{P_{h\lambda}(t)})_{h,\lambda}$. By Lemma 5, $J = [2^{-(L+1)^m/n}(p+1)^{1+1/n}] - 1$ satisfies the inequality and

$$\operatorname{index} \Big(\sum_{h=0}^{p} P_h(z;t) F(z;t)^h \Big) \ge J \ge c_8 (p+1)^{1+1/n}.$$

If $\operatorname{index} P_0(z;t) < \infty$, the proof is complete. Otherwise, we set

$$r = \min\{h \mid \operatorname{index} P_h(z;t) < \infty\}, \quad E_p(z;t) = \sum_{h=r}^{p} P_h(z;t) F(z;t)^{h-r}.$$

Since $\operatorname{index} E_p(z;t) F(z;t)^r \ge J$, we have

$$\operatorname{index} E_p(z;t) \ge J - r \operatorname{index} F(z;t) \ge c_7 (p+1)^{1+1/n}.$$

Now we can complete the proof of Theorem 2. Let $\operatorname{index} E_p(z;t) = I$ and $\gamma_1, \gamma_2, \dots$ denote positive constants depending on $E_p(z;t)$. Let $k \ge \gamma_1$, where γ_1 will be determined below. Let

$$E_p(z;t) = \sum_{\nu} g_{\nu}(z) t^{\nu}, \quad g_{\nu}(z) = \sum_{\lambda} g_{\nu\lambda} z^{\lambda}.$$

Then $g_{\nu}(z)$ converges in the n-polydisc with radius 1 around the origin. Since

$$\lim_{k\to\infty} f_i^{(k)}(\Omega^{(k)}\alpha) = f_i(0),$$

we have

$$|b_i^{(k)}|, |T_{\mu}(\tau; b^{(k)})| \le c_9.$$

Thus by Lemma 1,

$$|E_p(\Omega^{(k)}\alpha;T(\tau;b^{(k)}))| \le \sum_{\nu}\Big(\sum_{|\lambda|\ge I}|g_{\nu\lambda}|\cdot|(\Omega^{(k)}\alpha)^{\lambda}|\Big)|T(\tau;b^{(k)})^{\nu}|$$
$$\le \gamma_2\max_i|\alpha_i^{(k)}|^I.$$

We choose a positive number θ with $e^{-c_2c_7}<\theta<1$. By the property (I) we have

$$|E_p(\Omega^{(k)}\alpha;T(\tau;b^{(k)}))| \le \frac{1}{2}\theta^{r_k(p+1)^{1+1/n}}.$$

We put

$$E_p^{(k)}(z;t)=\sum_{h=0}^{p}P_h(z;t)F^{(k)}(z;t)^h,$$

and choose a large H satisfying

$$e^{-c_2H}\le\theta\cdot\theta^{(p+1)^{1+1/n}}.$$

If $f_i^{(k)}-f_i\in\chi^H$, by Lemma 2(i) we have

$$|E_p(\Omega^{(k)}\alpha;T(\tau;b^{(k)}))-E_p^{(k)}(\Omega^{(k)}\alpha;T(\tau;b^{(k)}))|\le\gamma_3e^{-c_2Hr_k}.$$

Then

$$|E_p^{(k)}(\Omega^{(k)}\alpha;T(\tau;b^{(k)}))|\le\gamma_3e^{-c_2Hr_k}+\frac{1}{2}\theta^{r_k(p+1)^{1+1/n}}\le\theta^{r_k(p+1)^{1+1/n}}.$$

On the other hand,

$$E_p^{(k)}(\Omega^{(k)}\alpha;T(\tau;b^{(k)}))=P_0(\Omega^{(k)}\alpha;T(\tau;b^{(k)}))=\text{(say) }\beta_k\in K.$$

By the properties (I) and (II), we easily see $\|\beta_k\|\le c_{10}^{r_kp}$. By the fact that $\operatorname{index}P_0(z;t)<\infty$, there are infinitely many k with $\beta_k\ne0$. For such k, we have

$$r_k(p+1)^{1+1/n}\log\theta\ge\log|\beta_k|\ge-2[K:\mathbb{Q}]\log\|\beta_k\|\ge-2[K:\mathbb{Q}]r_kp\log c_{10}.$$

Dividing both sides by $r_k(p+1)^{1+1/n}$ and letting p tend to ∞, we obtain $\log\theta\ge0$, a contradiction.

