On the tails of the limiting distribution function of the error term in the Dirichlet divisor problem

by

Yuk-Kam Lau (Hong Kong)

1. Introduction. Let $d(n) = \sum_{d|n} 1$ be the divisor function and define

$$\Delta(t) = \sum_{n \le t} d(n) - t(\log t + 2\gamma - 1).$$

We are concerned with the limiting distribution function $D(\cdot)$ of $t^{-1/4}\Delta(t)$, more explicitly,

$$(1.1) \qquad D(u) = \lim_{T \to \infty} T^{-1}\mu\{t \in [1,T] : t^{-1/4}\Delta(t) \le u\}$$

where $\mu\{\ldots\}$ denotes the Lebesgue measure on $\mathbb{R}$. The existence of $D(u)$ was first established by Heath-Brown [4]. In that paper, he proved that

$$D(u) = \int\limits_{-\infty}^{u} f(\alpha)\,d\alpha$$

where the density function $f(\alpha)$ can be extended to an entire function on the complex plane and satisfies the growth condition

$$\frac{d^k}{d\alpha^k}f(\alpha) \ll_{A,k} (1 + |\alpha|)^{-A}.$$

Concerning the properties of $f(\alpha)$, Heath-Brown [5] gave a further discussion and, in particular, he found that for any $\varepsilon > 0$,

$$(1.2) \qquad f(\alpha) \ll \exp(-|\alpha|^{4-\varepsilon}).$$

Define

$$\text{tail}\,D(u) = \begin{cases} D(u) & \text{if } u < 0, \\ 1 - D(u) & \text{if } u \ge 0. \end{cases}$$

Then (1.2) gives $\text{tail}\,D(u) \ll \exp(-|u|^{4-\varepsilon})$. (1.2) can also be shown by Theorem 4.1 of [2] due to Bleher, Cheng, Dyson and Lebowitz; part of their work in [2] is a generalization of [5]. The main focus of [2] is the error term $P_a(x)$

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in the shifted circle problem (centered at $a \in \mathbb{R}^2$). It is proved in [2] that the limiting distribution function $D_a(u)$ of the normalized $P_a(x)$ satisfies

$$\exp(-|u|^{4+\varepsilon}) \ll \operatorname{tail} D_a(u) \ll \exp(-|u|^{4-\varepsilon}).$$

Their argument for the lower bound can be applied to the case $\Delta(t)$ as well. So, one can have

(1.3) $$\exp(-|u|^{4+\varepsilon}) \ll \operatorname{tail} D(u) \ll \exp(-|u|^{4-\varepsilon}).$$

Our purpose is to refine (1.3). Throughout the paper, c, c_i and c_i' ($i = 1, 2, \ldots$) denote some unspecified positive constants, and the value of c may be different at each occurrence.

THEOREM 1. *Let $D(u)$ be defined as in* (1.1). *Then for $|u| \geq 2$,*

(1.4) $$\exp\left(-c_1 \frac{|u|^4}{(\log|u|)^\beta}\right) \ll \operatorname{tail} D(u) \ll \exp\left(-c_2 \frac{|u|^4}{(\log|u|)^\beta}\right)$$

where $\beta = 3(2^{4/3} - 1) \approx 4.5595$.

This is derived by using the limiting behaviour of the Laplace transform of $D(u)$ at infinity; see [3] and [7] for further information of this approach.

2. Some preparations. We need two lemmas to prove our result. Our first lemma is related to Balasubramanian and Ramachandra [1].

