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Quadratic forms and a product-to-sum formula

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1. Introduction. The set of positive integers is denoted by $\mathbb{N}$ and the set of nonnegative integers by $\mathbb{N}_0$ so that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The domain of all integers is denoted by $\mathbb{Z}$ and the field of complex numbers by $\mathbb{C}$. Throughout this paper $q \in \mathbb{C}$ is taken to satisfy $|q| < 1$. For such q we define

$$E_k = E_k(q) := \prod_{n\in\mathbb{N}} (1 - q^{kn}), \qquad k \in \mathbb{N}. \tag{1.1}$$

We note for later use that replacing q by $-q$ in (1.1) gives

$$E_k(-q) = \begin{cases} \dfrac{E_{2k}^3}{E_k E_{4k}} & \text{if } k \text{ is odd,} \\ E_k & \text{if } k \text{ is even.} \end{cases} \tag{1.2}$$

If $f(q) = \sum_{n=0}^{\infty} f_n q^n$ we write

$$[f(q)]_n = f_n, \qquad n \in \mathbb{N}_0.$$

Scattered throughout the mathematical literature there are a number of results of the form

$$[q^a E_{m_1}^{a_1} \cdots E_{m_\ell}^{a_\ell}]_n = \sum_{\substack{(x_1,\dots,x_m)\in\mathbb{Z}^m \\ Q(x_1,\dots,x_m)=n}} P(x_1, \dots, x_m), \qquad n \in \mathbb{N}_0, \tag{1.3}$$

where $a \in \mathbb{N}_0$, $\ell \in \mathbb{N}$, $m_1, \dots, m_\ell \in \mathbb{N}$ with $m_1 < \cdots < m_\ell$, $a_1, \dots, a_\ell \in \mathbb{Z} \setminus \{0\}$, $m \in \mathbb{N}$, P is a polynomial in $x_1, \dots, x_m$ with rational coefficients and Q is a positive-definite, diagonal, quadratic form in $x_1, \dots, x_m$ with integral coefficients. For example it is a classical result of Klein and Fricke

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[16, Vol. 2, p. 377] that

$$[qE_4^6]_n = \sum_{\substack{(x_1,x_2)\in\mathbb{Z}^2\\ x_1^2+4x_2^2=n}} \frac{1}{2}(x_1^2-4x_2^2), \qquad n\in\mathbb{N}_0; \tag{1.4}$$

see also Mordell [19, p. 122]. More recently Chan, Cooper and Liaw [6, Theorem 4.1, p. 309] have proved that

$$\big[qE_2^3E_6^3\big]_n = \sum_{\substack{(x_1,x_2)\in\mathbb{Z}^2\\ x_1^2+3x_2^2=n}} \frac{1}{2}(x_1^2-3x_2^2), \qquad n\in\mathbb{N}_0. \tag{1.5}$$

Our purpose is to give a fairly general result of the type (1.3) with $m_1,\dots,m_\ell \in \{1,2,3,4,6,8,12,16\}$, which includes (1.4), (1.5) and many other similar results as special cases. The following theorem is proved in Section 3 after some preliminary results are established in Section 2. Four examples of the theorem are given at the end of Section 3 and two applications in Section 4. The first application is to sums of squares and the second to the Ramanujan tau function.

Theorem 1.1. *Let $k\in\mathbb{N}_0$ and $\ell\in\mathbb{N}$. Let $r,s,t,u\in\mathbb{N}_0$ be such that*

$$r+s+t+u=k. \tag{1.6}$$

Let $v,w,x,y\in\mathbb{N}_0$ be such that

$$v+w+x+y=\ell. \tag{1.7}$$

Set

$$m=k+2\ell \tag{1.8}$$

so that $m\in\mathbb{N}$ and $m\geq 2$. Let

$$\begin{aligned} P(x_1,\dots,x_m) = \frac{1}{2^\ell}\prod_{g=r+1}^{r+v}(x_g^2-2x_{g+s+\ell+y}^2)\prod_{g=r+v+1}^{r+v+w}(x_g^2-3x_{g+s+t+\ell+y}^2) \\ \times\prod_{g=r+v+w+1}^{r+v+w+x}(x_g^2-4x_{g+s+t+\ell+y+u}^2)\prod_{g=r+v+w+x+1}^{r+\ell}(x_g^4-3x_g^2x_{g+y}^2) \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} Q(x_1,\dots,x_m) &= x_1^2+\cdots+x_{r+\ell+y}^2+2x_{r+\ell+y+1}^2+\cdots+2x_{r+s+\ell+v+y}^2 \\ &\quad + 3x_{r+s+\ell+v+y+1}^2+\cdots+3x_{r+s+t+\ell+v+w+y}^2 \\ &\quad + 4x_{r+s+t+\ell+v+w+y+1}^2+\cdots+4x_m^2. \end{aligned} \tag{1.10}$$

Let

$$(1.11)\qquad \begin{aligned} a_1 &= -2r+2v+4y, & a_6 &= 5t+3w,\\ a_2 &= 5r-2s+v+3w+2y, & a_8 &= -2s+5u+2v,\\ a_3 &= -2t, & a_{12} &= -2t,\\ a_4 &= -2r+5s-2u+v+6x+4y, & a_{16} &= -2u. \end{aligned}$$

