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A trio of Bernoulli relations, their implications for the Ramanujan polynomials and the special values of the Riemann zeta function

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1. Introduction. Let $B_0 = 1$ and define the sth Bernoulli number, B_s, and the sth Bernoulli polynomial, $B_s(x)$, in the usual fashion [9], [7], so that

$$B_s = -\frac{1}{s+1}\sum_{k=0}^{s-1}\binom{s+1}{k}B_k, \qquad B_s(x) = \sum_{k=0}^{s}\binom{s}{k}B_{s-k}x^k. \tag{1.1}$$

In recent papers [14], [13] we showed that the Bernoulli numbers satisfy the recurrence relation

$$2^{2s-1}B_{2s} = \frac{s}{2s+1} - \frac{1}{2s+1}\sum_{k=1}^{s-1}\binom{2s+1}{2k}2^{2k-1}B_{2k}, \tag{1.2}$$

and we applied the well-known Bernoulli-zeta even integer identity [4]

$$\zeta(2s) = \frac{(-1)^{s+1}2^{2s-1}\pi^{2s}B_{2s}}{(2s)!}, \tag{1.3}$$

to obtain the result

$$\zeta(2s) = (-1)^{s-1}\left(\frac{\pi^{2s}s}{(2s+1)!} + \sum_{k=1}^{s-1}\frac{(-1)^{s-k}\pi^{2k}}{(2k+1)!}\zeta(2s-2k)\right). \tag{1.4}$$

Our results are motivated partly by the relations of the title (given in Lemma 2.1) and partly by the connection with results obtained by Murty et al. concerning Ramanujan polynomials [17], [11].

The odd-indexed Ramanujan polynomials are defined by

$$R_{2s+1}(z) = \sum_{k=0}^{s+1}\frac{B_{2k}B_{2s+2-2k}}{(2k)!(2s+2-2k)!}z^{2k}. \tag{1.5}$$

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They satisfy the functional (reciprocal polynomial) equation

$$R_{2s+1}(z) = z^{2s+2} R_{2s+1}(1/z), \tag{1.6}$$

and occur in Ramanujan's renowned identity involving the odd zeta constants

$$\begin{aligned}\alpha^{-s}\left(\frac{1}{2}\zeta(2s+1) + \sum_{n=1}^{\infty} \frac{n^{-(2s+1)}}{e^{2\alpha n}-1}\right) &= \frac{1}{\beta^s}\left(\frac{1}{2}\zeta(2s+1) + \sum_{n=1}^{\infty} \frac{n^{-(2s+1)}}{e^{2\beta n}-1}\right) \\ &\quad - 2^{2s}\sum_{k=0}^{s+1}(-1)^k \frac{B_{2k}B_{2s+2-2k}}{(2k)!(2s+2-2k)!}\alpha^{s+1-k}\beta^k,\end{aligned} \tag{1.7}$$

where $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$. By definition (1.5), the sum involving Bernoulli numbers in (1.7) therefore corresponds to the Ramanujan polynomial

$$\alpha^{s+1} R_{2s+1}(i\sqrt{\beta/\alpha}) = \alpha^{s+1} R_{2s+1}(i\beta/\pi).$$

The following definition enables us to generalise the Ramanujan polynomials to include the even-indexed values $R_{2s}(z)$.

Definition. Let B_s^* and B_s' be defined for $s \geq 2$ and $s \geq 1$ respectively by the recurrences

$$B_s^* = -\frac{1}{s+1}\sum_{k=0}^{s-1}\binom{s+1}{k}2^{k-s}B_k, \qquad B_s' = -\frac{1}{s+1}\sum_{k=0}^{s-1}\binom{s+1}{k}2^{-s}B_k, \tag{1.8}$$

with initial values $B_0^* = B_0' = 1$ and $B_1^* = \frac{1}{4}$. For $r \geq 0$, we define the *generalised Ramanujan polynomial* $Q_r(z)$ such that

$$Q_r(z) = \sum_{k=0}^{[(r+1)/2]} \frac{B^*_{r+1-2k}B^*_{2k}}{(r+1-2k)!(2k)!} z^{2k}. \tag{1.9}$$

Theorem 1.1. *With the definition of $Q_r(z)$ in (1.9), for $r = 2s+1$ we have*

$$Q_{2s+1}(z) = R_{2s+1}(z). \tag{1.10}$$

When $r = 2s$ is even we have

$$\begin{aligned}Q_{2s}(z) &= 4z^{2s+2}\left(R_{2s+1}\left(\frac{1}{z}\right) - R_{2s+1}\left(\frac{1}{2z}\right)\right) \\ &= 4\left(R_{2s+1}(z) - \frac{1}{2^{2s+2}}R_{2s+1}(2z)\right),\end{aligned} \tag{1.11}$$

and defining $R_{2s}(z) = Q_{2s}(z)$ *we deduce the two-term reciprocal relation*

$$R_{2s}(z) - R_{2s}\left(\frac{z}{2}\right) = z^{2s+2}\left(R_{2s}\left(\frac{1}{z}\right) - R_{2s}\left(\frac{1}{2z}\right)\right). \tag{1.12}$$

Hence we can maintain the notation developed by Murty et al. and speak of the even-indexed Ramanujan polynomials, $R_{2s}(z)$*, as well as the odd-indexed Ramanujan polynomials* $R_{2s+1}(z)$*.*

Corollary 1. *For every integer* $s \geq 1$ *we have*

$$R_{2s+1}(2) = R_{2s+1}(1) = -\frac{(2s+1)B_{2s+2}}{(2s+2)!}, \qquad R_{2s}(1) = \frac{-B^*_{2s+1}}{(2s)!}, \tag{1.13}$$

so that

$$\begin{aligned} R_{2s+1}\left(\frac{1}{2}\right) &= \frac{1}{2^{2s+2}} R_{2s+1}(1), \\ R_{2s}\left(\frac{1}{2}\right) &= \sum_{k=0}^{s} \frac{B^*_{2s+1-2k} B'_{2k}}{(2s+1-2k)!(2k)!} = 0, \end{aligned} \tag{1.14}$$

and when $s = 2s_1$ *is even we have for the complex values*

$$R_{2s+1}(i) = 0, \qquad R_{2s}(i) = R_{2s}(i/2). \tag{1.15}$$

Corollary 2. *For every integer* $s \geq 1$ *at least one of*

$$\zeta(4s-1), \qquad \sum_{n=1}^{\infty} \frac{1}{n^{4s-1}(e^{2\pi n} - 1)} \tag{1.16}$$

is transcendental. Similarly, for every integer $s \geq 1$ *at least one of*

$$\zeta(4s+1), \qquad \sum_{n=1}^{\infty} \frac{1}{n^{4s+1}}\left(\frac{1}{e^{\pi n} - 1} - \frac{1}{2^{4s}(e^{4\pi n} - 1)}\right) \tag{1.17}$$

is transcendental.

We note here that the result analogous to (1.16) for the case $4s+1$ is contained in page 939 of [11].

From the first relation in (1.13) we obtain

$$\sum_{k=0}^{s+1} (2^{2k} - 1) \frac{B_{2s+2-2k} B_{2k}}{(2s+2-2k)!(2k)!} = R_{2s+1}(2) - R_{2s+1}(1) = 0, \tag{1.18}$$

which in turn relates to a quadratic recurrence relation for the even zeta numbers (stated in Theorem 1.3) similar to that discussed by Dilcher [8]. In Lemma 2.1 we derive the Bernoulli relations of the title, which enables us prove Theorems 1.1 and 1.2. We mention in passing that (1.18) implies that the odd-indexed Ramanujan polynomials have a root approaching 2 (from above) as $s \to \infty$.

In this paper we also show that recurrence relations of the type depicted in (1.4) are closely linked to functions related to $\zeta(2s)$ as well as to the Li equivalence for the Riemann Hypothesis. This type of recurrence relations can be expressed in determinant form, and in Theorem 1.4, we give a restatement of the Li equivalence in terms of determinant properties of a square matrix.

Further results concern the existence of pseudo-characteristic equations for $\zeta(s)$ and related functions on the interval $[1,\infty)$, where the approximations are exact at the end points $s=1$ and $s=\infty$, taking approximate values in between. In music a related type of problem is encountered when considering *open or natural tuning* versus *equal temperament tuning.* A harmoniously acceptable but inexact solution is obtained by dividing the interval $[1,2]$, representing the octave, into twelfths, by defining the frequency ratio of two adjacent notes (an equally tempered semitone) to be $2^{1/12}$. The approximation then agrees at the end points of the octave but takes approximate values in between.

For $1/\zeta(s)$ (and again related functions thereof) we also give a pseudo-characteristic equation with bounds for the accuracy of these approximations in Theorem 1.5.

We now introduce some more notation.

Definition (of functions related to $\zeta(s)$). Let

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\qquad \eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=\left(1-\frac{1}{2^{s-1}}\right)\zeta(s), \tag{1.19}$$

$$\theta(s)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^s}=\left(1-\frac{1}{2^s}\right)\zeta(s),\qquad \phi(s)=\sum_{n=1}^{\infty}\frac{1}{(2n)^s}=\frac{1}{2^s}\zeta(s). \tag{1.20}$$

Then

$$\zeta(s)=\theta(s)+\phi(s)\quad\text{and}\quad \eta(s)=\theta(s)-\phi(s). \tag{1.21}$$

Theorem 1.2 gives linear recurrence relations, similar to that in (1.4), for the functions $\eta(2s)$, $\theta(2s)$ and $\phi(2s)$.

Theorem 1.2. *We have*

$$\theta(2s)=(-1)^{s-1}\left(\frac{(2s-1)\pi^{2s}}{4(2s)!}+\sum_{k=1}^{s-1}\frac{(-1)^{s-k}\pi^{2k}}{2(2k)!}\zeta(2s-2k)\right), \tag{1.22}$$

$$\phi(2s)=(-1)^{s-1}\left(\frac{(2s-1)\pi^{2s}}{4(2s+1)!}+\sum_{k=1}^{s-1}\frac{(-1)^{s-k}\pi^{2k}}{(2k+1)!2^{2s-2k}}\zeta(2s-2k)\right). \tag{1.23}$$

Corollary. *We have*

$$\theta(2s) = (-1)^{s-1}\left(\frac{\pi^{2s}}{4(2s)!} + \sum_{k=1}^{s-1}\frac{(-1)^{s-k}\pi^{2k}}{(2k+1)!}\theta(2s-2k)\right), \tag{1.24}$$

$$\phi(2s) = (-1)^{s-1}\left(\frac{(2s-1)\pi^{2s}}{4(2s+1)!} + \sum_{k=1}^{s-1}\frac{(-1)^{s-k}\pi^{2k}}{(2k+1)!}\phi(2s-2k)\right), \tag{1.25}$$

$$\eta(2s) = (-1)^{s-1}\left(\frac{\pi^{2s}}{2(2s+1)!} + \sum_{k=1}^{s-1}\frac{(-1)^{s-k}\pi^{2k}}{(2k+1)!}\eta(2s-2k)\right). \tag{1.26}$$

The recurrence relation in (1.4) was originally deduced by studying determinants [12] and the leading coefficients of the *geometric polynomials* $b_q^{(s)}$ in m, defined for $r \geq 0$ and $q = 1, \ldots, m$, by the polynomial recurrence relations

$$b_q^{(2r+1)} = \binom{m+r-q+1}{2r+1} - \sum_{k=0}^{r-1}\frac{1}{2r-2k+1}\binom{m+r-k}{2r-2k}b_q^{(2k+1)}, \tag{1.27}$$

$$b_q^{(2r)} = -\binom{m+r-q}{2r} - \sum_{k=0}^{r-1}\frac{1}{2r-2k+1}\binom{m+r-k}{2r-2k}b_q^{(2k)}, \tag{1.28}$$

and also for $b_0^{(2r)}$ with $r \geq 0$ in (1.28) with $b_0^{(0)} = m$. When $q = m$ in (1.27), the leading coefficients of the polynomials then follow the Dirichlet eta function recurrence relation given in (1.26).