Consider a positive multiplicative function $g(n)$ satisfying (i) $g(p) = 1/\gamma$ for all primes p, where γ is a positive constant and (ii) $g(n) \gg n^{-1/16}$. Define $\varepsilon_k(n) = 1$ if n is k-free and 0 otherwise. (n is k-free if there is no prime p such that $p^k \mid n$.) Then we have

(2.1) $$\sum_{ng(n) \leq x} \varepsilon_k(n) = c_k' x (\log x)^{\gamma-1} + O(x(\log x)^{\gamma-11/10}).$$

With $\varepsilon_k(n)$ replaced by the constant function 1, the result (2.1) is proved in [1] (see [1, Theorems 3 and 4]). But our case can be proved in the same way, with the function $f(s)$ in [1, p. 314] replaced by $f(s) = \sum_{n=1}^\infty \varepsilon_k(n)(ng(n))^{-s}$. The corresponding $\Phi(s)$ in our situation is

$$\Phi(s) = f(s)\zeta(s)^{-\gamma^s} = \prod_p \left(1 + \sum_{n=2}^\infty c_n(s)p^{-ns}\right)$$

where

$$c_n(s) = \sum_{\substack{r+l=n \\ 0 \le r < k}} \frac{(-1)^r}{g(p^l)^s} \binom{\gamma^s}{r}.$$

So, $|c_n(s)| \le p^{n\sigma/16}(1 - 2^{-\sigma/16})^{-\gamma^\sigma}$ ($\sigma = \operatorname{Re} s$) and the product in $\Phi(s)$ converges absolutely in $\sigma \ge 3/4$.

LEMMA 2.1. *Let* $g(n)$ *and* $\varepsilon_k(n)$ *be defined as above. Then*

$$\sum_{ng(n)>Y} \varepsilon_k(n)(ng(n))^{-3/2} \asymp Y^{-1/2}(\log Y)^{\gamma-1}$$

and

$$\sum_{ng(n)\le Y} (ng(n))^{-3/4} \ll Y^{1/4}(\log Y)^{\gamma-1}.$$

Proof. Stieltjes integration with (2.1) gives

$$\sum_{ng(n)>Y} \varepsilon_k(n)(ng(n))^{-3/2} = c'_k \int_Y^\infty \frac{(\log t)^{\gamma-1}}{t^{3/2}}\, dt + O(Y^{-1/2}(\log Y)^{\gamma-11/10})$$

and

$$\sum_{ng(n)\le Y} (ng(n))^{-3/4} = c'_k \int_0^Y \frac{(\log t)^{\gamma-1}}{t^{3/4}}\, dt + O(Y^{1/4}(\log Y)^{\gamma-11/10}).$$

Our result follows.

The next lemma is a weak form of the results in [7] but it is sufficient for our purpose.

LEMMA 2.2. *Let* X *be a real random variable with probability distribution* $D(x)$ *and let* $\phi(x)$ *be a regularly varying function with index* $0 < \alpha < 1$, *and* $\psi(x)$ *an asymptotic inverse of* $x/\phi(x)$. *Suppose that* $D(x) > 0$ *for any* $x > 0$. *There are positive constants* L_i, K_i $(i = 1, 2, 3, 4)$ *such that*

(a) *if* $\limsup_{\lambda\to\infty} \psi(\lambda)^{-1} \log E(\exp(-\lambda X)) \le L_1$, *then*

$$\limsup_{x\to\infty} x^{-1} \log D(-\phi(x)) \le -K_1;$$

furthermore, if also $\liminf_{\lambda\to\infty} \psi(\lambda)^{-1} \log E(\exp(-\lambda X)) \ge L_2$, *then*

$$\liminf_{x\to\infty} x^{-1} \log D(-\phi(x)) \ge -K_2;$$

(b) *if* $\limsup_{\lambda\to\infty} \psi(\lambda)^{-1} \log E(\exp(\lambda X)) \le L_3$, *then*

$$\limsup_{x\to\infty} x^{-1} \log(1 - D(\phi(x))) \le -K_3;$$

furthermore, if also $\liminf_{\lambda\to\infty} \psi(\lambda)^{-1} \log E(\exp(\lambda X)) \ge L_4$, *then*

$$\liminf_{x\to\infty} x^{-1} \log(1 - D(\phi(x))) \ge -K_4.$$

REMARKS. (i) A function $\phi(x)$ is called a *regularly varying function* with index α if it is a positive measurable function defined for all sufficiently large positive x and $\lim_{x\to\infty} \phi(\lambda x)/\phi(x) = \lambda^\alpha$ for any $\lambda > 1$. (Note that this is equivalent to the condition that the limit exists for any $\lambda > 0$.)