3. Proof of Theorem 1. The following lemma is proved in a similar way to the proof of Lemma A.1 in Masser [1].

LEMMA 7. *Let $b_1>\ldots>b_n\ge2$ be pairwise multiplicatively independent integers. Let $\theta=\log b_1$ and $\theta_i=\theta/\log b_i$. Suppose that for each α in a finite set A we are given real numbers $\lambda_{1\alpha},\ldots,\lambda_{n\alpha}$ not all zero, and define the sequence*

$$S_\alpha(k)=\sum_{i=1}^{n}\lambda_{i\alpha}b_i^{[\theta_ik]},\qquad k=0,1,2,\ldots$$

If $\{k_l\}_{l\ge1}$ *is an increasing sequence of positive integers with* $\{k_{l+1}-k_l\}_{l\ge1}$ *bounded*, *then there exists* $\delta>0$ *such that*

$$R(\delta)=\{k_l \mid \min_\alpha |S_\alpha(k_l)| \ge \delta e^{\theta k_l}\}=\{m_l\}_{l\ge1}, \qquad m_l<m_{l+1},$$

is an infinite set and $\{m_{l+1}-m_l\}_{l\ge1}$ *is bounded.*

Proof. Let $k_{l+1}-k_l\le K$, $l\ge1$. We prove the lemma by induction on n. If $n=1$, then $\{m_l\}_{l\ge1}=\{k_l\}_{l\ge1}$ is the required sequence. Assume that we have proved the result with n replaced by $n-1$ for some $n\ge2$ and the result is not true for n. Then for any $\delta>0$ and any positive integer M there is k_l such that for $k=k_l,k_{l+1},\dots,k_{l+M}$ we have

$$S(k)=\min_\alpha |S_\alpha(k)|<\delta e^{\theta k}.$$

We may assume that for each $\alpha\in A$ the numbers $\lambda_{1\alpha},\dots,\lambda_{n-1,\alpha}$ are not all zero. Let $L=(\max_i\theta_i)|A|K+1$ and

$$J=\{(p_1,\dots,p_n,q_1,\dots,q_n)\mid 0\le p_i,q_i\le L, p_n\ne q_n\}.$$

We take $B=A\times J$ and for each $\beta=(\alpha,p_1,\dots,p_n,q_1,\dots,q_n)\in B$ we define

$$\mu_{i\beta}=\lambda_{i\alpha}(b_n^{q_n}b_i^{p_i}-b_n^{p_n}b_i^{q_i}), \qquad 1\le i\le n-1.$$

Since $p_n\ne q_n$, the pairwise multiplicative independence shows that $\mu_{1\beta},\dots,$ $\mu_{n-1,\beta}$ are not all zero. We define

$$T_\beta(k)=\sum_{i=1}^{n-1}\mu_{i\beta}b_i^{[\theta_i k]}, \qquad k=0,1,\dots$$

For any positive integer k there is $\alpha=\alpha(k)\in A$ such that $S(k)=|S_\alpha(k)|$. By the Box Principle, for any j with $l\le j\le l+M-|A|$ there exist $\alpha\in A$ and integers l_1,l_2 such that $j\le l_1<l_2\le j+|A|$ and

$$S(k_{l_1})=|S_\alpha(k_{l_1})|, \qquad S(k_{l_2})=|S_\alpha(k_{l_2})|.$$

Put

$$p_i=[\theta_i k_{l_1}]-[\theta_i k_j], \qquad q_i=[\theta_i k_{l_2}]-[\theta_i k_j].$$

Then $0\le p_i,q_i\le L$. Since $\theta_n>1$ and $l_1<l_2$ imply $p_n<q_n$, we have $\beta=(\alpha,p_1,\dots,p_n,q_1,\dots,q_n)\in B$ and

$$T_\beta(k_j)=b_n^{q_n}S_\alpha(k_{l_1})-b_n^{p_n}S_\alpha(k_{l_2}).$$

By the assumption, for $j=l,l+1,\dots,l+M-|A|$ we have

$$|T_\beta(k_j)|<c\delta e^{\theta k_j},$$

where c is a positive constant. This contradicts the induction hypothesis.