Definition. Corresponding to the three infinite-dimensional vectors

$$\mathbf{h} = (h_1, h_2, h_3, \ldots), \qquad \mathbf{H} = (H_1, H_2, H_3, \ldots), \qquad \mathbf{G} = (G_1, G_2, G_3, \ldots),$$

we define

$$\mathit{\Delta}_s(\mathbf{h}) = (-1)^s \begin{vmatrix} h_1 & 1 & 0 & 0 & \ldots & 0 \\ h_2 & h_1 & 1 & 0 & \ldots & 0 \\ h_3 & h_2 & h_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{s-1} & h_{s-2} & h_{s-3} & h_{s-4} & \ldots & 1 \\ h_s & h_{s-1} & h_{s-2} & h_{s-3} & \ldots & h_1 \end{vmatrix}, \tag{1.29}$$

$$(1.30)\qquad \Psi_s(\mathbf{h},\mathbf{H}) = (-1)^s \begin{vmatrix} H_1 & 1 & 0 & 0 & \dots & 0 \\ H_2 & h_1 & 1 & 0 & \dots & 0 \\ H_3 & h_2 & h_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{s-1} & h_{s-2} & h_{s-3} & h_{s-4} & \dots & 1 \\ H_s & h_{s-1} & h_{s-2} & h_{s-3} & \dots & h_1 \end{vmatrix},$$

$$(1.31)\qquad \Lambda_s(\mathbf{h},\mathbf{H},\mathbf{G}) = (-1)^s \begin{vmatrix} H_1 & 1 & 0 & 0 & \dots & 0 \\ H_2 & h_1 & 1 & 0 & \dots & 0 \\ H_3 & h_2 & h_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{s-1} & h_{s-2} & h_{s-3} & h_{s-4} & \dots & 1 \\ H_s & G_{s-1} & G_{s-2} & G_{s-3} & \dots & G_1 \end{vmatrix}.$$

We refer to $\Delta_s(\mathbf{h})$ as an $s\times s$ *minor corner layered determinant*, or type 1 MCL determinant for short; to $\Psi_s(\mathbf{h},\mathbf{H})$ as a *half-weighted* $s\times s$ MCL determinant, or type 2 MCL determinant for short; and to $\Lambda_s(\mathbf{h},\mathbf{H},\mathbf{G})$ as a *fully-weighted* $s\times s$ MCL determinant, or type 3 MCL determinant for short. Furthermore, if $H_k = G_k$ for each $k = 1,\dots,s$ then we call $\Lambda(\mathbf{h},\mathbf{H},\mathbf{H})$ a *balanced fully-weighted MCL determinant*.

The closed forms for $b_q^{(t)}$ given in (1.27) and (1.28) were originally obtained by studying associated magic squares under matrix multiplication [14]. We will see later in Lemma 3.1 that all recurrence relations of this type can be expressed as one of the three types of minor corner layered determinants defined above.

We can now state Theorem 1.3, which expresses both $\theta(2s+2)$ and $\zeta(2s+2)$ as a quadratic recurrence relation, an "integer composition" based sum and as a type one MCL determinant.

Theorem 1.3. *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be the two infinite-dimensional vectors defined such that*

$$\mathbf{p} = 2(\phi(2),-\phi(4),\phi(6),-\phi(8),\dots),\qquad \mathbf{q} = 2(\zeta(2),-\zeta(4),\zeta(6),-\zeta(8),\dots).$$

Then

$$(1.32)\qquad \theta(2s+2) = 2\sum_{k=0}^{s-1}\phi(2s-2k)\theta(2k+2) = \frac{(-1)^s\pi^2}{8}\Delta_s(\mathbf{p})$$
$$= \frac{\pi^2}{8}\sum_{t=1}^{s}\sum_{\substack{d_i\ge 0\\ d_1+d_2+\dots+d_s=t\\ d_1+2d_2+\dots+sd_s=s}} 2^t\binom{t}{d_1,d_2,\dots,d_s}\phi^{d_1}(2)\phi^{d_2}(4)\dots\phi^{d_s}(2s),$$

and

(1.33)
$$\zeta(2s+2) = \frac{2}{2^{2s+2}-1}\sum_{k=0}^{s-1}(2^{2k+2}-1)\zeta(2s-2k)\zeta(2k+2) = \frac{(-1)^s\pi^2}{2}\Delta_s(\mathbf{q})$$
$$= \frac{\pi^2/2}{2^{2s+2}-1}\sum_{t=1}^{s}\sum_{\substack{d_i\ge 0\\ d_1+d_2+\cdots+d_s=t\\ d_1+2d_2+\cdots+sd_s=s}} 2^t\binom{t}{d_1,d_2,\ldots,d_s}\zeta^{d_1}(2)\zeta^{d_2}(4)\ldots\zeta^{d_s}(2s).$$

Remark. Similar sums have been considered by Dilcher in [8], where for $N \ge 1$, he defines the $S_N(n)$ such that

$$S_N(n) = \sum_{d_1+\cdots+d_s=n}\ \sum_{d_i\ge 0}\binom{2n}{2d_1,\ldots,2d_N}B_{2d_1}\cdots B_{2d_N}, \tag{1.34}$$

and the sequence $r_k^{(N)}$ of rational numbers recursively by $r_0^{(N)} = 1$ and

$$r_k^{(N+1)} = \frac{-1}{N}r_k^{(N)} + \frac{1}{4}r_{k-1}^{(N-1)},$$

with $r_k^{(N)} = 0$ for $k < 0$. For $2n > N$, Dilcher then shows that

$$S_N(n) = \frac{(2n)!}{(2n-N)!}\sum_{k=0}^{[(N-1)/2]} r_k^{(N)}\frac{B_{2n-2k}}{2n-2k} = \sum_{d_1+\cdots+d_s=n}\ \sum_{d_i\ge 0}\zeta(2d_1)\cdots\zeta(2d_N);$$

note that Dilcher's sum includes the non-elementary value $\zeta(0) = -1/2$ (see Titchmarsh [19, equation (2.4.3)]).

In Lemma 3.3 we show that the linear recurrence relations already stated for $\zeta(2s)$, $\eta(2s)$, $\theta(2s)$ and $\phi(2s)$, in (1.4), (1.24), (1.25) and (1.26), can also be expressed as MCL determinants and as "integer composition" based sums.

This seemingly fundamental link between the even zeta based constants, closed form recurrence relations, "integer composition" based sums and MCL determinants also extends to the *Li equivalence* for the Riemann Hypothesis.

The Li equivalence relies on the non-negativity of a sequence of real numbers $\{\lambda_n\}_{n=1}^{\infty}$ determined from the Riemann xi function as follows. Let

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$
$$\lambda_n = \frac{1}{(n-1)!}\frac{d^n}{ds^n}[s^{n-1}\log\xi(s)]_{s=1},$$

and

$$\varphi(z) = \xi\left(\frac{1}{1-z}\right) = 1 + \sum_{j=1}^{\infty} a_j z^j$$

for $|z| < 1/4$. Then $\xi(s)$ satisfies the functional equation $\xi(s) = \xi(1-s)$. By expressing λ_n as a sum over the non-trivial zeros of $\zeta(s)$ and utilising Jacobi theta functions, Li [15] shows that a_j is a positive real number for every positive integer j, that

$$\lambda_n = \sum_{t=1}^{s} \frac{(-1)^{t-1}}{t} \sum_{\substack{1 \le k_1, \ldots, k_t \le n \\ k_1 + \cdots + k_t = n}} a_{k_1} \ldots a_{k_t}, \tag{1.35}$$

and that the recurrence relation

$$\lambda_n = n a_n - \sum_{j=1}^{n-1} \lambda_j a_{n-j} \tag{1.36}$$

holds for every positive integer n.

Proposition (Li Criterion). *A necessary and sufficient condition for the non-trivial zeros of the Riemann zeta function $\zeta(s)$ to lie on the critical line is that λ_n is non-negative for every positive integer n.*

Li obtains a corresponding equivalence for the Dedekind zeta function $\zeta_k(s)$ of an algebraic number field k. There is an illuminating discussion of the Li Criterion in [5], [6].

We can reword the Li Criterion (and similarly for an algebraic number field) using half-weighted MCL determinants.

Theorem 1.4. *With λ_j and a_j defined as in* (1.35) *and* (1.36)*, let* $\mathbf{a}$ *and* $\mathbf{A}$ *be the two infinite-dimensional vectors defined by*

$$\mathbf{a} = (a_1, a_2, a_3, a_4, \ldots), \qquad \mathbf{A} = (a_1, 2a_2, 3a_3, 4a_4, \ldots).$$

Let L_n be the $n \times n$ matrix given by

$$L_n = \begin{pmatrix} -a_1 & 1 & 0 & 0 & \ldots & 0 \\ -2a_2 & a_1 & 1 & 0 & \ldots & 0 \\ -3a_3 & a_2 & a_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -(n-1)a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & \ldots & 1 \\ -na_n & a_{n-1} & a_{n-2} & a_{n-3} & \ldots & a_1 \end{pmatrix}, \tag{1.37}$$

and define M_n such that $M_n = (-1)^n |L_n|$. Then

$$M_n = \lambda_n = \Psi(\mathbf{a}, -\mathbf{A}) = -\Psi(\mathbf{a}, \mathbf{A}), \tag{1.38}$$

and a necessary and sufficient condition for the non-trivial zeros of the Riemann zeta function to lie on the critical line is that the $n \times n$ half-weighted MCL determinant M_n given in (1.38) *satisfies $M_n \geq 0$ for all $n = 1, 2, \ldots$.*

Theorem 1.5 below examines some 'pseudo-characteristic polynomials' that approximate $\zeta(s)$, $1/\zeta(s)$ and related functions.

Definition (of pseudo-characteristic polynomials). Let

$$p_s(x) = \sum_{k=1}^{s-1} \frac{(-1)^{k-1}\pi^{2k}}{(2k+1)!} x^{2k}, \qquad q_s(x) = \sum_{k=0}^{s-1} \frac{(-1)^{k}\pi^{2k}}{(2k+1)!} x^{2k}, \tag{1.39}$$

$$z_s(x) = \frac{(-1)^{s-1} s \pi^{2s}}{(2s+1)!} + p_s(x), \quad t_s(x) = \frac{(-1)^{s-1}\pi^{2s}}{4(2s)!} + p_s(x), \tag{1.40}$$

$$e_s(x) = \frac{(-1)^{s-1}\pi^{2s}}{2(2s+1)!} + p_s(x), \quad f_s(x) = \frac{(-1)^{s-1}(2s-1)\pi^{2s}}{4(2s+1)!} + p_s(x). \tag{1.41}$$

The polynomials above are all of a similar structure to that in (1.4).