(ii) $\psi(x)$ is called an *asymptotic inverse* of $\phi(x)$ if $\lim_{x\to\infty} \psi(\phi(x))/x = 1$.

Proof of Lemma 2.2. We consider the case (a) only. Its first part follows from [8, Lemma 3.1(b)]. To show the second part, we fix ξ such that $0 < \xi < (L_2/8)^{1/\alpha}$. Then

$$E\left(\exp\left(-\frac{\eta}{\phi(\eta)}X\right)\right) = \int_{-\infty}^{-\phi(\xi\eta)} \exp\left(-\frac{\eta}{\phi(\eta)}x\right) dD(x)$$

$$+ \int_{-\phi(\xi\eta)}^{\infty} \exp\left(-\frac{\eta}{\phi(\eta)}x\right) dD(x)$$

$$\leq \int_{-\infty}^{-\phi(\xi\eta)} \exp\left(-\frac{\eta}{\phi(\eta)}x\right) dD(x) + \exp\left(\eta\frac{\phi(\xi\eta)}{\phi(\eta)}\right).$$

(Note that $\int_{-\infty}^{\infty} dD(x) = 1$.) Since $\liminf_{\lambda\to\infty} \psi(\lambda)^{-1} \log E(\exp(-\lambda X)) \geq L_2$ and ϕ is of index α, for all large $\eta \geq \eta_0 = \eta_0(\xi)$ we have

$$\log E\left(\exp\left(-\frac{\eta}{\phi(\eta)}X\right)\right) \geq \frac{L_2}{2}\eta \geq 2\xi^\alpha\eta + \log 2 \geq \frac{\phi(\xi\eta)}{\phi(\eta)}\eta + \log 2.$$

(Here we need the fact that $\lambda = \eta/\phi(\eta) \to \infty$ as $\eta \to \infty$, which is assured by [9, Section 1.1].) Thus, when $\eta \geq \eta_0$,

$$\frac{1}{2}E\left(\exp\left(-\frac{\eta}{\phi(\eta)}X\right)\right) \leq \int_{-\infty}^{-\phi(\xi\eta)} \exp\left(-\frac{\eta}{\phi(\eta)}x\right) dD(x)$$

$$\leq E\left(\exp\left(-2\frac{\eta}{\phi(\eta)}X\right)\right)^{1/2} D(-\phi(\xi\eta))^{1/2}$$

by the Cauchy–Schwarz inequality. For any $\varepsilon > 0$, we get

$$2\eta^{-1} \log E\left(\exp\left(-\frac{\eta}{\phi(\eta)}X\right)\right) \leq \eta^{-1} \log D(-\phi(\xi\eta)) + 2^{1/(1-\alpha)}L_1 + \varepsilon,$$

for all $\eta \geq \eta_1(\varepsilon, \xi)$ ($\geq \eta_0$). Here we have used

$$\limsup_{\lambda\to\infty} \psi(\lambda)^{-1} \log E(\exp(-2\lambda X)) \leq 2^{1/(1-\alpha)}L_1.$$

Our result then follows after taking $\liminf$.

Finally, we quote a result [2, Theorem 1.2] about functions satisfying the following hypothesis.