Lemma 8. *Let* $b_1,\dots,b_n$ *be integers as in Lemma* 7. *Then there exist an infinite set* Λ *of positive integers*, *a sequence* $\{\delta(l)\}_{l\ge1}$ *of positive numbers* *and a total order in* $(\mathbb{N}_0)^n$ *such that if* $\lambda > \mu,\ |\lambda|, |\mu| \le l$, *then*

$$\sum_{i=1}^{n} \lambda_i b_i^{[\theta_i q]} - \sum_{i=1}^{n} \mu_i b_i^{[\theta_i q]} \ge \delta(l) e^{\theta q}$$

for every sufficiently large $q \in \Lambda$. *Moreover any subset of* $(\mathbb{N}_0)^n$ *has the least element.*

Proof. We put

$$A(l) = \{(\lambda, \mu) \mid \lambda, \mu \in (\mathbb{N}_0)^n,\ |\lambda|, |\mu| \le l,\ \lambda \ne \mu\}.$$

For $(\lambda, \mu) \in A(l)$ we set

$$S_{(\lambda,\mu)}(q) = \sum_{i=1}^{n} (\lambda_i - \mu_i) b_i^{[\theta_i q]}.$$

We inductively define $\delta(l)$ and $\Lambda(l)$ as follows. We put $\Lambda(0) = \mathbb{N}$. By Lemma 7 there exists a positive number $\delta(l)$ such that

$$\Lambda(l) = \{q \in \Lambda(l-1) \mid \min_{(\lambda,\mu)\in A(l)} |S_{(\lambda,\mu)}(q)| \ge \delta(l) e^{\theta q}\}$$

is an infinite set and the differences of two consecutive elements of $\Lambda(l)$ are bounded. We can choose a sequence $\{q_l\}_{l\ge 1}$ satisfying $q_l \in \Lambda(l)$ and $q_l < q_{l+1}$. There exists a subsequence $\{q_l^{(1)}\}_{l\ge1}$ of $\{q_l\}_{l\ge1}$ such that the signs of $S_{(\lambda,\mu)}(q_l^{(1)})$, $|\lambda|, |\mu| \le 1$, are fixed for all $l \ge 1$. There exists a subsequence $\{q_l^{(2)}\}_{l\ge2}$ of $\{q_l^{(1)}\}_{l\ge2}$ such that the signs of $S_{(\lambda,\mu)}(q_l^{(2)})$, $|\lambda|, |\mu| \le 2$, are fixed for all $l \ge 2$. Continuing this process, we obtain a sequence $\{q_l^{(m)}\}_{l\ge m}$ for every $m \ge 1$. We set

$$\Lambda = \{q_1^{(1)}, q_2^{(2)}, \ldots, q_l^{(l)}, \ldots\},$$

and for $\lambda, \mu \in (\mathbb{N}_0)^n$ we define $\lambda > \mu$ if and only if $S_{(\lambda,\mu)}(q) > 0$ for all large $q \in \Lambda$. Noting $\Lambda(l) \supset \Lambda(l+1)$ and $q_l^{(l)} \in \Lambda(l)$ completes the proof of the first part of the lemma. For the second part, we use the following fact (cf. [2], Lemma 2.6.4): if S is a subset of $(\mathbb{N}_0)^n$, then there is a finite subset T of S such that for any $(\lambda_1, \ldots, \lambda_n) \in S$, there is an element $(\mu_1, \ldots, \mu_n) \in T$ with $\mu_i \le \lambda_i,\ i = 1, \ldots, n$. If μ is the least element of T, we can easily see it is also the least element of S.