Theorem 1.5. *For positive integers s the polynomials $z_s(x)$ and $q_s(x)$, evaluated at either $k = 2s$ or $k = 2s-1$, satisfy the following inequalities; For $s \geq 17$,*

$$\zeta(k) - 3\{\zeta(k)\}^2 \leq z_s(\zeta(k)) \leq \zeta(k). \tag{1.42}$$

For $s \geq 38$,

$$\theta(k) - 3\{\theta(k)\}^2 \leq t_s(\theta(k)) \leq \theta(k). \tag{1.43}$$

For $s \geq 34$,

$$\frac{1}{\zeta(k)} - \{\zeta(k)\}^3 \leq 1 + q_s(\zeta(k)) \leq \frac{1}{\zeta(k)} + 11\{\zeta(k)\}^3. \tag{1.44}$$

For $s \geq 114$,

$$\frac{1}{\theta(k)} - \{\theta(k)\}^3 \leq 1 + q_s(\theta(k)) \leq \frac{1}{\theta(k)} + 11\{\theta(k)\}^3. \tag{1.45}$$

Here $\{\zeta(k)\} = \zeta(k) - 1$ is the fractional part of $\zeta(k)$. Similar results hold for $e_s(\eta(k))$, $f_s(\phi(k))$ and $1 + q_s(\eta(k))$.

2. Bernoulli relations. We now establish the Bernoulli relations of the title. Different relations of this type were obtained by Woon [20].

Lemma 2.1 (Bernoulli trio). *Let B'_s and B^*_s be defined as in* (1.8)*. Then for any natural number s, the following three identities hold:*

$$\text{(i)} \qquad B^*_{2s} = B_{2s}, \qquad B^*_{2s-1} = \left(1 - \frac{1}{2^{2s}}\right)\frac{2B_{2s}}{s},$$

$$\text{(ii)} \qquad B'_s = \frac{B_s}{2^s},$$

(iii) $$T(x)=\sum_{s=1}^{\infty}\frac{2^{2s}B^*_{2s-1}}{(2s-1)!}x^{2s-2}=\left(1+\sum_{s=1}^{\infty}\frac{2^{2s}B'_{2s}}{(2s)!}x^{2s}\right)^{-1}$$
$$=8\sum_{s=1}^{\infty}(-1)^{s-1}\frac{\theta(2s)}{\pi^{2s}}x^{2s-2}=\left(1+2\sum_{s=1}^{\infty}(-1)^{s-1}\frac{\phi(2s)}{\pi^{2s}}x^{2s}\right)^{-1}.$$

As an immediate consequence of the second identity in (i), we have

$$\theta(2s)=\frac{(-1)^{s+1}2^{2s-3}\pi^{2s}B^*_{2s-1}}{(2s-1)!}. \tag{2.1}$$

The first few values of B_s and B^*_s are given in the following table:

s	0	1	2	3	4	5	6	7	8	9	10	11	12
B_s	1	$\frac{-1}{2}$	$\frac{1}{6}$	0	$\frac{-1}{30}$	0	$\frac{1}{42}$	0	$\frac{-1}{30}$	0	$\frac{5}{66}$	0	$\frac{-691}{2730}$
$B^\star_s$	1	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{-1}{32}$	$\frac{-1}{30}$	$\frac{1}{64}$	$\frac{1}{42}$	$\frac{-17}{1024}$	$\frac{-1}{30}$	$\frac{31}{1024}$	$\frac{5}{66}$	$\frac{-691}{8192}$	$\frac{-691}{2730}$

Proof of Lemma 2.1. Let the sth Bernoulli number, B_s, and the sth Bernoulli polynomial, $B_s(x)$, be defined as in (1.1), where $B_0=1$.

The first expression in (i) follows directly from rearranging the identity in (1.2). We have

$$\begin{aligned}2^{2s-1}B_{2s}&=\frac{s}{2s+1}-\frac{1}{2s+1}\sum_{k=1}^{s-1}\binom{2s+1}{2k}2^{2k-1}B_{2k}\\&=\frac{s}{2s+1}+\frac{1}{2s+1}\left(\frac{1}{2}-\frac{2s+1}{2}\right)-\frac{1}{2s+1}\sum_{k=0}^{2s-2}\binom{2s+1}{k}2^{k-1}B_k\\&=-\frac{1}{2s+1}\sum_{k=0}^{2s-1}\binom{2s+1}{k}2^{k-1}B_k,\end{aligned}$$

so that

$$B_{2s}=-\frac{1}{2s+1}\sum_{k=0}^{2s-1}\binom{2s+1}{k}2^{k-2s}B_k=B^*_{2s},$$

as required.

To obtain the second part of (i) we consider $B_s(x)$ with $s>1$ and $x=1/2$. We then have

$$(2^{1-s}-1)B_s=B_s\left(\frac{1}{2}\right)=\sum_{k=0}^{s}\binom{s}{k}\left(\frac{1}{2}\right)^{s-k}B_k,$$

yielding

$$(2^{1-2s}-1)B_{2s}=\sum_{k=0}^{2s}\binom{2s}{k}\left(\frac{1}{2}\right)^{2s-k}B_k.$$

Thus we get

$$\frac{(2^{2s-1}-1)}{2^{2s-1}}B_{2s} = -\sum_{k=0}^{2s}\binom{2s}{k}2^{k-2s}B_k,$$

$$\frac{(2^{2s-1}-1)}{2^{2s-1}}B_{2s} + B_{2s} = -\sum_{k=0}^{2s-2}\binom{2s}{k}2^{k-2s}B_k,$$

so that

$$(2^{2s}-1)B_{2s} = -\sum_{k=0}^{2s-2}\binom{2s}{k}2^{k-1}B_k.$$

It therefore follows that

$$\left(1-\frac{1}{2^{2s}}\right)\frac{2B_{2s}}{s} = -\frac{2}{2^{2s}s}\sum_{k=0}^{2s-2}\binom{2s}{k}2^{k-1}B_k = -\frac{1}{2s}\sum_{k=0}^{2s-2}\binom{2s}{k}2^{k-2s+1}B_k,$$

and replacing s with $2s-1$ in (1.8) we deduce the result. The identity (2.1) then follows from the definition of $\theta(2s)$.

Part (ii) of the lemma can be obtained by simply multiplying through by 2^{-s} in the definition for B_s, although for part (iii), we need to consider the series expansions of both $\coth x$ and $\tanh x$. It is known from [1] that

$$\coth x = x^{-1} + \sum_{s=1}^{\infty}\frac{2^{2s}B_{2s}}{(2s)!}x^{2s-1}, \qquad |x|<\pi,$$

and that

$$\tanh x = \sum_{s=1}^{\infty}\frac{2^{2s}(2^{2s}-1)B_{2s}}{(2s)!}x^{2s-1}, \qquad |x|<\frac{\pi}{2}.$$

Writing

$$T(x) = \frac{2}{x}\tanh\frac{x}{2} = \left(\frac{x}{2}\coth\frac{x}{2}\right)^{-1}, \qquad |x|<\pi,$$

then gives

$$\sum_{s=1}^{\infty}\frac{2^{2s+1}(2^{2s}-1)B_{2s}}{(2s)!}\,\frac{x^{2s-2}}{2^{2s-1}} = \left(1+\sum_{s=1}^{\infty}\frac{2^{2s}B_{2s}}{(2s)!}\,\frac{x^{2s}}{2^{2s}}\right)^{-1},$$

so that

$$\sum_{s=1}^{\infty}\frac{4(2^{2s}-1)B_{2s}}{(2s)!}x^{2s-2} = \left(1+\sum_{s=1}^{\infty}\frac{B_{2s}}{(2s)!}x^{2s}\right)^{-1},$$

$$\sum_{s=1}^{\infty}\frac{2}{s}\,\frac{(2^{2s}-1)B_{2s}}{(2s-1)!}x^{2s-2} = \left(1+\sum_{s=1}^{\infty}\frac{B_{2s}}{(2s)!}x^{2s}\right)^{-1}.$$

We then apply the first two parts of this lemma to obtain the polynomial result

$$T(x) = \sum_{s=1}^{\infty} \frac{2^{2s} B^*_{2s-1}}{(2s-1)!} x^{2s-2} = \left(1 + \sum_{s=1}^{\infty} \frac{2^{2s} B'_{2s}}{(2s)!} x^{2s}\right)^{-1}, \tag{2.2}$$

and from the definitions of $\theta(2s)$ and $\phi(2s)$ we obtain the final display in (iii). ∎

With the aid of Lemma 2.1, we are now in a position to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. To see (1.10) we have

$$\begin{aligned} Q_{2s+1}(z) &= \sum_{k=0}^{[(2s+2)/2]} \frac{B^*_{2s+2-2k} B^*_{2k}}{(2s+2-2k)!(2k)!} z^{2k} \\ &= \sum_{k=0}^{s+1} \frac{B_{2s+2-2k} B_{2k}}{(2s+2-2k)!(2k)!} z^{2k} = R_{2s+1}(z), \end{aligned}$$

and for (1.11), when $r = 2s$,

$$\begin{aligned} Q_{2s}(z) = R_{2s}(z) &= \sum_{k=0}^{s} \frac{B^*_{2s+1-2k} B^*_{2k}}{(2s+1-2k)!(2k)!} z^{2k} \\ &= \sum_{k=1}^{s+1} \frac{B_{2s+2-2k} B^*_{2k-1}}{(2s+2-2k)!(2k-1)!} z^{2s+2-2k} \\ &= \sum_{k=1}^{s+1} \frac{B_{2s+2-2k}}{(2s+2-2k)!(2k-1)!} \left(1 - \frac{1}{2^{2k}}\right) \frac{2B_{2k}}{k} z^{2s+2-2k} \\ &= \sum_{k=1}^{s+1} (2^{2k}-1) \frac{4B_{2s+2-2k} B_{2k}}{(2s+2-2k)!(2k)!2^{2k}} z^{2s+2-2k} \\ &= 4z^{2s+2} \sum_{k=0}^{s+1} (2^{2k}-1) \frac{B_{2s+2-2k} B_{2k}}{(2s+2-2k)!(2k)!} \left(\frac{1}{2z}\right)^{2k} \\ &= 4z^{2s+2} \left(R_{2s+1}\left(\frac{1}{z}\right) - R_{2s+1}\left(\frac{1}{2z}\right)\right). \end{aligned}$$