HYPOTHESIS (H_0). *Let* $a_1(t), a_2(t), \ldots$ *be continuous* $\mathbb{R}$-*valued periodic functions of period* 1 *such that*

$$\int_0^1 a_n(t)\, dt = 0 \quad and \quad \sum_{n=1}^{\infty} \int_0^1 a_n(t)^2\, dt < \infty.$$

Suppose that there are positive constants $\gamma_1, \gamma_2, \ldots$ *which are linearly independent over* $\mathbb{Q}$ *such that*

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_1^T \min\left(1, \left| F(t) - \sum_{n\leq N} a_n(\gamma_n t) \right|\right) dt = 0.$$

THEOREM [BCDL]. *Let* $F(t)$ *satisfy Hypothesis* (H_0). *Then for every bounded continuous function* $g(x)$ *on* $\mathbb{R}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_1^T g(F(t))\, dt = \int_{-\infty}^{\infty} g(x)\, \nu(dx)$$

where $\nu(dx)$ *is the distribution of the random series* $\eta = \sum_{n=1}^{\infty} a_n(t_n)$ *and* t_n's *are independent random variables uniformly distributed on* $[0,1]$.

Now take $F(t) = t^{-1/2} \Delta(t^2)$ and

$$a_n(t) = \frac{1}{\pi\sqrt{2}} \cdot \frac{\mu(n)^2}{n^{3/4}} \sum_{r=1}^{\infty} \frac{d(nr^2)}{r^{3/2}} \cos(2\pi rt - \pi/4)$$

where $\mu(\cdot)$ is the Möbius function. Moreover, we choose $\gamma_n = 2\sqrt{n}$ for square-free n, and for other n we take any positive numbers such that $\{\gamma_1, \gamma_2, \ldots\}$ is linearly independent over $\mathbb{Q}$. Then, by Heath-Brown's result and Theorem [BCDL], we see that $D(u)$ (defined in (1.1)) is the probability distribution of the random variable $X = \sum_{n=1}^{\infty} a_n(t_n)$ where t_n's are independent random variables uniformly distributed on $[0,1]$. (Note that the infinite sum converges almost surely by Kolmogorov's theorem.)

3. Proof of Theorem 1. Since $X = \sum_{n=1}^{\infty} a_n(t_n)$ and t_n's are independent random variables uniformly distributed on $[0,1]$, we have

$$E(\exp(\pm\lambda X)) = \prod_{n=1}^{\infty} E(\exp(\pm\lambda a_n(t))) = \prod_{n=1}^{\infty} \int_0^1 \exp(\pm\lambda a_n(t))\, dt.$$

Our task is to bound $\log E(\exp(\pm\lambda X))$ from above and below for $\lambda \geq 1$. We first recall some simple inequalities: $e^x \leq 1 + x + x^2$ if $x \leq 1$, $e^x \geq 1 + x$ and $e^x \geq 1 + x + x^2/2 + x^3/6$ for all x. Also, for squarefree n, we have

$$|a_n(t)| \leq cd(n)n^{-3/4},$$

$$\int\limits_0^1 a_n(t)^2\,dt = \frac{\mu(n)^2}{4\pi^2 n^{3/2}} \sum_{r=1}^\infty \frac{d(nr^2)^2}{r^3} \asymp \left(\frac{d(n)}{n^{3/4}}\right)^2,$$

$$\int\limits_0^1 a_n(t)^3\,dt = \frac{3\mu(n)^2}{16\pi^3 n^{9/4}} \sum_{r,s=1}^\infty \frac{d(nr^2)d(ns^2)d(n(r+s)^2)}{r^{3/2}s^{3/2}(r+s)^{3/2}} \ll \left(\frac{d(n)}{n^{3/4}}\right)^3$$

where all the implied constants are absolute. Let $\varepsilon_0 > 0$ be sufficiently small. When $\lambda d(n)/n^{3/4} < \varepsilon_0$, from $\int_0^1 a_n(t)\,dt = 0$ we have

$$1 + \frac{\lambda^2}{4}\int\limits_0^1 a_n(t)^2\,dt \le 1 \pm \lambda\int\limits_0^1 a_n(t)\,dt + \frac{\lambda^2}{2}\int\limits_0^1 a_n(t)^2\,dt \pm \frac{\lambda^3}{6}\int\limits_0^1 a_n(t)^3\,dt$$

$$\le \int\limits_0^1 \exp(\pm\lambda a_n(t))\,dt \le 1 + \lambda^2\int\limits_0^1 a_n(t)^2\,dt.$$

