Algebraic independence of Fredholm series

by

KUMIKO NISHIOKA (Yokohama)

1. Introduction. Let $K$ be an algebraic number field and $d \geq 2$ be an integer. We call

$$f(z) = \sum_{h=0}^{\infty} \sigma_h z^{dh}, \quad \sigma_h \in K^\times, \log \|\sigma_h\| = o(d^h),$$

a Fredholm series. The convergence radius of $f(z)$ is 1. By Hadamard’s gap theorem, the unit circle is the natural boundary of $f(z)$. If $\alpha$ is an algebraic number with $0 < |\alpha| < 1$, then $f(\alpha)$ is transcendental (cf. Theorem 2.10.1 in Nishioka [2]). Let

$$f_d(z) = \sum_{h=0}^{\infty} \sigma_{dh} z^{dh}, \quad \sigma_{dh} \in K^\times, \log \|\sigma_{dh}\| = o(d^h), \quad d = 2, 3, \ldots$$

Then we may expect that $f_d(\alpha), d = 2, 3, \ldots$, are algebraically independent. When $\sigma_{dh} = 1$ for all $d, h$, this is proved in Nishioka [3]. Here we will prove the following.

**Theorem 1.** If for every $d$, the $\sigma_{dh}$ ($h = 0, 1, \ldots$) are in a finite set of nonzero algebraic numbers, then $f_d(\alpha), d = 2, 3, \ldots$, are algebraically independent for any algebraic number $\alpha$ with $0 < |\alpha| < 1$.

2. Mahler’s method. By $\mathbb{N}$ and $\mathbb{N}_0$ we denote the set of positive integers and the set of nonnegative integers respectively. If $\alpha$ is an algebraic number, we denote by $|\alpha|$ the maximum of the absolute values of the conjugates of $\alpha$ and by $\text{den}(\alpha)$ the least positive integer such that $\text{den}(\alpha)\alpha$ is an algebraic integer, and we set $||\alpha|| = \max\{|\alpha|, \text{den}(\alpha)\}$. Then we have the inequalities

$$|\alpha| \geq ||\alpha||^{2[\mathbb{Q}(\alpha) : \mathbb{Q}]} \quad \text{and} \quad ||\alpha^{-1}|| \leq ||\alpha||^{2[\mathbb{Q}(\alpha) : \mathbb{Q}]}$$

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(cf. Lemma 2.10.2 in [2]). If $\Omega = (\omega_{ij})$ is an $n \times n$ matrix with nonnegative integer entries and $z = (z_1, \ldots, z_n)$ is a point of $\mathbb{C}^n$, we define a transformation $\Omega : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\Omega z = \left( \prod_{j=1}^{n} z_j^{\omega_{1j}}, \ldots, \prod_{j=1}^{n} z_j^{\omega_{nj}} \right).$$

Let $\{\Omega^{(k)}\}_{k \geq 0}$ be a sequence of matrices with nonnegative integer entries. We put

$$\Omega^{(k)} = (\omega_{ij}^{(k)}) \quad \text{and} \quad \Omega^{(k)} z = (z_1^{(k)}, \ldots, z_n^{(k)}).$$

For $\lambda = (\lambda_1, \ldots, \lambda_n)$, we define $z^\lambda = z_1^{\lambda_1} \ldots z_n^{\lambda_n}$ and $|\lambda| = \lambda_1 + \ldots + \lambda_n$. Let $\{f_1^{(k)}(z)\}_{k \geq 0}, \ldots, \{f_m^{(k)}(z)\}_{k \geq 0}$ be sequences of power series in $K[[z_1, \ldots, z_n]]$. Let $\chi = (z_1, \ldots, z_n)$ be the ideal generated by $z_1, \ldots, z_n$ in $K[[z_1, \ldots, z_n]]$. We assume

$$f_i^{(k)} \to f_i \quad (k \to \infty), \quad i = 1, \ldots, m,$$

under the topology defined by $\chi$. In what follows, $c_1, c_2, \ldots$ denote positive constants independent of $k$.

**Theorem 2.** Suppose that the coefficients of $f_i^{(k)}$ are in a finite set $S \subseteq K$ for all $i$ and $k$. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in K^n$, $0 < |\alpha_i| < 1$, $i = 1, \ldots, n$, and the following three properties are satisfied, then $f_1^{(0)}(\alpha), \ldots, f_m^{(0)}(\alpha)$ are algebraically independent.

1. **(I)** There exists a sequence $\{r_k\}_{k \geq 0}$ of positive numbers such that

$$\lim_{k \to \infty} r_k = \infty, \quad \omega_{ij}^{(k)} \leq c_1 r_k, \quad \log |\alpha_i^{(k)}| \leq -c_2 r_k.$$