Using the odd-indexed reciprocal relationship of (1.6) then gives

$$\begin{aligned} z^{2s+2} R_{2s+1}\left(\frac{1}{z}\right) &= R_{2s+1}(z), \\ z^{2s+2} R_{2s+1}\left(\frac{1}{2z}\right) &= \frac{1}{2^{2s+2}} R_{2s+1}(2z), \end{aligned}$$

from which we obtain

$$\begin{aligned} Q_{2s}(z) = R_{2s}(z) &= 4z^{2s+2}\left(R_{2s+1}\left(\frac{1}{z}\right) - R_{2s+1}\left(\frac{1}{2z}\right)\right) \\ &= 4\left(R_{2s+1}(z) - \frac{1}{2^{2s+2}}R_{2s+1}(2z)\right), \end{aligned}$$

which is the relationship given in (1.11). Setting

$$P_{2s}(z) = R_{2s}(z) - R_{2s}(z/2),$$

and then applying (1.6) and (1.11) we deduce the final statement of the theorem

$$P_{2s}(z) = z^{2s+2}P_{2s}\left(\frac{1}{z}\right) = z^{2s+2}\left(R_{2s}\left(\frac{1}{z}\right) - R_{2s}\left(\frac{1}{2z}\right)\right).$$

It was proven in [17] that

$$R_{2s+1}(2) = -\frac{(2s+1)B_{2s+2}}{(2s+2)!},$$

so to obtain the left hand identity in (1.13) we only need to show that

$$R_{2s+1}(2) = R_{2s+1}(1).$$

We recall that the generating function for the Bernoulli numbers is

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty}\frac{B_n t^n}{n!},$$

and that $B_{2j+1} = 0$ for $j \geq 1$. Hence

$$\frac{t}{e^t-1}\cdot\frac{t}{e^t-1} = \frac{t^2}{(e^t-1)^2} = \left(\sum_{k=0}^{\infty}\frac{B_k t^k}{k!}\right)\left(\sum_{n=0}^{\infty}\frac{B_n t^n}{n!}\right),$$

and for $s \geq 1$, the coefficient of t^{2s+2} in this product of sums is

$$\begin{aligned} \sum_{k=0}^{2s+2}\frac{B_{2s+2-k}B_k}{(2s+2-k)!(k)!}1^k &= \sum_{k\text{ even}}^{2s+2}\frac{B_{2s+2-k}B_k}{(2s+2-k)!k!}1^k \\ &= \sum_{k=0}^{s+1}\frac{B_{2s+2-2k}B_{2k}}{(2s+2-2k)!(2k)!}1^{2k} = R_{2s+1}(1). \end{aligned}$$

We notice also that

$$\frac{d}{dt}\left(\frac{t^2}{e^t-1}\right) = \frac{2t}{e^t-1} - \frac{t^2}{e^t-1} - \frac{t^2}{(e^t-1)^2}.$$

Hence

$$\frac{t^2}{(e^t-1)^2} = \frac{2t}{e^t-1} - \frac{t^2}{e^t-1} - \frac{d}{dt}\left(\frac{t^2}{e^t-1}\right)$$

and this implies

$$\begin{aligned}\frac{t^2}{(e^t-1)^2} &= \sum_{n=0}^{\infty}\frac{B_n t^n}{n!}(2-t)-\frac{d}{dt}\left(\sum_{n=0}^{\infty}\frac{B_n t^{n+1}}{n!}\right)\\ &= \sum_{n=0}^{\infty}\frac{B_n}{n!}(2t^n-t^{n+1}-(n+1)t^n)=\sum_{n=0}^{\infty}\frac{B_n}{n!}(-(n-1)t^n-t^{n+1}).\end{aligned}$$

So for $n=2s+2$ with $s\geq 1$, the coefficient of t^{2s+2} in the above sum is given by

$$-\frac{(n-1)B_n}{n!}=-\frac{(2s+1)B_{2s+2}}{(2s+2)!},$$

and equating the two different expressions for the coefficients of t^{2s+2} gives

$$R_{2s+1}(1)=R_{2s+1}(2)=-\frac{(2s+1)B_{2s+2}}{(2s+2)!},$$

where the right-hand identity in (1.12) is obtained in a similar fashion. Hence $R_{2s}(1/2)=0$ and applying (1.11) we deduce the two expressions in (1.14).

It was shown in [17] that when s is even $R_{2s+1}(i)=0$. To prove the remaining identity in (1.15) and also Corollary 2 we need to introduce Grosswald's generalisation [10] of Ramanujan's formula given in (1.7).

Grosswald defines

$$\sigma_t(n)=\sum_{d|n}d^t \quad\text{and}\quad F_s(z)=\sum_{n=1}^{\infty}\sigma_{-s}(n)e^{2\pi inz}=\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^s}e^{2\pi inz},$$

so that we may also write

$$\begin{aligned}F_s(z) &= \sum_{n,\,m=1}^{\infty}n^{-s}e^{2\pi inmz}=\sum_{n=1}^{\infty}\frac{1}{n^s}\frac{e^{2\pi inz}}{1-e^{2\pi inz}}\\ &= -\zeta(s)-\sum_{n=1}^{\infty}\frac{1}{n^s(e^{2\pi inz}-1)}=-\zeta(s)-F_s(-z).\end{aligned}$$

For any z lying in the upper half-plane, Grosswald obtained

$$F_{2s+1}(z)-z^{2s}F_{2s+1}\left(\frac{-1}{z}\right)=\frac{1}{2}\zeta(2s+1)(z^{2s}-1)+\frac{(2\pi i)^{2s+1}}{2z}R_{2s+1}(z), \tag{2.3}$$

and he set $z=i\sqrt{\beta/\alpha}=i\beta/\pi$ in (2.3) to get Ramanujan's formula (1.7).

Substituting (2.3) into (1.11) we have

$$\begin{aligned}(2.4)\qquad \frac{(2\pi i)^{2s+1}}{8z}R_{2s}(z) &= -\frac{1}{2}\zeta(2s+1)\left(\frac{z^{2s}}{2}-1+\frac{1}{2^{2s+1}}\right)\\ &+F_{2s+1}(z)-z^{2s}F_{2s+1}\left(\frac{-1}{z}\right)-\frac{1}{2^{2s+1}}F_{2s+1}(2z)+\frac{z^{2s}}{2}F_{2s+1}\left(\frac{-1}{2z}\right),\end{aligned}$$

and setting $z = i/2$ in (2.4) when s is even gives

$$(2.5)\qquad \frac{(2\pi)^{2s+1}}{4}R_{2s}(i/2) = F_{2s+1}(i/2)-\frac{1}{2^{2s}}F_{2s+1}(2i)+\frac{1}{2}\eta(2s+1),$$

and when s is odd

$$\begin{aligned}(2.6)\qquad &-\frac{(2\pi)^{2s+1}}{4}R_{2s}(i/2)\\ &\quad = F_{2s+1}(i/2)-\frac{1}{2^{2s}}F_{2s+1}(i)+\frac{1}{2^{2s}}F_{2s+1}(2i)+\frac{1}{2}\zeta(2s+1).\end{aligned}$$

Similarly for $z = i$ in (2.4) when s is even we have

$$(2.7)\qquad \frac{(2\pi)^{2s+1}}{4}R_{2s}(i) = F_{2s+1}(i/2)-\frac{1}{2^{2s}}F_{2s+1}(2i)+\frac{1}{2}\eta(2s+1),$$

and when s is odd

$$\begin{aligned}(2.8)\qquad &-\frac{(2\pi)^{2s+1}}{4}R_{2s}(i)\\ &\quad = -F_{2s+1}(i/2)+4F_{2s+1}(i)-\frac{1}{2^{2s}}F_{2s+1}(2i)+\frac{1}{2}\left(3-\frac{1}{2^{2s}}\right)\zeta(2s+1).\end{aligned}$$

When s is even, (2.5) and (2.7) together imply that $R_{2s}(i) = R_{2s}(i/2)$, which is the final expression of Corollary 1.

It can be deduced from (2.6) and (2.8), using a similar approach to Murty et al. [11], that (1.16) is true for every integer $s \geq 1$. Applying the same method to (2.5) (or (2.7)) in order that we may prove (1.17), we argue as follows. Let s be even. Then

$$\begin{aligned}\frac{(2\pi)^{2s+1}}{4}R_{2s}(i/2) &= F_{2s+1}(i/2)-\frac{1}{2^{2s}}F_{2s+1}(2i)+\frac{1}{2}\eta(2s+1)\\ &= \sum_{n=1}^{\infty}\frac{1}{n^{2s+1}}\frac{e^{-\pi n}}{1-e^{-\pi n}}-\frac{1}{2^{2s}}\sum_{n=1}^{\infty}\frac{1}{n^{2s+1}}\frac{e^{-4\pi n}}{1-e^{-4\pi n}}+\frac{1}{2}\eta(2s+1)\\ &= \sum_{n=1}^{\infty}\frac{1}{n^{2s+1}}\left(\frac{1}{e^{\pi n}-1}-\frac{1}{2^{2s}(e^{4\pi n}-1)}\right)+\frac{1}{2}\eta(2s+1).\end{aligned}$$

The right-hand side of this equation is a sum of positive terms, and so is non-zero. The left-hand side is therefore a non-zero rational multiple of π^{2s+1}, where s is an even integer. Consequently, for every integer $s \geq 1$, at least one of

$$\zeta(4s+1), \quad \sum_{n=1}^{\infty} \frac{1}{n^{4s+1}}\left(\frac{1}{e^{\pi n}-1} - \frac{1}{2^{4s}(e^{4\pi n}-1)}\right)$$

is transcendental. ■

Proof of Theorem 1.2. From Lemma 2.1(i) we have

$$\left(1-\frac{1}{2^{2s}}\right)\frac{2B_{2s}}{s} = -\frac{1}{2s}\sum_{k=0}^{2s-2}\binom{2s}{k}2^{k-2s+1}B_k,$$

so that

$$\left(1-\frac{1}{2^{2s}}\right)B_{2s} = -\frac{s}{4s}\left(\binom{2s}{1}2^{2-2s}B_1 + \sum_{k=0}^{s-1}\binom{2s}{2k}2^{2k-2s+1}B_{2k}\right).$$

Hence

$$\begin{aligned}\theta(2s) &= \left(1-\frac{1}{2^{2s}}\right)\frac{(-1)^{s-1}2^{2s-1}\pi^{2s}B_{2s}}{(2s)!}\\ &= \frac{(-1)^{s-1}\pi^{2s}}{4}\left(\frac{2s}{(2s)!} - \sum_{k=0}^{s-1}\frac{2^{2k}B_{2k}}{(2s-2k)!(2k)!}\right)\\ &= (-1)^{s-1}\pi^{2s}\left(\frac{2s-1}{4(2s)!} - \sum_{k=1}^{s-1}\frac{2^{2k}B_{2k}}{4(2s-2k)!(2k)!}\right),\end{aligned}$$

yielding

$$\theta(2s) = (-1)^{s-1}\left(\frac{(2s-1)\pi^{2s}}{4(2s)!} + \sum_{k=1}^{s-1}\frac{(-1)^k\pi^{2s-2k}}{2(2s-2k)!}\zeta(2k)\right),$$

which is the required identity given in (1.22).