Thus, as $\log(1+y) \le y$ for $y \ge 0$ and $\log(1+y) \ge y/2$ if $0 \le y \le 1$, we get

$$\frac{\lambda^2}{8}\int\limits_0^1 a_n(t)^2\,dt \le \log\int\limits_0^1 \exp(\pm\lambda a_n(t))\,dt \le \lambda^2\int\limits_0^1 a_n(t)^2\,dt.$$

If $\lambda d(n)/n^{3/4} \ge \varepsilon_0$, then

$$1 = \int\limits_0^1 (1 \pm \lambda a_n(t))\,dt \le \int\limits_0^1 \exp(\pm\lambda a_n(t))\,dt \le \exp\left(c\lambda\frac{d(n)}{n^{3/4}}\right).$$

Hence,

$$\lambda^2 \sum_{\lambda d(n)/n^{3/4} < \varepsilon_0} \mu(n)^2\left(\frac{d(n)}{n^{3/4}}\right)^2 \ll \log E(\exp(\pm\lambda X))$$

and

$$\log E(\exp(\pm\lambda X)) \ll \lambda^2 \sum_{\lambda d(n)/n^{3/4} < \varepsilon_0} \mu(n)^2\left(\frac{d(n)}{n^{3/4}}\right)^2 + \lambda \sum_{\lambda d(n)/n^{3/4} \ge \varepsilon_0} \frac{d(n)}{n^{3/4}}$$

where the implied constants depend on ε_0. Now, we take

$$\phi(x) = (x\log^{3(\gamma-1)}x)^{1/4}, \quad \psi(x) = x^{4/3}\log^{\gamma-1}x \quad \text{where} \quad \gamma = 2^{4/3}.$$

Applying Lemma 2.1 with $g(n) = d(n)^{-4/3}$ and $Y = (\varepsilon_0^{-1}\lambda)^{4/3}$ (note that $d(n) \ll n^\varepsilon$ and so $g(n) \gg n^{-\varepsilon}$), we see that $\log E(\exp(\pm\lambda X)) \asymp \lambda^{4/3}(\log\lambda)^{\gamma-1} = \psi(\lambda)$. Our result follows from Lemma 2.2.

4. Final remark. The above argument can be applied to $E(t)$, $\Delta_3(t)$ and $P(t)$ which are the error terms in the mean square formula of the Riemann zeta-function on the critical line, in the Piltz divisor problem and the circle problem respectively.

THEOREM 2. *Write* $D_h(u) = \lim_{T \to \infty} T^{-1} \mu\{t \in [1, T] : h(t) \leq u\}$ *for the limiting distribution function of a real-valued function* h. *Then*

1. *when* $h(t) = t^{-1/4} E(t)$, (1.4) *holds with* $D(u)$ *replaced by* $D_h(u)$;
2. *when* $h(t) = t^{-1/3} \Delta_3(t)$, *we have, with* $\kappa = 2(3^{3/2} - 1)$,

$$\exp(-c_3 |u|^3 / (\log |u|)^\kappa) \ll \operatorname{tail} D_h(u) \ll \exp(-c_4 |u|^3 / (\log |u|)^\kappa);$$

3. *when* $h(t) = t^{-1/4} P(t)$, *we have, with* $\theta = 3(2^{1/3} - 1)$,

$$\exp(-c_5 |u|^4 / (\log |u|)^\theta) \ll \operatorname{tail} D_h(u) \ll \exp(-c_6 |u|^4 / (\log |u|)^\theta).$$

Proof. For $E(t)$, we take $\gamma_n = \sqrt{2n/\pi}$ when n is squarefree and any suitable value otherwise; and the function $a_n(t)$ is given by

$$a_n(t) = \left(\frac{2}{\pi}\right)^{1/4} \frac{\mu(n)^2}{n^{3/4}} \sum_{r=1}^{\infty} (-1)^{nr} \frac{d(nr^2)}{r^{3/2}} \cos(2\pi rt - \pi/4).$$

(See [4, Section 6].) It is clear from the proof that the factor $(-1)^{nr}$ will not affect the argument and hence the result.