2. **(II)** If we put

$$f_i^{(0)}(\alpha) = f_i^{(k)}(\Omega^{(k)} \alpha) + b_i^{(k)},$$

then $b_i^{(k)} \in K$ and

$$\log \|b_i^{(k)}\| \leq c_3 r_k.$$

3. **(III)** For any power series $F(z)$ represented as a polynomial in $z_1, \ldots, z_n$, $f_1, \ldots, f_m$ with complex coefficients,

$$F(z) = \sum_{\lambda, \mu = (\mu_1, \ldots, \mu_m)} a_{\lambda \mu} z^\lambda f_1(z)^{\mu_1} \ldots f_m(z)^{\mu_m},$$

where $a_{\lambda \mu}$ are not all zero, there exists $\lambda_0 \in (\mathbb{N}_0)^n$ such that if $k$ is sufficiently large, then

$$|F(\Omega^{(k)} \alpha)| \geq c_4 |(\Omega^{(k)} \alpha)^{\lambda_0}|.$$

**Proof of Theorem 2.** The following lemma is easy to prove.
Lemma 1. Let \( f(z) = \sum_{\lambda_1, \ldots, \lambda_n} c_{\lambda_1 \ldots \lambda_n} z_1^{\lambda_1} \ldots z_n^{\lambda_n} \in \mathbb{C}[[z_1, \ldots, z_n]] \) converge around the origin. If \( z \) is sufficiently close to the origin, then
\[
\sum_{|\lambda| \geq H} |c_{\lambda_1 \ldots \lambda_n}| \cdot |z_1|^{\lambda_1} \ldots |z_n|^{\lambda_n} \leq \gamma^{H+1} \max_i |z_i|^H,
\]
where \( \gamma \) is a positive constant depending on \( f(z) \).

Lemma 2. (i) If \( f_i^{(k)} - f_i \in \chi^H \), then
\[
|f_i^{(k)}(\Omega^{(k)} \alpha) - f_i(\Omega^{(k)} \alpha)| \leq c_5^{H+1} e^{-c_2 r_k H}.
\]

(ii) For \( F(z) \) in (III) we put
\[
F^{(k)}(z) = \sum_{\lambda, \mu=(\mu_1, \ldots, \mu_m)} a_{\lambda \mu} z^{\lambda} f_1^{(k)}(z)^{\mu_1} \ldots f_m^{(k)}(z)^{\mu_m}.
\]
Then \( F^{(k)}(\Omega^{(k)} \alpha) \neq 0 \) if \( k \) is sufficiently large.

Proof. The assumption (I) and Lemma 1 imply (i). We choose a large \( H \) satisfying
\[
e^{-c_2 H} < \left( \prod_{i=1}^n |\alpha_i| \right)^{c_1 |\lambda_0|}.
\]
Using (i) we have
\[
|F^{(k)}(\Omega^{(k)} \alpha) - F(\Omega^{(k)} \alpha)| \leq c_6 e^{-c_2 H r_k} \leq \frac{1}{2} c_4 \left( \prod_{i=1}^n |\alpha_i| \right)^{c_1 |\lambda_0| r_k}
\]
if \( k \) is sufficiently large. On the other hand, by (I) and (III),
\[
|F(\Omega^{(k)} \alpha)| \geq c_4 |(\Omega^{(k)} \alpha)^{\lambda_0}| \geq c_4 \left( \prod_{i=1}^n |\alpha_i| \right)^{c_1 |\lambda_0| r_k}.
\]
This implies the lemma.

We assume \( f_1^{(0)}(\alpha), \ldots, f_m^{(0)}(\alpha) \) are algebraically dependent and deduce a contradiction. There exist a positive integer \( L \) and integers \( \tau_\mu \), not all zero, for \( \mu = (\mu_1, \ldots, \mu_m) \) with \( 0 \leq \mu_i \leq L \) such that
\[
\sum_{\mu} \tau_\mu f_1^{(0)}(\alpha)^{\mu_1} \ldots f_m^{(0)}(\alpha)^{\mu_m} = 0.
\]
Let \( w_1, \ldots, w_m, y_1, \ldots, y_m \) and \( t_\mu (\mu = (\mu_1, \ldots, \mu_m), 0 \leq \mu_i \leq L) \) be variables and put
\[
F^{(k)}(z; t) = \sum_\mu t_\mu f_1^{(k)}(z)^{\mu_1} \ldots f_m^{(k)}(z)^{\mu_m},
\]
\[
F(z; t) = \sum_\mu t_\mu f_1(z)^{\mu_1} \ldots f_m(z)^{\mu_m}.
\]
and
\[ \sum_{\mu} t_{\mu}(w_1 + y_1)^{\mu_1} \ldots (w_m + y_m)^{\mu_m} = \sum_{\mu} T_{\mu}(t; y) w_1^{\mu_1} \ldots w_m^{\mu_m}. \]

Then we obtain
\[ 0 = F^{(0)}(\alpha; \tau) = \sum_{\mu} T_{\mu}(\tau; b^{(k)}) f_1^{(k)}(\Omega^{(k)} \alpha) + b_1^{(k)} \ldots f_m^{(k)}(\Omega^{(k)} \alpha) + b_m^{(k)} \]
\[ = \sum_{\mu} T_{\mu}(\tau; b^{(k)}) f_1^{(k)}(\Omega^{(k)} \alpha)^{\mu_1} \ldots f_m^{(k)}(\Omega^{(k)} \alpha)^{\mu_m} \]
\[ = F^{(k)}(\Omega^{(k)} \alpha; T(\tau; b^{(k)})). \]

We put \( R = K[t] = K[\{t_{\mu}\}_{\mu=(\mu_1, \ldots, \mu_m)}, 0 \leq \mu_i \leq L] \) and
\[ V(\tau) = \{Q(t) \in R \mid Q(T(\tau; y)) = 0\}. \]

Then \( V(\tau) \) is a prime ideal of \( R \).