To obtain (1.23) and (1.25) we use the definition in (1.8) so that

$$\phi(2s) = \frac{(-1)^{s-1}\pi^{2s}B'_{2s}}{2(2s)!} = \frac{(-1)^s\pi^{2s}}{(2s+1)!}\sum_{k=0}^{2s-1}\binom{2s+1}{k}2^{-1}B_k.$$

$$\begin{aligned}&= \frac{(-1)^s\pi^{2s}}{(2s+1)!}\left(\frac{(2s+1)B_1}{2} + \sum_{k=0}^{s-1}\binom{2s+1}{2k}\frac{B_{2k}}{2}\right)\\ &= (-1)^{s-1}\pi^{2s}\left(\frac{(2s+1)}{4(2s+1)!} - \frac{1}{2(2s+1)!}\right.\\ &\qquad\qquad\left. + \sum_{k=1}^{s-1}\frac{2^{2k-1}\pi^{2k}B_{2k}}{(2s-2k+1)!(2k)!2^{2k}\pi^{2k}}\right),\end{aligned}$$

giving the required expressions

$$\phi(2s) = (-1)^{s-1}\left(\frac{(2s-1)\pi^{2s}}{4(2s+1)!} + \sum_{k=1}^{s-1} \frac{(-1)^k \pi^{2s-2k}}{(2s-2k+1)!2^{2k}}\zeta(2k)\right)$$
$$= (-1)^{s-1}\left(\frac{(2s-1)\pi^{2s}}{4(2s+1)!} + \sum_{k=1}^{s-1} \frac{(-1)^k \pi^{2k}}{(2k+1)!2^{2s-2k}}\zeta(2s-2k)\right),$$

and

$$\phi(2s) = (-1)^{s-1}\left(\frac{(2s-1)\pi^{2s}}{4(2s+1)!} + \sum_{k=1}^{s-1} \frac{(-1)^{s-k} \pi^{2s-2k}}{(2s-2k+1)!}\phi(2k)\right)$$
$$= (-1)^{s-1}\left(\frac{(2s-1)\pi^{2s}}{4(2s+1)!} + \sum_{k=1}^{s-1} \frac{(-1)^{s-k} \pi^{2k}}{(2k+1)!}\phi(2s-2k)\right).$$

We substitute (1.4) and (1.23) into the first identity in (1.21) to obtain

$$\theta(2s) = \zeta(2s) - \phi(2s)$$
$$= (-1)^{s-1}\pi^{2s}\left(\frac{(4s-2s+1)}{4(2s+1)!} + \sum_{k=1}^{s-1} \frac{(-1)^k}{(2s-2k+1)!\pi^{2k}}(\zeta(2k) - \phi(2k))\right)$$
$$= (-1)^{s-1}\pi^{2s}\left(\frac{(4s-2s+1)}{4(2s+1)!} + \sum_{k=1}^{s-1} \frac{(-1)^k}{(2s-2k+1)!\pi^{2k}}\theta(2k)\right)$$
$$= (-1)^{s-1}\left(\frac{\pi^{2s}}{4(2s)!} + \sum_{k=1}^{s-1} \frac{(-1)^{s-k}\pi^{2k}}{(2k+1)!}\theta(2s-2k)\right),$$

which is the expression in (1.24).

Finally, to obtain (1.26) we substitute (1.24) and (1.25) into the identity

$$\eta(2s) = \theta(2s) - \phi(2s),$$

given by the second relation in (1.21). ∎

3. Families of determinant equations. This section describe some of the fundamental relationships between the three types of MCL determinant and certain recurrence relations.

LEMMA 3.1. *Let $h_1, \ldots, h_s$, $H_1, \ldots, H_s$ and $G_1, \ldots, G_s$ be given. For $k = 1, \ldots, s$, let $\Delta_k(\mathbf{h})$ be the $k \times k$ type* 1 *MCL determinant in* (1.29)*. Let $\Psi_k(\mathbf{h}, \mathbf{H})$ be the $k \times k$ type* 2 *MCL determinant in* (1.30) *and let $\Lambda_k(\mathbf{h}, \mathbf{H}, \mathbf{G})$ be the $k \times k$ type* 3 *MCL determinant in* (1.31)*. Let $\Delta_0(\mathbf{h}) = \Psi_0(\mathbf{h}, \mathbf{H}) =$*

$\Lambda_0(\mathbf{h},\mathbf{H},\mathbf{G}) = 1$. *Then*

$$\Delta_s(\mathbf{h}) = -\sum_{k=0}^{s-1} h_{s-k}\Delta_k(\mathbf{h}), \tag{3.1}$$

$$\Psi_s(\mathbf{h},\mathbf{H}) = -\sum_{k=0}^{s-1} H_{s-k}\Delta_k(\mathbf{h}), \tag{3.2}$$

$$\Psi_s(\mathbf{h},\mathbf{H}) = -H_s - \sum_{k=1}^{s-1} h_{s-k}\Psi_k(\mathbf{h},\mathbf{H}), \tag{3.3}$$

$$\Lambda_s(\mathbf{h},\mathbf{H},\mathbf{G}) = -\sum_{k=0}^{s-1} H_{s-k}\Psi_k(\mathbf{h},\mathbf{G}). \tag{3.4}$$

Conversely, if $\Delta_0(\mathbf{h}) = \Psi_0(\mathbf{h},\mathbf{H}) = \Lambda_0(\mathbf{h},\mathbf{H},\mathbf{G}) = 1$ *and* $\Delta_1(\mathbf{h}),\dots,\Delta_s(\mathbf{h})$, $h_1,\dots,h_s$ *satisfy* (3.1), *then* $\Delta_s(\mathbf{h})$ *is given in terms of* $h_1,\dots,h_s$ *by the MCL determinant* (1.29). *In addition, if* $\Psi_1(\mathbf{h},\mathbf{H}),\dots,\Psi_s(\mathbf{h},\mathbf{H})$ *and* $H_1,\dots,H_s$ *satisfy either* (3.2) *or* (3.3) (*one implies the other*), *then* $\Psi_s(\mathbf{h},\mathbf{H})$ *is given in terms of* $h_1,\dots,h_s$ *and* $H_1,\dots,H_s$ *by the half-weighted MCL determinant* (1.30). *As a further addition, if* $\Lambda_1(\mathbf{h},\mathbf{H},\mathbf{G}),\dots,\Lambda_s(\mathbf{h},\mathbf{H},\mathbf{G})$ *and* $G_1,\dots,G_s$ *also satisfy* (3.4) *then* $\Lambda_s(\mathbf{h},\mathbf{H},\mathbf{G})$ *is given in terms of* $h_1,\dots,h_s$, $H_1,\dots,H_{s-1}$ *and* $G_1,\dots,G_s$, *by the fully-weighted MCL determinant in* (1.31).

We refer to the above recurrence relations according to which type of MCL determinant they relate to: *type* 1 recurrence relations are of the form (3.1), *type* 2 recurrence relations are of the form (3.2) or (3.3), and *type* 3 recurrence relations, where Ψ_s satisfies a type 2 recurrence relation, are of the form (3.4).

Corollary. *Let* U_s, V_s *and* W_s *be the respective* $s\times s$ *matrices corresponding to the determinants* $\Delta_s(\mathbf{h})$, $\Psi_s(\mathbf{h},\mathbf{H})$ *and* $\Lambda_s(\mathbf{h},\mathbf{H},\mathbf{G})$, *i.e.*

$$\Delta_s(\mathbf{h}) = |U_s|, \quad \Psi_s(\mathbf{h},\mathbf{H}) = |V_s|, \quad \Lambda_s(\mathbf{h},\mathbf{H},\mathbf{G}) = |W_s|.$$

Then denoting the characteristic polynomials of U_s, V_s *and* W_s *by*

$$\Delta_s^{(\mu)}(\mathbf{h}) = |U_s-\mu I_s|, \quad \Psi_s^{(\mu)}(\mathbf{h},\mathbf{H}) = |V_s-\mu I_s|, \quad \Lambda_s^{(\mu)}(\mathbf{h},\mathbf{H},\mathbf{G}) = |W_s-\mu I_s|,$$

we find that the characteristic polynomials of U_s, V_s *and* W_s *also satisfy the recurrence relations of the lemma, but with* h_1 *replaced with* $h_1-\mu$, H_1 *replaced with* $H_1-\mu$, G_1 *replaced with* $G_1-\mu$ *and* $\Delta_s(\mathbf{h})$, $\Psi_s(\mathbf{h},\mathbf{H})$, $\Lambda_s(\mathbf{h},\mathbf{H},\mathbf{G})$ *respectively replaced by* $\Delta_s^{(\mu)}(\mathbf{h})$, $\Psi_s^{(\mu)}(\mathbf{h},\mathbf{H})$, $\Lambda_s^{(\mu)}(\mathbf{h},\mathbf{H},\mathbf{G})$.

Proof of Lemma 3.1. To obtain (3.1), we expand the determinant in (1.29) along its first column starting at the sth row so that

$$\begin{aligned}(-1)^s \Delta_s(\mathbf{h}) &= (-1)^{s-1}1^{s-1}h_s(-1)^0\Delta_0(\mathbf{h}) + (-1)^{s-2}1^{s-2}h_{s-1}(-1)^1\Delta_1(\mathbf{h})\\ &\quad + (-1)^{s-3}1^{s-3}h_{s-2}(-1)^2\Delta_2(\mathbf{h}) + \cdots + h_1(-1)^{s-1}\Delta_{s-1}(\mathbf{h})\\ &= (-1)^{s-1}\sum_{k=0}^{s-1} h_{s-k}\Delta_k(\mathbf{h}),\end{aligned}$$

and hence the result. Similarly, for (3.2), we expand the determinant in (1.30) along its first column starting at the sth row, which yields

$$\begin{aligned}(-1)^s \Psi_s(\mathbf{h},\mathbf{H}) &= (-1)^{s-1}1^{s-1}H_s(-1)^0\Delta_0(\mathbf{h}) + (-1)^{s-2}1^{s-2}H_{s-1}(-1)^1\Delta_1(\mathbf{h})\\ &\quad + (-1)^{s-3}1^{s-3}H_{s-2}(-1)^2\Delta_2(\mathbf{h}) + \cdots + H_1(-1)^{s-1}\Delta_{s-1}(\mathbf{h})\\ &= (-1)^{s-1}\sum_{k=0}^{s-1} H_{s-k}\Delta_k(\mathbf{h}).\end{aligned}$$

To obtain (3.3), we expand the determinant in (1.30) along its sth row starting at the first column, which gives

$$\begin{aligned}&(-1)^s \Psi_s(\mathbf{h},\mathbf{H})\\ &\quad= (-1)^{s-1}1^{s-1}H_s + (-1)^{s-2}1^{s-2}h_{s-1}(-1)^1\Psi_1(\mathbf{h},\mathbf{H})\\ &\qquad + (-1)^{s-3}1^{s-3}h_{s-2}(-1)^2\Psi_2(\mathbf{h},\mathbf{H}) + \cdots + h_1(-1)^{s-1}\Psi_{s-1}(\mathbf{h},\mathbf{H})\\ &\quad= (-1)^{s-1}\Big(H_s + \sum_{k=0}^{s-1} h_{s-k}\Psi_k(\mathbf{h},\mathbf{H})\Big).\end{aligned}$$

Finally, to deduce (3.4), we expand the determinant in (1.31) along its first column starting at the sth row, which yields

$$\begin{aligned}(-1)^s \Lambda_s(\mathbf{h},\mathbf{H},\mathbf{G}) &= (-1)^{s-1}1^{s-1}H_s + (-1)^{s-2}1^{s-2}H_{s-1}|G_1|\\ &\quad + (-1)^{s-3}1^{s-3}H_{s-2}\begin{vmatrix} h_1 & 1\\ G_2 & G_1\end{vmatrix} + \cdots + H_1\begin{vmatrix} h_1 & 1 & \ldots & 0\\ h_2 & h_1 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ h_{s-2} & h_{s-1} & \ldots & 1\\ G_{s-1} & G_{s-2} & \ldots & G_1\end{vmatrix},\end{aligned}$$

and by comparing the above determinants with those of the form (1.30) we obtain the result. The converse follows by showing inductively that each $\Delta_s(\mathbf{h})$, $\Psi_s(\mathbf{h},\mathbf{H})$ and $\Lambda_s(\mathbf{h},\mathbf{H},\mathbf{G})$ can be expressed as a determinant of the required form and then re-packing the original determinants expanded above.