Lemma 9. *Let* $d \ge 2$ *and*

$$f_j(z) = \sum_{h=0}^{\infty} s_{j,jh} z^{d^{jh}}, \qquad s_{j,jh} \in \mathbb{C}^{\times},\ j = 1, 2, \ldots$$

Then $f_j,\ j = 1, 2, \ldots$, *are algebraically independent over* $\mathbb{C}(z)$.

Proof. If $f_1, \dots, f_t$ are algebraically dependent over $\mathbb{C}(z)$, then there exist $a_{\lambda\mu} \in \mathbb{C}$, not all zero, such that

$$F(z) = \sum_{\lambda,\, \mu=(\mu_1,\dots,\mu_t)} a_{\lambda\mu} z^\lambda f_1(z)^{\mu_1} \dots f_t(z)^{\mu_t} = 0.$$

We choose a positive integer l satisfying

$$\max\{\lambda \mid a_{\lambda\mu} \neq 0 \text{ for some } \mu\} < d^l.$$

We define

$$M = \max\{|\mu| \mid a_{\lambda\mu} \neq 0 \text{ for some } \lambda\} \geq 1,$$
$$A = \{\mu \mid |\mu| = M,\ a_{\lambda\mu} \neq 0 \text{ for some } \lambda\}.$$

Let $\nu = (\nu_1, \dots, \nu_t)$ be the largest element of A for the lexicographical order and κ be the largest integer such that $a_{\kappa\nu} \neq 0$. Letting

$$p = \kappa + d^{t!l+1} + d^{t!2l+1} + \dots + d^{t!\nu_1 l+1} + d^{t!(\nu_1+1)l+2} + \dots + d^{t!(\nu_1+\nu_2)l+2}$$
$$+ d^{t!(\nu_1+\dots+\nu_{t-1}+1)l+t} + \dots + d^{t!(\nu_1+\dots+\nu_t)l+t},$$

we will show that the Taylor coefficient of z^p in $F(z)$ is not zero. This contradicts $F(z) = 0$ and completes the proof.

The d-adic expansion of p has the form

$$* \dots * 0\ e_{l-1} \dots e_0, \qquad 0 \leq e_i < d.$$

If a positive integer n has the d-adic expansion

$$e_L \dots e_{l+1} e_l \dots e_1 e_0, \qquad 0 \leq e_i < d,$$

we denote by $w(n)$ the number of nonzero elements among $e_L, \dots, e_{l+1}$. Then $w(p) = \nu_1 + \dots + \nu_t = M$. For any $a, b \in \mathbb{N}_0$, we see $w(a + d^b) \leq w(a) + 1$. If q is the degree of a term appearing in the development of

$$a_{\lambda\mu} z^\lambda f_1(z)^{\mu_1} \dots f_t(z)^{\mu_t},$$

then

$$q = \lambda + d^{h_1} + \dots + d^{h_{\mu_1}} + d^{2h_{\mu_1+1}} + \dots + d^{2h_{\mu_1+\mu_2}}$$
$$+ d^{th_{\mu_1+\dots+\mu_{t-1}+1}} + \dots + d^{th_{\mu_1+\dots+\mu_t}},$$

where $h_i \in \mathbb{N}_0$. If $p = q$,

$$M = w(p) = w(q) \leq w(\lambda) + \mu_1 + \dots + \mu_t = |\mu| \leq M,$$

and so $|\mu| = M$. If $w(\lambda + d^{jh_i}) = 0$, $w(q) \leq M - 1$. Therefore $w(\lambda + d^{jh_i}) = 1$ and so $jh_i \geq l + 1$ since $\lambda < d^l$. Hence we have $\lambda = \kappa$. If $jh_i = j'h_{i'}$ for some $(i, j) \neq (i', j')$, then $w(\lambda + d^{jh_i} + d^{j'h_{i'}}) = 1$. This implies $w(q) \leq M - 1$, a contradiction. Therefore jh_i are distinct and