**Definition.** For \( P(z; t) = \sum_{\lambda} P_\lambda(t) z^\lambda \in R[[z_1, \ldots, z_n]] \), we define
\[ \text{index } P(z; t) = \min\{ |\lambda| \mid P_\lambda \notin V(\tau) \}. \]

If \( P_\lambda(t) \in V(\tau) \) for any \( \lambda \), then we define \( \text{index } P(z; t) = \infty \).

Since \( R/V(\tau) \) is an integral domain, we have
\[ \text{index } P_1(z; t) P_2(z; t) = \text{index } P_1(z; t) + \text{index } P_2(z; t). \]

**Lemma 3.** The following two properties are equivalent for any \( P(z; t) \in R[z] \).

(i) \( P(\Omega^{(k)} \alpha; T(\tau; b^{(k)})) = 0 \) for all large \( k \).

(ii) \( \text{index } P(z; t) = \infty \).

**Proof.** We put
\[ P(z; t) = \sum_{\lambda} Q_\lambda(t) z^\lambda, \quad Q_\lambda(t) \in R, \]
and
\[ Q_\lambda(T(\tau; f^{(0)}(\alpha) - w)) = \sum_{\mu} a_{\lambda \mu} w_1^{\mu_1} \ldots w_m^{\mu_m}. \]

We assume (i). Since \( b_i^{(k)} = f_i^{(0)}(\alpha) - f_i^{(k)}(\Omega^{(k)} \alpha) \), we have
\[ 0 = P(\Omega^{(k)} \alpha; T(\tau; b^{(k)})) \]
\[ = \sum_{\lambda} \sum_{\mu} a_{\lambda \mu}(\Omega^{(k)} \alpha)^{\lambda} f_1^{(k)}(\Omega^{(k)} \alpha)^{\mu_1} \ldots f_m^{(k)}(\Omega^{(k)} \alpha)^{\mu_m}, \]
for all large \( k \). Lemma 2 implies \( a_{\lambda \mu} = 0 \) for all \( \lambda, \mu \). Hence
\[ Q_\lambda(T(\tau; f^{(0)}(\alpha) - w)) = 0. \]
Since \( w_1, \ldots, w_m \) are variables, \( Q_\lambda(T(\tau; y)) = 0 \), which implies (ii). The opposite is trivial.

**Lemma 4.** \( \text{index } F(z; t) < \infty \).

**Proof.** By the property (III), there exists \( k_0 \) such that \( F(\Omega^{(k_0)} \alpha; \tau) \neq 0 \). If index \( F(z; t) = \infty \), then
\[
F(z; t) = \sum_\lambda P_\lambda(t) z^\lambda, \quad P_\lambda(t) \in V(\tau).
\]
Noting \( T_\mu(\tau; 0) = \tau_\mu \), we have
\[
F(\Omega^{(k_0)} \alpha; \tau) = \sum_\lambda P_\lambda(\tau)(\Omega^{(k_0)} \alpha)^\lambda = 0,
\]
which is a contradiction.

For a positive integer \( p \), we define
\[
R(p) = \{ g(t) \in R \mid \deg_{\tau_\mu} g(t) \leq p \},
\]
\[
\overline{R(p)} = R(p)/R(p) \cap V(\tau),
\]
\[
d(p) = \dim_K \overline{R(p)}.
\]

**Lemma 5.** \( d(2p) \leq 2^{(L+1)m} d(p) \).

**Proof.** If \( P(t) \in R(2p) \), it can be expressed as
\[
P(t) = \sum_\varepsilon Q_\varepsilon(t) \prod_\mu t^{\varepsilon(\mu)p}_\mu,
\]
where \( Q_\varepsilon(t) \in R(p) \), \( \varepsilon \) is a mapping from the set of \( \mu \) to \( \{0, 1\} \) and the sum is taken over all such mappings. If \( \{ Q_1(t), \ldots, Q_{d(p)}(t) \} \) is a base of \( \overline{R(p)} \), then the set
\[
\left\{ Q_i(t) \prod_\mu t^{\varepsilon(\mu)p}_\mu \right\}_{1 \leq i \leq d(p), \varepsilon}
\]
generates \( \overline{R(2p)} \) and the lemma is proved.

**Lemma 6.** Let \( p \) be a sufficiently large integer. Then there exist polynomials \( P_0(z; t), \ldots, P_p(z; t) \in K[z; t] \) with degree at most \( p \) in each variable such that the following properties are satisfied.

(i) \( \text{index } P_0(z; t) < \infty \).