To see the Corollary one simply replaces h_1, H_1 and G_1 with $h_1-\mu$, $H_1-\mu$ and $G_1-\mu$ respectively in the recurrence relations of the lemma. ■

Remark. We note that the symmetric structure of the $n\times n$ MCL determinant $\Delta_s(\mathbf{h})=|U_s|$, in (1.29), leads to a symmetry in the cofactors of the matrix $U=(u_{i,j})$. Specifically, let $M_{i,j}$ be the cofactor or minor of $u_{i,j}$. Then for $i-j\geq 0$ we have

$$M_{i,j}=(-1)^{i+j}\Delta_{n-i}(\mathbf{h})\times\Delta_{j-1}(\mathbf{h}).$$

Lemma 3.2. *With $\Delta_s(\mathbf{h})$ as defined in the previous lemma we have*

$$\Delta_s(\mathbf{h})=\sum_{t=1}^{s}\sum_{\substack{d_i\geq 0\\ d_1+d_2+\cdots+d_s=t\\ d_1+2d_2+\cdots+sd_s=s}}(-1)^t\binom{t}{d_1,d_2,\ldots,d_s}h_1^{d_1}h_2^{d_2}\ldots h_s^{d_s},$$

where the above sum consists of 2^{s-1} monomial terms.

Proof. Repeated use of (3.1) gives

$$\Delta_s(\mathbf{h})=(-1)^w\sum_{k_1=0}^{s-1}\sum_{k_2=0}^{k_1-1}\cdots\sum_{k_w=0}^{k_{w-1}-1}h_{s-k_1}h_{k_1-k_2}\ldots h_{k_{w-1}-k_w}\Delta_{k_w}(\mathbf{h}),$$

with $k_w=k_{w-1}-1=0$, so that $\Delta_{k_w}(\mathbf{h})=\Delta_0(\mathbf{h})=1$. Hence we can write

$$\Delta_s(\mathbf{h})=(-1)^w\sum_{k_1=0}^{s-1}\sum_{k_2=0}^{k_1-1}\cdots\sum_{k_w=0}^{k_{w-1}-1}h_{s-k_1}h_{k_1-k_2}\ldots h_{k_{w-1}-k_w},$$

which is just a sum of products of h_k, where the subscripts in each product sum to s. Therefore we have established that $\Delta_s(\mathbf{h})$ is a sum of monomials of the form

$$\pm h_1^{d_1}h_2^{d_2}\ldots h_s^{d_s} \tag{3.5}$$

with

$$d_i\geq 0,\qquad d_1+2d_2+\cdots+sd_s=s.$$

We note that for a given $d_1+2d_2+\cdots+sd_s=s$ with $d_1+\cdots+d_s=t$, the coefficient of the product in (3.5) is the same (ignoring sign) as that in the multinomial expansion of

$$(h_1+h_2+\cdots+h_s)^t.$$

Hence we can write

$$\Delta_s(\mathbf{h})=\sum_{t=1}^{s}\sum_{\substack{d_i\geq 0\\ d_1+d_2+\cdots+d_s=t\\ d_1+2d_2+\cdots+sd_s=s}}(-1)^t\binom{t}{d_1,d_2,\ldots,d_s}h_1^{d_1}h_2^{d_2}\ldots h_s^{d_s}.$$

To see that the number of monomials in this expression for $\Delta_s(\mathbf{h})$ is equal to 2^{s-1}, we note that for fixed values of t, the rules of summation give the number of compositions of the integer s into t parts, which is known to be $\binom{s-1}{t-1}$. Explicitly we have

$$\sum_{\substack{d_i \geq 0 \\ d_1+d_2+\cdots+d_s=t \\ d_1+2d_2+\cdots+sd_s=s}} \binom{t}{d_1, d_2, \ldots, d_s} = \binom{s-1}{t-1},$$

so that summing over the values $1 \leq t \leq s$ gives the required total of 2^{s-1} monomials. For example, when $s = 5$, we have

$$\Delta_5(\mathbf{h}) = -h_1^5 + 4h_1^3h_2 - (3h_1h_2^2 + 3h_1^2h_3) + (2h_2h_3 + 2h_1h_4) - h_5. \ \blacksquare$$

Definition. Define five infinite-dimensional vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{U}_1$, $\mathbf{U}_2$ and $\mathbf{U}_3$ by

$$\begin{aligned}
\mathbf{u} &= \left(\frac{1}{3!}, \frac{1}{5!}, \frac{1}{7!}, \ldots, \frac{1}{(2s+1)!}, \ldots\right), \\
\mathbf{v} &= \left(\frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \ldots, \frac{1}{(s+1)!}, \ldots\right), \\
\mathbf{U}_1 &= \left(\frac{1}{3!}, \frac{2}{5!}, \frac{3}{7!}, \ldots, \frac{s}{(2s+1)!}, \ldots\right), \\
\mathbf{U}_2 &= \left(\frac{1}{3!}, \frac{3}{5!}, \frac{5}{7!}, \ldots, \frac{2s-1}{(2s+1)!}, \ldots\right), \\
\mathbf{U}_3 &= \left(\frac{1}{2!}, \frac{1}{4!}, \frac{1}{6!}, \ldots, \frac{1}{(2s)!}, \ldots\right).
\end{aligned}$$

We now give two well-known MCL determinant identities for the Bernoulli numbers [20]. In the first instance we have

$$B(s) = s!\Delta_s(\mathbf{v}),$$

which by Lemma 3.1 can be written as the recurrence relation

$$B_s = -\frac{1}{s+1}\sum_{k=0}^{s-1}\binom{s+1}{k}B_k,$$

given in (1.1), and by Lemma 3.2, as the double sum

$$\frac{B(s)}{s!} = \sum_{t=1}^{s}\sum_{\substack{d_i \geq 0 \\ d_1+d_2+\cdots+d_s=t \\ d_1+2d_2+\cdots+sd_s=s}} \binom{t}{d_1, d_2, \ldots, d_s}\frac{(-1)^t}{2!^{d_1}3!^{d_2}\ldots(s+1)!^{d_s}}.$$

The second identity states that

$$B(2s) = -\frac{-(2s)!}{2(2^{2s-1}-1)}\Delta_s(\mathbf{u}),$$

the equivalent forms of which we express in terms of $\eta(2s)$ in Lemma 3.3.

Lemmas 3.1 and 3.2 effectively give us two alternative ways to express an MCL determinant: as a recurrence relation and as a double sum over compositions of s into t parts. For a half-weighted MCL determinant we get the two alternatives just stated, along with an extra recurrence relation obtained by expanding the determinant along the sth row instead of the first column.

Combining these equivalent methods of expression for $\eta(2s)$, $\zeta(2s)$, $\theta(2s)$ and $\phi(2s)$ we obtain the following lemma.

LEMMA 3.3. *Let $\eta(2s)$, $\zeta(2s)$, $\theta(2s)$ and $\phi(2s)$ be defined as in Theorem 1.1. Then*

$$\begin{aligned} 2\eta(2s) = (-1)^s \pi^{2s} \varDelta_s(\mathbf{u}) &= \frac{(-1)^{s-1}\pi^{2s}}{(2s+1)!} + \sum_{k=1}^{s-1} \frac{(-1)^{k-1}\pi^{2k}}{(2k+1)!} 2\eta(2s-2k) \\ &= \pi^{2s} \sum_{t=1}^{s} \sum_{\substack{d_i \geq 0 \\ d_1+d_2+\cdots+d_s=t \\ d_1+2d_2+\cdots+sd_s=s}} \binom{s}{d_1, d_2, \ldots, d_s} \frac{(-1)^{t+s}}{3!^{d_1} 5!^{d_2} \ldots (2s+1)!^{d_s}}, \end{aligned}$$

$$\begin{aligned} \zeta(2s) = (-1)^s \pi^{2s} \varPsi_s(\mathbf{u}, \mathbf{U}_1) &= \frac{(-1)^{s-1} s\pi^{2s}}{(2s+1)!} + \sum_{k=1}^{s-1} \frac{(-1)^{k-1} k\pi^{2k}}{(2k+1)!} 2\eta(2s-2k) \\ &= \frac{(-1)^{s-1} s\pi^{2s}}{(2s+1)!} + \sum_{k=1}^{s-1} \frac{(-1)^{k-1}\pi^{2k}}{(2k+1)!} \zeta(2s-2k) \\ &= \frac{2^{2s-2}\pi^{2s}}{(2^{2s-1}-1)} \sum_{t=1}^{s} \sum_{\substack{d_i \geq 0 \\ d_1+d_2+\cdots+d_s=t \\ d_1+2d_2+\cdots+sd_s=s}} \binom{t}{d_1, d_2, \ldots, d_s} \frac{(-1)^{t+s}}{3!^{d_1} 5!^{d_2} \ldots (2s+1)!^{d_s}}, \end{aligned}$$

$$\begin{aligned} 4\phi(2s) &= (-1)^s \pi^{2s} \varPsi_s(\mathbf{u}, \mathbf{U}_2) \\ &= \frac{(-1)^{s-1}(2s-1)\pi^{2s}}{(2s+1)!} + \sum_{k=1}^{s-1} \frac{(-1)^{k-1}(2k-1)\pi^{2k}}{(2k+1)!} 2\eta(2s-2k) \\ &= \frac{(-1)^{s-1}(2s-1)\pi^{2s}}{(2s+1)!} + \sum_{k=1}^{s-1} \frac{(-1)^{k-1}\pi^{2k}}{(2k+1)!} 4\phi(2s-2k) \\ &= \frac{\pi^{2s}}{(2^{2s-1}-1)} \sum_{t=1}^{s} \sum_{\substack{d_i \geq 0 \\ d_1+d_2+\cdots+d_s=t \\ d_1+2d_2+\cdots+sd_s=s}} \binom{s}{d_1, d_2, \ldots, d_s} \frac{(-1)^{t+s}}{3!^{d_1} 5!^{d_2} \ldots (2s+1)!^{d_s}}, \end{aligned}$$

$$
\begin{aligned}
4\theta(2s) = (-1)^s\pi^{2s}\Psi_s(\mathbf{u},\,\mathbf{U}_3) &= \frac{(-1)^{s-1}\pi^{2s}}{(2s)!} + \sum_{k=1}^{s-1}\frac{(-1)^{k-1}\pi^{2k}}{(2k)!}2\eta(2s-2k)\\
&= \frac{(-1)^{s-1}\pi^{2s}}{(2s)!} + \sum_{k=1}^{s-1}\frac{(-1)^{k-1}\pi^{2k}}{(2k+1)!}4\theta(2s-2k)\\
&= \frac{(2^{2s}-1)\pi^{2s}}{(2^{2s-1}-1)} \sum_{t=1}^{s}\sum_{\substack{d_i\ge 0\\ d_1+d_2+\cdots+d_s=t\\ d_1+2d_2+\cdots+sd_s=s}} \binom{s}{d_1, d_2, \ldots, d_s}\frac{(-1)^{t+s}}{3!^{d_1}5!^{d_2}\ldots(2s+1)!^{d_s}}.
\end{aligned}
$$