$$\{t!l + 1, \dots, t!\nu_1 l + 1, t!(\nu_1 + 1)l + 2, \dots, t!(\nu_1 + \nu_2)l + 2,$$
$$\dots, t!(\nu_1 + \dots + \nu_{t-1} + 1)l + t, \dots, t!|\nu|l + t\}$$

$$= \{h_1,\dots,h_{\mu_1},2h_{\mu_1+1},\dots,2h_{\mu_1+\mu_2},\dots,th_{\mu_1+\ldots+\mu_{t-1}+1},\dots,th_{|\mu|}\}.$$

There are exactly ν_1 elements which are not divided by any of $2,\dots,t$ on both sides above. Therefore $\mu_1 \ge \nu_1$, which implies $\mu_1 = \nu_1$ since $\mu \le \nu$. Then

$$\{t!(\nu_1+1)l+2,\dots,t!(\nu_1+\nu_2)l+2,\dots,t!(\nu_1+\ldots+\nu_{t-1}+1)l+t,\dots,t!|\nu|l+t\} \\ = \{2h_{\mu_1+1},\dots,2h_{\mu_1+\mu_2},\dots,th_{\mu_1+\ldots+\mu_{t-1}+1},\dots,th_{|\mu|}\}.$$

There are exactly ν_2 elements which are not divided by any of $3,\dots,t$ on both sides above. Therefore $\mu_2 \ge \nu_2$, which implies $\mu_2 = \nu_2$ since $\mu \le \nu$ and $\mu_1 \le \nu_1$. Continuing, we obtain $\mu = \nu$. Therefore the Taylor coefficient of z^p in $F(z)$ is

$$a_{\kappa\nu}\nu_1!\dots\nu_t!s_{1,t!l+1}\dots s_{1,t!\nu_1 l+1}\dots s_{t,t!(\nu_1+\ldots+\nu_{t-1}+1)l+t}\dots s_{t,t!|\nu|l+t} \ne 0.$$

Proof of Theorem 1. Let

$$D = \{d \in \mathbb{N} \mid d \ne a^n \text{ for any } a,n \in \mathbb{N}, n > 1\}.$$

Then

$$\mathbb{N}\setminus\{1\} = \bigcup_{d\in D}\{d,d^2,\dots\} \quad \text{(disjoint union)}$$

and any two elements of D are multiplicatively independent. Let $d_1 > \ldots > d_n$ be elements of D, $z = (z_1,\dots,z_n)$ and

$$f_{ij}^{(0)}(z) = \sum_{h=0}^{\infty}\sigma_{ijh}z_i^{d_i^{jh}}, \quad i=1,\dots,n,\ j=1,\dots,t,$$

where for any h, $\sigma_{ijh} \in S_{ij}$, which is a finite set of nonzero algebraic numbers. We will show that

$$\sum_{h=0}^{\infty}\sigma_{ijh}\alpha^{d_i^{jh}}, \quad i=1,\dots,n,\ j=1,\dots,t,$$

are algebraically independent for any algebraic α with $0 < |\alpha| < 1$. This implies Theorem 1. Put $b_i = d_i^{t!}$, $\theta = \log b_1$, $\theta_i = \theta/\log b_i$, $i=1,\dots,n$, and

$$\Sigma_q = (\sigma_{ijh})_{i=1,\dots,n,\,j=1,\dots,t,\,h\ge(t!/j)[\theta_i q]} \in \prod_{i=1}^{n}\prod_{j=1}^{t}S_{ij}^{\mathbb{N}}$$

for $q \in \Lambda$ (in Lemma 8). Since the right hand side is a compact set, there exists a converging subsequence $\{\Sigma_{q_k}\}_{k\ge1}$ of $\{\Sigma_q\}_{q\in\Lambda}$. Let

$$\lim_{k\to\infty}\Sigma_{q_k} = (s_{ijh})_{i=1,\dots,n,\,j=1,\dots,t,\,h\ge0}$$

and

$$f_{ij}^{(k)}(z) = \sum_{h=(t!/j)[\theta_i q_k]}^{\infty}\sigma_{ijh}z_i^{d_i^{j(h-(t!/j)[\theta_i q_k])}}, \quad f_{ij}(z) = \sum_{h=0}^{\infty}s_{ijh}z_i^{d_i^{jh}}.$$