(ii) If we put \( E_p(z; t) = \sum_{h=0}^p P_h(z; t) F(z; t)^h \), then
\[
\text{index } E_p(z; t) \geq c_7(p + 1)^{1+1/n}.
\]

**Proof.** If we express
\[
P_h(z; t) = \sum_\lambda P_{h\lambda}(t) z^\lambda, \quad h = 0, \ldots, p,
\]
\[ F(z; t)^h = \sum_{\lambda} Q_{h\lambda}(t)z^\lambda, \quad h = 0, \ldots, p, \]

then
\[ \sum_{h=0}^{p} P_h(z; t)F(z; t)^h = \sum_{\nu} \left( \sum_{h,\lambda,\mu,\lambda+\mu=\nu} P_{h\lambda}(t)Q_{h\mu}(t) \right)z^\nu. \]

We will choose \( P_{h\lambda}(t) \) satisfying
\[ \sum_{h,\lambda,\mu,\lambda+\mu=\nu} P_{h\lambda}(t)Q_{h\mu}(t) = 0 \quad \text{in} \ R(2p), \]
for any \( \nu = (\nu_1, \ldots, \nu_n) \) (\( \nu_i \leq J - 1 \)), where \( J \) will be defined below. We define a linear map from \( R(p)^{(p+1)n+1} \) to \( R(2p)^J \) by
\[
(P_{h\lambda}(t))_{h,\lambda} \mapsto \left( \sum_{h,\lambda,\mu,\lambda+\mu=\nu} P_{h\lambda}(t)Q_{h\mu}(t) \right)_{\nu}.
\]
Since
\[
\dim_K \overline{R(p)^{(p+1)n+1}} = d(p)(p+1)^{n+1}, \quad \dim_K \overline{R(2p)^J} = d(2p)J^n,
\]
if \( d(p)(p+1)^{n+1} > d(2p)J^n \), then there is a nontrivial solution \( (P_{h\lambda}(t))_{h,\lambda} \).

By Lemma 5, \( J = [2^{-(L+1)/n}(p+1)^{1+1/n}] - 1 \) satisfies the inequality and
\[ \text{index} \left( \sum_{h=0}^{p} P_h(z; t)F(z; t)^h \right) \geq J \geq c_8(p+1)^{1+1/n}. \]

If \( \text{index} P_0(z; t) < \infty \), the proof is complete. Otherwise, we set
\[ r = \min \{ h \mid \text{index} P_h(z; t) < \infty \}, \quad E_p(z; t) = \sum_{h=r}^{p} P_h(z; t)F(z; t)^{h-r}. \]

Since \( \text{index} E_p(z; t)F(z; t)^r \geq J \), we have
\[ \text{index} E_p(z; t) \geq J - r \text{ index} F(z; t) \geq c_7(p+1)^{1+1/n}. \]

Now we can complete the proof of Theorem 2. Let index \( E_p(z; t) = I \) and \( \gamma_1, \gamma_2, \ldots \) denote positive constants depending on \( E_p(z; t) \). Let \( k \geq \gamma_1 \), where \( \gamma_1 \) will be determined below. Let
\[ E_p(z; t) = \sum_{\nu} g_\nu(z)t^\nu, \quad g_\nu(z) = \sum_{\lambda} g_{\nu\lambda}z^\lambda. \]

Then \( g_\nu(z) \) converges in the \( n \)-polydisc with radius 1 around the origin. Since
\[ \lim_{k \to \infty} f_i^{(k)}(Q^{(k)}\alpha) = f_i(0), \]
we have
\[ |b_i^{(k)}|, |T_\mu(\tau; b^{(k)})| \leq c_9. \]
Thus by Lemma 1,
\[
|E_p(\Omega^{(k)}\alpha; T(\tau; b^{(k)}))| \leq \sum_{\nu} \left( \sum_{|\lambda| \geq I} |g_{\nu\lambda}| \cdot \left| (\Omega^{(k)}\alpha)^{\lambda} \right| \right) T(\tau; b^{(k)})^\nu \leq \gamma_2 \max_i |\alpha_i^{(k)}|^J.
\]
We choose a positive number \( \theta \) with \( e^{-c_2 \theta} < \theta < 1 \). By the property (I) we have
\[
|E_p(\Omega^{(k)}\alpha; T(\tau; b^{(k)}))| \leq \frac{1}{2} \theta^{r_k(p+1)^{1+1/n}}.
\]
We put
\[
E_p^{(k)}(z; t) = \sum_{h=0}^{p} P_h(z; t) F^{(k)}(z; t)^h,
\]
and choose a large \( H \) satisfying
\[
e^{-c_2 H} \leq \theta \cdot \theta^{(p+1)^{1+1/n}}.
\]
If \( f_i^{(k)} - f_i \in \chi^H \), by Lemma 2(i) we have
\[
|E_p(\Omega^{(k)}\alpha; T(\tau; b^{(k)})) - E_p^{(k)}(\Omega^{(k)}\alpha; T(\tau; b^{(k)}))| \leq \gamma_3 e^{-c_2 H r_k}.
\]
Then
\[
|E_p^{(k)}(\Omega^{(k)}\alpha; T(\tau; b^{(k)}))| \leq \gamma_3 e^{-c_2 H r_k} + \frac{1}{2} \theta^{r_k(p+1)^{1+1/n}} \leq \theta^{r_k(p+1)^{1+1/n}}.
\]
On the other hand,
\[
E_p^{(k)}(\Omega^{(k)}\alpha; T(\tau; b^{(k)})) = P_0(\Omega^{(k)}\alpha; T(\tau; b^{(k)})) = (\text{say}) \beta_k \in K.
\]
By the properties (I) and (II), we easily see \( \|\beta_k\| \leq c_{10}^{r_k p} \). By the fact that index \( P_0(z; t) < \infty \), there are infinitely many \( k \) with \( \beta_k \neq 0 \). For such \( k \), we have
\[
r_k(p+1)^{1+1/n} \log \theta \geq \log |\beta_k| \geq -2[K : \mathbb{Q}] \log \|\beta_k\| \geq -2[K : \mathbb{Q}] r_k p \log c_{10}.
\]
Dividing both sides by \( r_k(p+1)^{1+1/n} \) and letting \( p \) tend to \( \infty \), we obtain \( \log \theta \geq 0 \), a contradiction.