Proof. The formulas follow directly by applying Lemmas 3.1 and 3.2 to the recurrence relations stated in Theorem 1.2. ∎

We are now in a position to prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. From Lemma 2.1, we can write

$$
\begin{aligned}
8\sum_{s=1}^{\infty}(-1)^{s-1}\frac{\theta(2s)}{\pi^{2s}}x^{2s-2}\left(1+2\sum_{s=1}^{\infty}(-1)^{s-1}\frac{\phi(2s)}{\pi^{2s}}x^{2s}\right)^{-1} &= (1+2S)^{-1}\\
&= 1-2S+(2S)^2-(2S)^3+(2S)^4-\cdots,
\end{aligned}
$$

and comparing terms in x enables us to obtain the double-sum expression for $\theta(2s)$ in (1.32). The determinant and single-sum identities of $\theta(2s)$ in (1.32) are then deduced by applying Lemmas 3.1 and 3.2. To obtain the expressions in (1.33) we rearrange the double sum in (1.32) and again apply Lemmas 3.1 and 3.2 to get the determinant and single-sum identities. ∎

Proof of Theorem 1.4. Substituting $\Psi_n = \lambda_n$, $H_n = -na_n$ and $h_{n-k} = a_{n-k}$ into the type 2 recurrence relation (3.3) of Lemma 3.1 gives

$$
\lambda_n = na_n - \sum_{j=1}^{n-1}\lambda_j a_{n-j},
$$

which is just (1.36). Hence by Lemma 3.1 we have

$$
\begin{aligned}
\lambda_n = M_n = (-1)^{n-1}&\begin{vmatrix}
a_1 & 1 & 0 & 0 & \ldots & 0\\
2a_2 & a_1 & 1 & 0 & \ldots & 0\\
3a_3 & a_2 & a_1 & 1 & \ldots & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
(n-1)a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & \ldots & 1\\
na_n & a_{n-1} & a_{n-2} & a_{n-3} & \ldots & a_1
\end{vmatrix}\\
= -\Psi_n(\mathbf{a},\,\mathbf{A}),
\end{aligned}
$$

so that

$$\lambda_n \geq 0 \;\Leftrightarrow\; M_n \geq 0.$$

Therefore a necessary and sufficient condition for λ_n to be non-negative for $n = 1, 2, \ldots$ is for M_n to be non-negative, and the equivalence of the theorem follows. ∎

Remark. An important point to note with the sums in (1.32) and (1.33) is that they consist entirely of positive terms, whereas the sums in Lemma 3.3 are all alternating. We now give the examples for $\zeta(14) = 2\pi^{14}/18243225$, firstly by using the single-sum and double-sum composition expressions from (1.33), and secondly by employing the double-sum expression for $\zeta(2s)$ given in Lemma 3.3.

Taking $s = 6$ in (1.33), we have the strictly positive sums

$$\zeta(14) = \frac{2}{2^{14}-1}(4098\zeta(2)\zeta(12) + 1038\zeta(4)\zeta(10) + 318\zeta(6)\zeta(8)),$$

and

$$\begin{aligned}\zeta(14) = \frac{\pi^2}{2^{14}-1}(\zeta(12) &+ 2(2\zeta(2)\zeta(10) + 2\zeta(4)\zeta(8) + \zeta^2(6)) \\ &+ 2^2(3\zeta^2(2)\zeta(8) + 6\zeta(2)\zeta(4)\zeta(6) + \zeta(4)^3) \\ &+ 2^3(4\zeta^3(2)\zeta(6) + 6\zeta^2(2)\zeta^2(4)) \\ &+ 2^4 5\zeta^4(2)\zeta(4) + 2^5\zeta^6(2)),\end{aligned}$$

whereas using the double-sum expression for $\zeta(2s)$ in Lemma 3.3 with $s = 7$ gives the alternating expansion

$$\begin{aligned}\zeta(14) = \frac{\pi^{14}2^{12}}{2^{13}-1}\Bigg(\frac{1}{15!} &- \left(\frac{2}{3!13!} + \frac{2}{5!11!} + \frac{2}{7!9!}\right) \\ &+ \left(\frac{3}{3!^2 11!} + \frac{6}{3!5!9!} + \frac{3}{3!7!^2} + \frac{3}{5!^2 7!}\right) \\ &- \left(\frac{4}{3!^3 9!} + \frac{12}{3!^2 5!7!} + \frac{4}{3!5!^3}\right) \\ &+ \left(\frac{5}{3!^4 7!} + \frac{10}{3!^3 5!^2}\right) \\ &- \frac{6}{3!^5 5!} + \frac{1}{3!^7}\Bigg).\end{aligned}$$

As expected, the numbers of terms in the double sums from (1.33) and Lemma 3.3 are given by

$${}^{s-1}\mathrm{C}_{t-1}2^{t-1} = {}^{5}\mathrm{C}_{t-1}2^{t-1}, \qquad 1 \leq t \leq 6,$$

and

$${}^{s-1}\mathrm{C}_{t-1} = {}^{6}\mathrm{C}_{t-1}, \qquad 1 \leq t \leq 7,$$

respectively. Hence the sum of the coefficients in the double sums of Lemma 3.3 is simply 2^{s-1}, and for (1.33), taking into account the extra 2^{t-1} terms, the coefficients sum to 3^{s-1}.

For comparison we give the more common (alternating) recurrence identity [4] for $\zeta(2s)$, which states that

$$\zeta(2s) = \sum_{k=1}^{s} \frac{(-1)^k \pi^{2k}}{(2k+1)!}(1 - 2^{2k-2s+1})\zeta(2s-2k) = 0. \tag{3.6}$$

4. Approximating equations. The Riesz, Hardy–Littlewood and Báez-Duarte equivalences to the Riemann Hypothesis all rely on bounding sums involving inverse zeta constants. Hence the ability to approximate $1/\zeta(s)$ for $s \in \mathbb{N}$ is of interest. The simplest well-known bound for $\zeta(s)$ and its inverse can be deduced as follows.

From Euler's product [2], [18], and the identity

$$2\theta(s) - 1 = 2\left(1 - \frac{1}{2^s}\right)\zeta(s) - 1 < \zeta(s),$$

we obtain

$$1 + \frac{1}{2^s - 1} < \zeta(s) < 1 + \frac{1}{2^{s-1} - 1} \tag{4.1}$$

and

$$1 - \frac{1}{2^{s-1}} < \frac{1}{\zeta(s)} < 1 - \frac{1}{2^s}.$$

In a similar vein, using (1.20) and (1.19) gives

$$1 - \frac{1}{2^s - 1} < \eta(s) < 1 < \theta(s) < 1 + \frac{1}{2^s - 2},$$

from which we obtain the consecutive integer bounds

$$2^s - 2 < (2^s - 1)\eta(s) < 2^s - 1, \qquad 2^s - 2 < (2^s - 2)\theta(s) < 2^s - 1.$$

For $\zeta(s)$ and $\phi(s)$ we have

$$2^s - 2\zeta(s) < (2^s - 3)\zeta(s) < 2^s - \zeta(s), \qquad 1 - 2\phi(s) < (2^s - 3)\phi(s) < 1 - \phi(s),$$

so that the respective intervals here are $\zeta(s)$ and $\phi(s)$ themselves.

Although the upper bound in (4.1) has been improved by Murty et al. [17] to $1 + 2^{-s}(s+1)/(s-1)$, the interval bounding $\zeta(s)$ is still $O(1/2^{s-1})$. The bounds in Theorem 1.5 for the pseudo-characteristic polynomials that approximate $\zeta(2s)$ and $\zeta(2s-1)$ are more accurate. We need two lemmas before we prove Theorem 1.5.

LEMMA 4.1. *Let*

$$F_0(s) = \frac{\pi^s}{s!}, \qquad F_1(s) = \frac{F_0(s)}{\{\zeta(s)\}^2},$$
$$F_2(s) = \frac{F_0(s)}{\{\theta(s)\}^2}, \qquad F_3(s) = \frac{F_0(s)}{\{\zeta(s)\}^3}, \qquad F_4(s) = \frac{F_0(s)}{\{\theta(s)\}^3},$$

and $(t_0, t_1, t_2, t_3, t_4) = (9, 34, 76, 68, 228)$. *Then for integers* $k \ge 1$ *and* $s \ge t_i$ *we have*

$$F_i(s) \le \frac{1}{(2k)^{s-t_i}}, \qquad i = 1, \ldots, 4.$$

COROLLARY. *As* $s \to \infty$ *we have*

$$F_0(s) = o(\{\zeta(s)\}^3) \quad \textit{and} \quad F_0(s) = o(\{\theta(s)\}^3).$$

Proof of Lemma 4.1. For $i = 0$ and $s \ge 9$ we can write $s = 9k + r$ with $k \ge 1$, $r \ge 0$. Then $4^{9k}/(9k)! < 1$, which gives

$$\begin{aligned}\frac{\pi^s}{s!} < \frac{4^s}{s!} &\le \frac{4^{9k+r}}{(9k)!(9k+1)\cdots(9k+r)} \\ &\le \frac{4^r}{(9k+1)\cdots(9k+r)} \le \frac{4^r}{(9k)^r} \le \frac{1}{(2k)^r} \le \frac{1}{(2k)^{s-9k}},\end{aligned}$$

and taking $k = 1$ leads to the result.

Similarly, for $i = 1$ and $s \ge 34$ we can write $s = 34k + r$ with $k \ge 1$, $r > 0$. Then $13^{34k}/(34k)! < 1$ and we have

$$\begin{aligned}\frac{F_1(s)}{\{\zeta(s)\}^2} < \frac{(4\pi)^s}{s!} < \frac{13^s}{s!} &\le \frac{13^{34k+r}}{(34k)!(34k+1)\cdots(34k+r)} \\ &\le \frac{13^r}{(34k+1)\cdots(34k+r)} \le \frac{13^r}{(34k)^r} \le \frac{1}{(2k)^r} \le \frac{1}{(2k)^{s-34k}} \le \frac{1}{(2k)^{s-34}},\end{aligned}$$

when $k = 1$. The proofs are similar for the remaining three cases when $i = 2, 3, 4$.