In the case of $\Delta_3(t)$, from [4, Section 7], we choose $\gamma_n = n^{1/3}$ when $\varepsilon_3(n) = 1$ and

$$a_n(t) = \frac{1}{\pi\sqrt{3}} \cdot \frac{\varepsilon_3(n)}{n^{2/3}} \sum_{r=1}^{\infty} \frac{d_3(nr^3)}{r^2} \cos(6\pi rt).$$

Accordingly, from (2.1) we obtain

$$\lambda^2 \sum_{\lambda d_3(n)/n^{2/3} < \varepsilon_0} \varepsilon_3(n) \left(\frac{d_3(n)}{n^{2/3}}\right)^2 \ll \log E(\exp(\pm \lambda X))$$

$$\ll \lambda^2 \sum_{\lambda d_3(n)/n^{2/3} < \varepsilon_0} \varepsilon_3(n) \left(\frac{d_3(n)}{n^{2/3}}\right)^2$$

$$+ \lambda \sum_{\lambda d_3(n)/n^{2/3} \geq \varepsilon_0} \varepsilon_3(n) \frac{d_3(n)}{n^{2/3}}.$$

We see that both sides are $\asymp \lambda^{3/2} (\log \lambda)^\varrho$ where $\varrho = 3^{3/2} - 1$. Taking $\psi(\lambda) = \lambda^{3/2} (\log \lambda)^\varrho$ and $\phi(x) = x^{1/3} (\log x)^{2\varrho/3}$ and applying Lemma 2.2 yields the result.

Finally, from [4, Section 6] again, the choice of γ_n for $P(t)$ is $\sqrt{n}$ (n squarefree) and

$$a_n(t) = -\frac{1}{\pi} \cdot \frac{\mu(n)^2}{n^{3/4}} \sum_{l=1}^{\infty} \frac{r(nl^2)}{l^{3/2}} \cos(2\pi lt + \pi/4).$$

From [6, Section 6.7], we know that $\delta(n) = r(n)/4$ is multiplicative and

$$(4.1) \qquad \frac{r(p^m)}{4} = \begin{cases} 1 & \text{if } p = 2, \\ (1 + (-1)^m)/2 & \text{if } p \equiv 3 \pmod 4, \\ m + 1 & \text{if } p \equiv 1 \pmod 4. \end{cases}$$

Hence $\delta(p^{m+2k}) \leq \delta(p^m)\delta(p^{2k})$. Together with $r(n) \ll n^\varepsilon$, we obtain $r(n) \leq r(nl^2) \ll l^\varepsilon r(n)$, and therefore,

$$(4.2) \quad \lambda^2 \sum_{\lambda r(n)/n^{3/4} < \varepsilon_0} \mu(n)^2 \left(\frac{r(n)}{n^{3/4}}\right)^2 \ll \log E(\exp(\pm \lambda X))$$

$$\ll \lambda^2 \sum_{\lambda r(n)/n^{3/4} < \varepsilon_0} \mu(n)^2 \left(\frac{r(n)}{n^{3/4}}\right)^2$$

$$+ \lambda \sum_{\lambda r(n)/n^{3/4} \geq \varepsilon_0} \mu(n)^2 \frac{r(n)}{n^{3/4}}.$$