3. Proof of Theorem 1. The following lemma is proved in a similar way to the proof of Lemma A.1 in Masser [1].

**Lemma 7.** Let \( b_1 > \ldots > b_n \geq 2 \) be pairwise multiplicatively independent integers. Let \( \theta = \log b_1 \) and \( \theta_i = \theta / \log b_i \). Suppose that for each \( \alpha \) in a finite set \( A \) we are given real numbers \( \lambda_{1\alpha}, \ldots, \lambda_{n\alpha} \) not all zero, and define the sequence
\[
S_\alpha(k) = \sum_{i=1}^{n} \lambda_{i\alpha} b_i^{[\theta_i k]}, \quad k = 0, 1, 2, \ldots
\]
If \( \{k_l\}_{l \geq 1} \) is an increasing sequence of positive integers with \( \{k_{l+1} - k_l\}_{l \geq 1} \) bounded, then there exists \( \delta > 0 \) such that

\[
R(\delta) = \{k_l \mid \min_\alpha |S_\alpha(k_l)| \geq \delta e^{\theta k_l} \} = \{m_l\}_{l \geq 1}, \quad m_l < m_{l+1},
\]

is an infinite set and \( \{m_{l+1} - m_l\}_{l \geq 1} \) is bounded.

Proof. Let \( k_{l+1} - k_l \leq K, l \geq 1 \). We prove the lemma by induction on \( n \). If \( n = 1 \), then \( \{m_l\}_{l \geq 1} = \{k_l\}_{l \geq 1} \) is the required sequence. Assume that we have proved the result with \( n \) replaced by \( n - 1 \) for some \( n \geq 2 \) and the result is not true for \( n \). Then for any \( \delta > 0 \) and any positive integer \( M \) there is \( k_l \) such that for \( k = k_l, k_{l+1}, \ldots, k_{l+M} \) we have

\[
S(k) = \min_\alpha |S_\alpha(k)| < \delta e^{\theta k}.
\]

We may assume that for each \( \alpha \in A \) the numbers \( \lambda_{1\alpha}, \ldots, \lambda_{n-1,\alpha} \) are not all zero. Let \( L = (\max_i \theta_i)|A|K + 1 \) and

\[
J = \{(p_1, \ldots, p_n, q_1, \ldots, q_n) \mid 0 \leq p_i, q_i \leq L, p_n \neq q_n\}.
\]

We take \( B = A \times J \) and for each \( \beta = (\alpha, p_1, \ldots, p_n, q_1, \ldots, q_n) \in B \) we define

\[
\mu_{i\beta} = \lambda_{i\alpha}(b^{p_{i}}_n b^{q_{i}}_i - b^{p_{i}}_n b^{q_{i}}_i), \quad 1 \leq i \leq n - 1.
\]

Since \( p_n \neq q_n \), the pairwise multiplicative independence shows that \( \mu_{1\beta}, \ldots, \mu_{n-1,\beta} \) are not all zero. We define

\[
T_{\beta}(k) = \sum_{i=1}^{n-1} \mu_{i\beta}b^{[\theta_i k]}_i, \quad k = 0, 1, \ldots
\]

For any positive integer \( k \) there is \( \alpha = \alpha(k) \in A \) such that \( S(k) = |S_\alpha(k)| \). By the Box Principle, for any \( j \) with \( l \leq j \leq l + M - |A| \) there exist \( \alpha \in A \) and integers \( l_1, l_2 \) such that \( j \leq l_1 < l_2 \leq j + |A| \) and

\[
S(k_{l_1}) = |S_\alpha(k_{l_1})|, \quad S(k_{l_2}) = |S_\alpha(k_{l_2})|.
\]

Put

\[
p_i = [\theta_i k_{l_1}] - [\theta_i k_{l_2}], \quad q_i = [\theta_i k_{l_2}] - [\theta_i k_{j}].
\]

Then \( 0 \leq p_i, q_i \leq L \). Since \( \theta_n > 1 \) and \( l_1 < l_2 \) imply \( p_n < q_n \), we have \( \beta = (\alpha, p_1, \ldots, p_n, q_1, \ldots, q_n) \in B \) and

\[
T_{\beta}(k_j) = b^{p_n}_n S_\alpha(k_{l_1}) - b^{p_n}_n S_\alpha(k_{l_2}).
\]

By the assumption, for \( j = l, l + 1, \ldots, l + M - |A| \) we have

\[
|T_{\beta}(k_j)| < c\delta e^{\theta k_j},
\]

where \( c \) is a positive constant. This contradicts the induction hypothesis.