The Corollary follows by considering the limit as $s \to \infty$ by either fixing k and increasing r or vice versa. ∎

LEMMA 4.2 (Approximate sine lemma). *In the notation of* (1.39) *we have*

$$p_s(x) = 1 - \frac{(-1)^{[x]} \sin \pi\{x\}}{\pi x} + \sum_{k=s}^{\infty} \frac{(-1)^{k-1}(\pi x)^{2k}}{(2k+1)!},$$
$$q_s(x) = \frac{(-1)^{[x]} \sin \pi\{x\}}{\pi x} + \sum_{k=s}^{\infty} \frac{(-1)^{k}(\pi x)^{2k}}{(2k+1)!}.$$

Proof. We have

$$
\begin{aligned}
p_s(x) &= \frac{1}{\pi x}\sum_{k=1}^{s-1}\frac{(-1)^{k-1}(\pi x)^{2k+1}}{(2k+1)!} = 1-\frac{1}{\pi x}\sum_{k=0}^{s-1}\frac{(-1)^{k}(\pi x)^{2k+1}}{(2k+1)!}\\
&= 1-\frac{1}{\pi x}\sum_{k=0}^{\infty}\frac{(-1)^{k}(\pi x)^{2k+1}}{(2k+1)!}+\sum_{k=s}^{\infty}\frac{(-1)^{k-1}(\pi x)^{2k}}{(2k+1)!}\\
&= 1-\frac{\sin\pi x}{\pi x}+\sum_{k=s}^{\infty}\frac{(-1)^{k-1}(\pi x)^{2k}}{(2k+1)!}\\
&= 1-\frac{\sin\pi([x]+\{x\})}{\pi x}+\sum_{k=s}^{\infty}\frac{(-1)^{k-1}(\pi x)^{2k}}{(2k+1)!}\\
&= 1-\frac{\sin\pi[x]\cos\pi\{x\}+\cos\pi[x]\sin\pi\{x\}}{\pi x}+\sum_{k=s}^{\infty}\frac{(-1)^{k-1}(\pi x)^{2k}}{(2k+1)!}\\
&= 1-\frac{\cos\pi[x]\sin\pi\{x\}}{\pi x}+\sum_{k=s}^{\infty}\frac{(-1)^{k-1}(\pi x)^{2k}}{(2k+1)!}\\
&= 1-\frac{(-1)^{[x]}\sin\pi\{x\}}{\pi x}+\sum_{k=s}^{\infty}\frac{(-1)^{k-1}(\pi x)^{2k}}{(2k+1)!},
\end{aligned}
$$

as required. ∎

We are now in a position to prove Theorem 1.5. As with the previous lemma we give the proof for $z_s(\zeta(k))$, with $k=2s$ or $k=2s-1$. The proofs for $t_s(\theta(k))$, $e_s(\eta(k))$, $f_s(\phi(k))$, $1+q_s(\theta(k))$ and $1+q_s(\eta(k))$ are similar.

Proof of Theorem 1.5. By (1.40) and Lemma 4.2, for any integer $k\geq 1$, we can write

$$
\begin{aligned}
&z_s(\zeta(k)) = \frac{(-1)^{s-1}\pi^{2s}s}{(2s+1)!}+p_s(\zeta(k))\\
&= \frac{(-1)^{s-1}\pi^{2s}s}{(2s+1)!}+1-\frac{(-1)^{1}\sin\pi\{\zeta(k)\}}{\pi\zeta(k)}+\sum_{k=s}^{\infty}\frac{(-1)^{k-1}(\pi\zeta(k))^{2k}}{(2k+1)!}\\
&= \frac{(-1)^{s-1}\pi^{2s}}{(2s+1)!}\left(s+\zeta(k)^{2s}-\frac{\pi^2\zeta(k)^{2s+2}}{(2s+2)(2s+3)}+\frac{\pi^4\zeta(k)^{2s+4}}{(2s+2)\cdots(2s+5)}-\cdots\right)\\
&\quad +1+\frac{\sin\pi\{\zeta(k)\}}{\pi\zeta(k)}
\end{aligned}
$$

$$
\begin{aligned}
&\le \left|\frac{(-1)^{s-1}\pi^{2s}}{(2s+1)!}\right|\Bigg(s+\zeta(k)^{2s}\Bigg(1+\frac{(\pi\zeta(k))^2}{(2s+2)(2s+3)} \\
&\qquad\qquad\qquad\qquad\qquad + \frac{(\pi\zeta(k))^4}{(2s+2)\cdots(2s+5)}+\cdots\Bigg)\Bigg) \\
&\quad +1+\frac{\{\zeta(k)\}}{\zeta(k)}-\frac{\pi^2\{\zeta(k)\}^3}{\zeta(k)}+\cdots \\
&\le \frac{\pi^{2s}}{(2s+1)!}\left(s+\zeta(k)^{2s}\left(1+\frac{(\pi\zeta(k))^2}{(2s+2)^2}+\frac{(\pi\zeta(k))^4}{(2s+2)^4}+\cdots\right)\right) \\
&\quad +1+\frac{\{\zeta(k)\}}{\zeta(k)}-\frac{\pi^2\{\zeta(k)\}^3}{\zeta(k)}+\cdots \\
&\le \frac{\pi^{2s}}{(2s+1)!}\left(s+\zeta(k)^{2s}\left(1-\frac{(\pi\zeta(k))^2}{(2s+2)^2}\right)^{-1}\right)+1+\frac{\{\zeta(k)\}}{\zeta(k)}-\frac{\pi^2\{\zeta(k)\}^3}{\zeta(k)}+\cdots \\
&\le \frac{\pi^{2s}}{(2s)!}+1+\frac{\{\zeta(k)\}}{\zeta(k)}-\frac{\pi^2\{\zeta(k)\}^3}{\zeta(k)}+\frac{\pi^4\{\zeta(k)\}^5}{\zeta(k)}-\cdots \qquad \text{for } k\ge 4 \\
&\le \frac{\pi^{2s}}{(2s)!}+1+(\{\zeta(k)\}-\pi^2\{\zeta(k)\}^3+\pi^4\{\zeta(k)\}^5-\cdots)(1+\{\zeta(k)\})^{-1}.
\end{aligned}
$$

Expanding out the brackets and collecting terms we find that

$$
\begin{aligned}
z_s(\zeta(k)) &\le 1+\{\zeta(k)\}+\frac{\pi^{2s}}{(2s)!}-\{\zeta(k)\}^2+(1-\pi^2)\{\zeta(k)\}^3+O(\{\zeta(k)\}^4) \\
&\le \zeta(k)+\frac{\pi^{2s}}{(2s)!}-\{\zeta(k)\}^2,
\end{aligned}
$$

and with a slight adjustment of signs to the above argument we can deduce the lower bound

$$
z_s(\zeta(k)) \ge \zeta(k)-\frac{\pi^k}{(2s)!}-2\{\zeta(k)\}^2.
$$

Hence we have

$$
\zeta(k)-\frac{\pi^{2s}}{(2s)!}-2\{\zeta(k)\}^2 \le z_s(\zeta(k)) \le \zeta(k)+\frac{\pi^{2s}}{(2s)!}-\{\zeta(k)\}^2,
$$

and applying Lemma 4.1 with $k=2s$ or $2s-1$ we deduce (1.42) of Theorem 1.5.

To see (1.44) we use $q_s(\zeta(k))=1-p_s(\zeta(k))$ in the above proof, omitting the initial term in $z_s(\zeta(k))$. This yields

$$
q_s(\zeta(k)) \le \frac{2(\pi^{2s})}{(2s+1)!}-\frac{\{\zeta(k)\}}{\zeta(k)}+\frac{\pi^2\{\zeta(k)\}^3}{\zeta(k)}-\frac{\pi^4\{\zeta(k)\}^5}{\zeta(k)}+\cdots \qquad \text{for } k\ge 4,
$$

so that

$$
q_s(\zeta(k)) \le \frac{1}{\zeta(k)}-1+\frac{2(\pi^{2s})}{(2s+1)!}+\pi^2\{\zeta(k)\}^3.
$$

We again obtain a lower bound by considering the signs in the upper bound argument, and combining these results we have

$$\frac{1}{\zeta(k)} - 1 - \frac{2(\pi^{2s})}{(2s+1)!} \le q_s(\zeta(k)) \le \frac{1}{\zeta(k)} - 1 + \frac{2(\pi^{2s})}{(2s+1)!} + \pi^2\{\zeta(k)\}^3,$$

whence we apply Lemma 4.1 with $k = 2s$ or $2s-1$ to deduce the inequality (1.44). The proofs for the inequalities involving $\theta(k)$ and $1/\theta(k)$ in (1.43) and (1.45) are similar. ∎

Hence $1 + q_s(\zeta(2s))$ approximates $1/\zeta(2s)$ to an accuracy of $O(\{\zeta(2s)\}^3)$ on the interval $[1, \infty)$, where the approximation is exact at the end point $s = \infty$.

REMARK. The Báez–Duarte equivalence to the Riemann Hypothesis ([3], [16]), using coefficients c_t defined by

$$c_t := \sum_{s=0}^{t} (-1)^s \binom{t}{s} \frac{1}{\zeta(2s+2)},$$

asserts that the Riemann Hypothesis is true if and only if for integers $t \ge 0$,

$$c_t = O(t^{-3/4+\epsilon}) \quad \text{for all } \epsilon > 0.$$

Our approximation to $1/\zeta(2s)$ is probably not strong enough to use in the Báez-Duarte equivalence to the Riemann Hypothesis in terms of restating the equivalence as sums of both $\zeta(2s)$ and $1/\zeta(2s)$.

5. Roots of the Ramanujan polynomials. We conclude this paper with a brief look at the roots of the Ramanujan polynomials $R_r(z)$. In [11], it was shown that $R_{2s+1}(z)$ is a polynomial in z of degree $2s+2$ whose four real roots are z_0, $1/z_0$, $-z_0$ and $-1/z_0$, where z_0 is the root of $R_{2s+1}(z)$ slightly greater than 2. It was also shown [17] that the $2s-2$ complex roots of $R_{2s+1}(z)$ lie on the unit circle and as $s \to \infty$ the distribution of these non-real roots on the unit circle becomes uniform. Specifically, the roots of unity that are zeros of $R_{2s+1}(z)$ are given by $\pm i$ when s is even; all four of $\pm\rho$, $\pm\overline{\rho}$ when s is a multiple of 3; and no others. Here ρ is a cube root of unity.

In contrast, the even-indexed Ramanujan polynomials $R_{2s}(z)$ are of degree $2s$ in z (as by Theorem 1.1 it can be seen that the leading terms cancel) and appear to only have the two real roots $\pm 1/2$, as detailed in Corollary 1 of Theorem 1.1. Explicit calculation suggests that for $s \ge 1$, $R_{2s}(z)$ has $2s-2$ complex roots, which all lie just outside the unit circle and whose distribution also becomes uniform as $s \to \infty$.

The zeros of the Ramanujan polynomials [17] are important because they occur in expressions for the odd zeta values and as such the roots of $R_{2s}(z)$ may well be worth investigating further.

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