We now need an asymptotic formula for $\sum_{n/r(n)^{4/3} \leq x} \mu(n)^2$ where the sum is restricted to $r(n) > 0$. Similarly to the proof of (2.1), we consider $f(s) = \sum_{n=1}^{\infty} \mu(n)^2 r(n)^{4s/3} n^{-s}$; (4.1) yields that

$$f(s) = 4^{4s/3}(1 + 2^{-s}) \prod_{p \equiv 1\,(4)} (1 + 2^{4s/3} p^{-s}).$$

Define the Dirichlet character $\chi_4(n) = 1$ or -1 according as $n \equiv 1$ or 3 (mod 4), and $\chi_4(n) = 0$ otherwise. The associated L-function is

$$L(s, \chi_4) = \prod_{p \equiv 1\,(4)} (1 - p^{-s})^{-1} \prod_{p \equiv 3\,(4)} (1 + p^{-s})^{-1}$$

for $\operatorname{Re} s > 1$. Thus, we have

$$\zeta(s)L(s, \chi_4) = (1 - 2^{-s})^{-1} \prod_{p \equiv 1\,(4)} (1 - p^{-s})^{-2} \prod_{p \equiv 3\,(4)} (1 - p^{-2s})^{-1}.$$

Define

$$\Phi(s) = f(s)(\zeta(s)L(s, \chi_4))^{-2^{(4s-3)/3}}$$

$$= 4^{4s/3}(1 + 2^{-s})(1 - 2^{-s})^{2^{(4s-3)/3}} \prod_{p \equiv 3\,(4)} (1 - p^{-2s})^{2^{(4s-3)/3}}$$

$$\times \prod_{p \equiv 1\,(4)} (1 + 2^{4s/3} p^{-s})(1 - p^{-s})^{2^{4s/3}}.$$

Then $\Phi(s)$ is holomorphic on the half-plane $\operatorname{Re} s \geq 1 - \varepsilon_0$ for some small

constant $\varepsilon_0 > 0$. Repeating the argument of [1], we obtain

$$\sum_{n/r(n)^{4/3} \le x} \mu(n)^2 = cx(\log x)^\alpha + O(x(\log x)^{\alpha - \varepsilon})$$

with $\alpha = 2^{1/3} - 1$. By partial summation, (4.2) yields $\log E(\exp(\pm\lambda X)) \asymp \lambda^{4/3}(\log \lambda)^\alpha$ which will be our $\psi(\lambda)$ (so $\phi(x) = x^{1/4}(\log x)^{3\alpha/4}$).

References

[1] R. Balasubramanian and K. Ramachandra, *On the number of integers n such that $nd(n) \le x$*, Acta Arith. 49 (1988), 313–322.
[2] P. M. Bleher, Z. Cheng, F. J. Dyson and J. L. Lebowitz, *Distribution of the error term for the number of lattice points inside a shifted circle*, Comm. Math. Phys. 154 (1993), 433–469.
[3] T. Hattori and K. Matsumoto, *A limit theorem for Bohr–Jessen's probability measures of the Riemann zeta-function*, J. Reine Angew. Math. 507 (1999), 219–232.
[4] D. R. Heath-Brown, *The distribution and moments of the error term in the Dirichlet divisor problem*, Acta Arith. 60 (1992), 389–415.
[5] —, *The Dirichlet divisor problem*, in: Advances in Number Theory, F. Q. Gouvêa and N. Yui (eds.), Clarendon Press, Oxford, 1993, 31–35.
[6] L. K. Hua, *Introduction to Number Theory*, Springer, 1982.
[7] Y. Kasahara, *Tauberian theorems of exponential type*, J. Math. Kyoto Univ. 18 (1978), 209–219.
[8] Y. K. Lau, *On the limiting distribution of a generalized divisor problem for the case $-1/2 < a < 0$*, Acta Arith. 98 (2001), 229–236.
[9] E. Seneta, *Regularly Varying Functions*, Lecture Notes in Math. 508, Springer, 1976.

Department of Mathematics
The University of Hong Kong
Pokfulam Road, Hong Kong
E-mail: yklau@maths.hku.hk