**Lemma 8.** Let \( b_1, \ldots, b_n \) be integers as in Lemma 7. Then there exist an infinite set \( \Lambda \) of positive integers, a sequence \( \{\delta(l)\}_{l \geq 1} \) of positive numbers
and a total order in \((\mathbb{N}_0)^n\) such that if \(\lambda > \mu, \ |\lambda|, |\mu| \leq l\), then
\[
\sum_{i=1}^{n} \lambda_i b_i^{[\theta_i, q]} - \sum_{i=1}^{n} \mu_i b_i^{[\theta_i, q]} \geq \delta(l)e^{\theta q}
\]
for every sufficiently large \(q \in \Lambda\). Moreover any subset of \((\mathbb{N}_0)^n\) has the least element.

**Proof.** We put
\[
A(l) = \{ (\lambda, \mu) \mid \lambda, \mu \in (\mathbb{N}_0)^n, \ |\lambda|, |\mu| \leq l, \ \lambda \neq \mu \}.
\]

For \((\lambda, \mu) \in A(l)\) we set
\[
S_{(\lambda, \mu)}(q) = \sum_{i=1}^{n} (\lambda_i - \mu_i)b_i^{[\theta_i, q]}.
\]

We inductively define \(\Lambda(l)\) and \(\Lambda(l)\) as follows. We put \(\Lambda(0) = \mathbb{N}\). By Lemma 7 there exists a positive number \(\delta(l)\) such that
\[
\Lambda(l) = \{ q \in \Lambda(l - 1) \mid \min_{(\lambda, \mu) \in A(l)} |S_{(\lambda, \mu)}(q)| \geq \delta(l)e^{\theta q} \}
\]
is an infinite set and the differences of two consecutive elements of \(\Lambda(l)\) are bounded. We can choose a sequence \(\{q_l\}_{l \geq 1}\) satisfying \(q_l \in A(l)\) and \(q_l < q_{l+1}\). There exists a subsequence \(\{q_l^{(1)}\}_{l \geq 1}\) of \(\{q_l\}_{l \geq 1}\) such that the signs of \(S_{(\lambda, \mu)}(q_l^{(1)}), |\lambda|, |\mu| \leq 1\), are fixed for all \(l \geq 1\). There exists a subsequence \(\{q_l^{(2)}\}_{l \geq 2}\) of \(\{q_l^{(1)}\}_{l \geq 2}\) such that the signs of \(S_{(\lambda, \mu)}(q_l^{(2)}), |\lambda|, |\mu| \leq 2\), are fixed for all \(l \geq 2\). Continuing this process, we obtain a sequence \(\{q_l^{(m)}\}_{l \geq m}\) for every \(m \geq 1\). We set
\[
\Lambda = \{ q_1^{(1)}, q_2^{(2)}, \ldots, q_l^{(l)}, \ldots \},
\]
and for \(\lambda, \mu \in (\mathbb{N}_0)^n\) we define \(\lambda > \mu\) if and only if \(S_{(\lambda, \mu)}(q) > 0\) for all large \(q \in \Lambda\). Noting \(A(l) \supset A(l+1)\) and \(q_l^{(l)} \in A(l)\) completes the proof of the first part of the lemma. For the second part, we use the following fact (cf. [2], Lemma 2.6.4): if \(S\) is a subset of \((\mathbb{N}_0)^n\), then there is a finite subset \(T\) of \(S\) such that for any \((\lambda_1, \ldots, \lambda_n) \in S\), there is an element \((\mu_1, \ldots, \mu_n) \in T\) with \(\mu_i \leq \lambda_i, \ i = 1, \ldots, n\). If \(\mu\) is the least element of \(T\), we can easily see it is also the least element of \(S\).

**Lemma 9.** Let \(d \geq 2\) and
\[
f_j(z) = \sum_{h=0}^{\infty} s_{j,h} z^{dh}, \quad s_{j,h} \in \mathbb{C}^x, \ j = 1, 2, \ldots
\]
Then \(f_j, \ j = 1, 2, \ldots, \) are algebraically independent over \(\mathbb{C}(z)\).
Proof. If \(f_1, \ldots, f_t\) are algebraically dependent over \(\mathbb{C}(z)\), then there exist \(a_{\lambda} \in \mathbb{C}\), not all zero, such that
\[
F(z) = \sum_{\lambda, \mu=(\mu_1, \ldots, \mu_t)} a_{\lambda} z^\lambda f_1(z)^{\mu_1} \cdots f_t(z)^{\mu_t} = 0.
\]
We choose a positive integer \(l\) satisfying
\[
\max\{\lambda \mid a_{\lambda} \neq 0 \text{ for some } \mu\} < d^l.
\]
We define
\[
M = \max\{|\mu| \mid a_{\lambda} \neq 0 \text{ for some } \lambda\} \geq 1,
\]
\[
A = \{\mu \mid |\mu| = M, \ a_{\lambda} \neq 0 \text{ for some } \lambda\}.
\]
Let \(\nu = (\nu_1, \ldots, \nu_t)\) be the largest element of \(A\) for the lexicographical order and \(\kappa\) be the largest integer such that \(a_{\kappa \nu} \neq 0\). Letting
\[
p = \kappa + d^{t!l+1} + d^{t!2l+1} + \ldots + d^{t!\nu_1 l+1} + d^{t! (\nu_1+1)l+2} + \ldots + d^{l!(\nu_1+\nu_2)l+2}
\]
\[
+ d^{l!(\nu_1+\ldots+\nu_{t-1}+1)l+t} + \ldots + d^{l!(\nu_1+\ldots+\nu_t)l+t},
\]
we will show that the Taylor coefficient of \(z^p\) in \(F(z)\) is not zero. This contradicts \(F(z) = 0\) and completes the proof.