Then

$$\lim_{k\to\infty} f_{ij}^{(k)}(z) = f_{ij}(z).$$

We define

$$\Omega^{(k)} = \begin{pmatrix} b_1^{[\theta_1 q_k]} & & 0 \\ & \ddots & \\ 0 & & b_n^{[\theta_n q_k]} \end{pmatrix}.$$

Then $\Omega^{(k)}$, $(\alpha_1, \dots, \alpha_n) = (\alpha, \dots, \alpha)$ and $r_k = b_1^{q_k}$ satisfy the assumptions (I) and (II). If the assumption (III) is also satisfied, the assertion follows. Noting $z_1, \dots, z_n$ are distinct variables, by Lemma 9 we see

$$f_{11}, \dots, f_{1t}, \dots, f_{n1}, \dots, f_{nt}$$

are algebraically independent over $\mathbb{C}(z_1, \dots, z_n)$. Let

$$F(z) = \sum_{\lambda,\mu} a_{\lambda\mu} z^\lambda f_{11}^{\mu_{11}} \dots f_{1t}^{\mu_{1t}} \dots f_{n1}^{\mu_{n1}} \dots f_{nt}^{\mu_{nt}} = \sum_{\lambda\in(\mathbb{N}_0)^n} c_\lambda z^\lambda$$

and λ_0 be the least element in $(\mathbb{N}_0)^n$ in the order defined in Lemma 8 among λ with $c_\lambda \neq 0$. Let $B = \max\{b_1, \dots, b_n\}$ and $l = (|\lambda_0| + 1)B$. Then

$$B^{-1} b_1^{q_k} \le b_i^{-1} b_1^{q_k} < b_i^{[\theta_i q_k]} \le b_1^{q_k}.$$

If k is sufficiently large, then by Lemma 1,

$$\sum_{|\lambda|\ge l} |c_\lambda| \cdot |\alpha|^{b_1^{[\theta_1 q_k]}\lambda_1} \dots |\alpha|^{b_n^{[\theta_n q_k]}\lambda_n} \le \gamma^{l+1} |\alpha|^{b_1^{q_k}(|\lambda_0|+1)}.$$

Since

$$\lambda_{01} b_1^{[\theta_1 q_k]} + \dots + \lambda_{0n} b_n^{[\theta_n q_k]} \le |\lambda_0| b_1^{q_k},$$

we have

$$\frac{|\sum_{|\lambda|\ge l} c_\lambda (\Omega^{(k)}\alpha)^\lambda|}{|(\Omega^{(k)}\alpha)^{\lambda_0}|} \le \gamma^{l+1} |\alpha|^{b_1^{q_k}}$$

if k is sufficiently large. If $|\lambda| < l$ and $\lambda \neq \lambda_0$, then by Lemma 8,

$$\frac{|c_\lambda (\Omega^{(k)}\alpha)^\lambda|}{|(\Omega^{(k)}\alpha)^{\lambda_0}|} \le |c_\lambda| \cdot |\alpha|^{\delta(l) e^{\theta q_k}}$$

for all large k. Therefore

$$|F(\Omega^{(k)}\alpha)/(\Omega^{(k)}\alpha)^{\lambda_0} - c_{\lambda_0}| \to 0 \quad (k \to \infty).$$

This implies (III).

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Mathematics, Hiyoshi Campus
Keio University
4-1-1 Hiyoshi, Kohoku-ku
Yokohama 223-8521, Japan
E-mail: nishioka@math.hc.keio.ac.jp

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