The \(d\)-adic expansion of \(p\) has the form
\[
* \ldots * 0 e_{l-1} \ldots e_0, \quad 0 \leq e_i < d.
\]
If a positive integer \(n\) has the \(d\)-adic expansion
\[
e_L \ldots e_{l+1} e_l \ldots e_1 e_0, \quad 0 \leq e_i < d,
\]
we denote by \(w(n)\) the number of nonzero elements among \(e_L, \ldots, e_{l+1}\). Then \(w(p) = \nu_1 + \ldots + \nu_t = M\). For any \(a, b \in \mathbb{N}_0\), we see \(w(a + d^b) \leq w(a) + 1\).
If \(q\) is the degree of a term appearing in the development of
\[
a_{\lambda} z^\lambda f_1(z)^{\mu_1} \cdots f_t(z)^{\mu_t},
\]
then
\[
q = \lambda + d^{h_1} + \ldots + d^{h_{\mu_1}} + d^{2h_{\mu_1+1}} + \ldots + d^{2h_{\mu_1+\mu_2}}
\]
\[
+ d^{h_{\mu_1+\ldots+\mu_{t-1}+1}} + \ldots + d^{h_{\mu_1+\ldots+\mu_t}},
\]
where \(h_i \in \mathbb{N}_0\). If \(p = q\),
\[
M = w(p) = w(q) \leq w(\lambda) + \mu_1 + \ldots + \mu_t = |\mu| \leq M,
\]
and so \(|\mu| = M\). If \(w(\lambda + d^{j h_i}) = 0, w(q) \leq M - 1\). Therefore \(w(\lambda + d^{j h_i}) = 1\) and so \(j h_i \geq l+1 \) since \(\lambda < d^l\). Hence we have \(\lambda = \kappa\). If \(j h_i = j'h_{i'}\) for some \((i, j) \neq (i', j')\), then \(w(\lambda + d^{j h_i} + d^{j' h_{i'}}) = 1\). This implies \(w(q) \leq M - 1\), a contradiction. Therefore \(j h_i\) are distinct and
\[
\{t!l+1, \ldots, t!\nu_1 l+1, t!(\nu_1+1)l+2, \ldots, t!(\nu_1+\nu_2)l+2,
\]
\[
\ldots, t!(\nu_1 + \ldots + \nu_{t-1} + 1)l+t, \ldots, t!|\nu|l+t\}
= \{h_1, \ldots, h_{\mu_1}, 2h_{\mu_1+1}, \ldots, 2h_{\mu_1+\mu_2}, \ldots, th_{\mu_1+\ldots+\mu_{t-1}+1}, \ldots, th_{|\mu|}\}.

There are exactly \(\nu_1\) elements which are not divided by any of 2, \ldots, \(t\) on both sides above. Therefore \(\mu_1 \geq \nu_1\), which implies \(\mu_1 = \nu_1\) since \(\mu \leq \nu\). Then
\[
\{t!(\nu_1+1)l+2, \ldots, t!(\nu_1+\nu_2)l+2, \ldots, t!(\nu_1+\ldots+\nu_{t-1}+1)l+t, \ldots, t!|\nu|l+t\}
= \{2h_{\mu_1+1}, \ldots, 2h_{\mu_1+\mu_2}, \ldots, th_{\mu_1+\ldots+\mu_{t-1}+1}, \ldots, th_{|\mu|}\}.
\]

There are exactly \(\nu_2\) elements which are not divided by any of 3, \ldots, \(t\) on both sides above. Therefore \(\mu_2 \geq \nu_2\), which implies \(\mu_2 = \nu_2\) since \(\mu \leq \nu\) and \(\mu_1 \leq \nu_1\). Continuing, we obtain \(\mu = \nu\). Therefore the Taylor coefficient of \(z^p\) in \(F(z)\) is
\[
a_{\kappa \nu_1! \ldots \nu_1! s_1 t! l+1 \ldots s_1 t! l+1 \ldots s_t t! (\nu_1+\ldots+\nu_{t-1}+1)l+t \ldots s_t t! |\nu| l+t \neq 0.
\]

Proof of Theorem 1. Let
\[
D = \{d \in \mathbb{N} \mid d \neq a^n \text{ for any } a, n \in \mathbb{N}, n > 1\}.
\]
Then
\[
\mathbb{N} \setminus \{1\} = \bigcup_{d \in D} \{d, d^2, \ldots\} \quad \text{(disjoint union)}
\]
and any two elements of \(D\) are multiplicatively independent. Let \(d_1 > \ldots > d_n\) be elements of \(D\), \(z = (z_1, \ldots, z_n)\) and
\[
f_{ij}^{(0)}(z) = \sum_{h=0}^{\infty} \sigma_{ijh} z_i^{d_{ij}^h}, \quad i = 1, \ldots, n, \ j = 1, \ldots, t,
\]
where for any \(h\), \(\sigma_{ijh} \in S_{ij}\), which is a finite set of nonzero algebraic numbers. We will show that
\[
\sum_{h=0}^{\infty} \sigma_{ijh} \alpha^{d_{ij}^h}, \quad i = 1, \ldots, n, \ j = 1, \ldots, t,
\]
are algebraically independent for any algebraic \(\alpha\) with \(0 < |\alpha| < 1\). This implies Theorem 1. Put \(b_i = d_i^{(1)}, \theta = \log b_1, \theta_i = \theta/\log b_i, \ i = 1, \ldots, n, \) and
\[
\Sigma_q = (\sigma_{ijh})_{i=1,\ldots,n, j=1,\ldots,t, h \geq (t!/j)[\theta_i q]} \in \prod_{i=1}^{n} \prod_{j=1}^{t} S_{ij}^{\mathbb{N}}\]
for \(q \in \Lambda\) (in Lemma 8). Since the right hand side is a compact set, there exists a converging subsequence \(\{\Sigma_{q_k}\}_{k \geq 1}\) of \(\{\Sigma_q\}_{q \in \Lambda}\). Let
\[
\lim_{k \to \infty} \Sigma_{q_k} = (s_{ijh})_{i=1,\ldots,n, j=1,\ldots,t, h \geq 0}
\]
and
\[
f_{ij}^{(k)}(z) = \sum_{h=(t!/j)[\theta_i q_k]}^{\infty} \sigma_{ijh} z_i^{d_{ij}^{(h-(t!/j)[\theta_i q_k])}}, \quad f_{ij}(z) = \sum_{h=0}^{\infty} s_{ijh} z_i^{d_{ij}^h}.
\]
Then
\[ \lim_{k \to \infty} f^{(k)}_{ij}(z) = f_{ij}(z). \]

We define
\[ \Omega^{(k)} = \begin{pmatrix} b^{[\theta_1 q_k]}_1 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b^{[\theta_n q_k]}_n \end{pmatrix}. \]

Then \( \Omega^{(k)}, (\alpha_1, \ldots, \alpha_n) = (\alpha, \ldots, \alpha) \) and \( r_k = b^{q_k}_1 \) satisfy the assumptions (I) and (II). If the assumption (III) is also satisfied, the assertion follows. Noting \( z_1, \ldots, z_n \) are distinct variables, by Lemma 9 we see
\[ f_{11}, \ldots, f_{1t}, \ldots, f_{n1}, \ldots, f_{nt} \]
are algebraically independent over \( C(z_1, \ldots, z_n) \). Let
\[ F(z) = \sum_{\lambda, \mu} a_{\lambda \mu} z^\lambda f^{\mu_{11}}_{11} \cdots f^{\mu_{1t}}_{1t} \cdots f^{\mu_{n1}}_{n1} \cdots f^{\mu_{nt}}_{nt} = \sum_{\lambda \in (\mathbb{N}_0)^n} c_\lambda z^\lambda \]
and \( \lambda_0 \) be the least element in \( (\mathbb{N}_0)^n \) in the order defined in Lemma 8 among \( \lambda \) with \( c_\lambda \neq 0 \). Let \( B = \max\{b_1, \ldots, b_n\} \) and \( l = (|\lambda_0| + 1)B \). Then
\[ B^{-1}b^{q_k}_1 \leq b^{-1}_i b^{q_k}_i < b^{[\theta_i q_k]}_i \leq b^{q_k}_1. \]

If \( k \) is sufficiently large, then by Lemma 1,
\[ \sum_{|\lambda| \geq l} |c_\lambda| \cdot |\alpha|^{b^{[\theta_1 q_k]}_1 \lambda_1} \cdots |\alpha|^{b^{[\theta_n q_k]}_n \lambda_n} \leq \gamma^{l+1} |\alpha|^{b^{q_k}_1 (|\lambda_0|+1)}. \]

Since
\[ \lambda_0 b^{[\theta_1 q_k]}_1 + \ldots + \lambda_n b^{[\theta_n q_k]}_n \leq |\lambda_0| b^{q_k}_1, \]
we have
\[ \frac{|\sum_{|\lambda| \geq l} c_\lambda (\Omega^{(k)} \alpha)^\lambda|}{|(\Omega^{(k)} \alpha)^{\lambda_0}|} \leq \gamma^{l+1} |\alpha|^{b^{q_k}_1} \]
if \( k \) is sufficiently large. If \( |\lambda| < l \) and \( \lambda \neq \lambda_0 \), then by Lemma 8,
\[ \frac{|c_\lambda (\Omega^{(k)} \alpha)^\lambda|}{|(\Omega^{(k)} \alpha)^{\lambda_0}|} \leq |c_\lambda| \cdot |\alpha|^\delta(l) e^{q_k} \]
for all large \( k \). Therefore
\[ |F(\Omega^{(k)} \alpha)/(\Omega^{(k)} \alpha)^{\lambda_0} - c_{\lambda_0}| \to 0 \quad (k \to \infty). \]
This implies (III).

References


Mathematics, Hiyoshi Campus
Keio University
4-1-1 Hiyoshi, Kohoku-ku
Yokohama 223-8521, Japan
E-mail: nishioka@math.hc.keio.ac.jp

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