Then, for $n \in \mathbb{N}$ *with* $n \ge \ell$, *we have*

$$(1.12)\qquad [q^\ell E_1^{a_1} E_2^{a_2} E_3^{a_3} E_4^{a_4} E_6^{a_6} E_8^{a_8} E_{12}^{a_{12}} E_{16}^{a_{16}}]_n = \sum_{\substack{(x_1,\ldots,x_m)\in\mathbb{Z}^m\\ Q(x_1,\ldots,x_m)=n}} P(x_1,\ldots,x_m)$$

and

$$(1.13)\qquad a_1+2a_2+3a_3+4a_4+6a_6+8a_8+12a_{12}+16a_{16} = 24\ell.$$

We remark that the first product on the right hand side of (1.9) contains v factors, the second w factors, the third x factors and the fourth y factors. Also, on the right hand side of (1.10) there are $r+\ell+y$ squares with coefficient 1, $s+v$ squares with coefficient 2, $t+w$ squares with coefficient 3 and $u+x$ squares with coefficient 4. We observe that (1.13) follows easily from (1.11) and (1.7).

We note that the choice $x=1$, $r=s=t=u=v=w=y=0$ gives, by (1.6)–(1.11), $k=0$, $\ell=1$, $m=2$, $a_1=a_2=a_3=a_6=a_8=a_{12}=a_{16}=0$, $a_4=6$, $P(x_1,x_2)=\frac{1}{2}(x_1^2-4x_2^2)$, $Q(x_1,x_2)=x_1^2+4x_2^2$, so that Theorem 1.1 gives Klein and Fricke's identity (1.4) in this case. Also the choice $w=1$, $r=s=t=u=v=x=y=0$ gives $k=0$, $\ell=1$, $m=2$, $a_1=a_3=a_4=a_8=a_{12}=a_{16}=0$, $a_2=3$, $a_6=3$, $P(x_1,x_2)=\frac{1}{2}(x_1^2-3x_2^2)$, $Q(x_1,x_2)=x_1^2+3x_2^2$, so that Theorem 1.1 reduces to the identity (1.5) of Chan, Cooper and Liaw. Thus identities (1.4) and (1.5) are indeed special cases of Theorem 1.1.

2. A two-dimensional theta function. For $k \in \mathbb{N}_0$ and $n \in \mathbb{Q}$ we define

$$(2.1)\qquad \tilde{\sigma}_k(n) := \begin{cases} \displaystyle\sum_{\substack{d\in\mathbb{N}\\ d\mid n\\ n/d\ \text{odd}}} d^k & \text{if } n\in\mathbb{N},\\ 0 & \text{if } n\in\mathbb{Q},\ n\notin\mathbb{N}. \end{cases}$$

We set $\tilde{\sigma}(n) := \tilde{\sigma}_1(n)$. The Eisenstein series $\xi_k(q)$ is defined for $k \in \mathbb{N}$ with $k \equiv 1 \pmod 2$ by

$$(2.2)\qquad \xi_k(q) := \sum_{n=1}^{\infty} \tilde{\sigma}_k(n) q^n = \sum_{n=1}^{\infty} \frac{n^k q^n}{1-q^{2n}}.$$

The one-dimensional theta function $\varphi_k(q)$ is defined for $k \in \mathbb{N}_0$ by

$$(2.3) \qquad \varphi_k(q) := \sum_{n=-\infty}^{\infty} n^{2k} q^{n^2}.$$

We set

$$(2.4) \qquad \varphi(q) := \varphi_0(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{E_2^5}{E_1^2 E_4^2},$$

where the infinite product representation is due to Jacobi. Replacing q by $-q$ in (2.4), and appealing to (1.2), we obtain

$$(2.5) \qquad \varphi(-q) = \frac{E_1^2}{E_2},$$

which is another classical result of Jacobi. We also require the theta function

$$(2.6) \qquad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{E_2^2}{E_1},$$

where again the infinite product representation is due to Jacobi. Basic identities satisfied by φ and ψ are

$$(2.7) \qquad \varphi(q)\varphi(-q) = \varphi^2(-q^2),$$

$$(2.8) \qquad \varphi(q)\psi(q^2) = \psi^2(q),$$

$$(2.9) \qquad \varphi(q) + \varphi(-q) = 2\varphi(q^4),$$

$$(2.10) \qquad \varphi(q) - \varphi(-q) = 4q\psi(q^8),$$

$$(2.11) \qquad \varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2),$$

$$(2.12) \qquad \varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^4)$$

(see for example Berndt [3, pp. 15, 71, 72]).

Some recent results of Toh [21] enable us to give $\varphi_1(q)$ and $\varphi_2(q)$ in terms of $\varphi(q)$ and Eisenstein series.

Theorem 2.1. *For $q \in \mathbb{C}$ with $|q| < 1$ we have*

(i) $\varphi_1(q) = -2\varphi(q)\xi_1(-q)$,

(ii) $\varphi_2(q) = 2\varphi(q)(6\xi_1^2(-q) - \xi_3(-q))$.

Proof. Take $j = 3$ in formulae (2.17a) and (2.17b) in Toh [21, p. 187]. ∎

We next define the two-dimensional theta function $\Phi_{k,\ell,m}(q)$ by

$$(2.13) \qquad \Phi_{k,\ell,m}(q) := \sum_{r,s=-\infty}^{\infty} (r\sqrt{\ell} + s\sqrt{-m})^{2k} q^{\ell r^2 + ms^2}, \qquad k,\ell,m \in \mathbb{N}.$$

It is easy to show that

$$(2.14) \qquad \Phi_{k,m,\ell}(q) = (-1)^k \Phi_{k,\ell,m}(q).$$

Taking $\ell = m$ in (2.14), we deduce

$$\Phi_{k,\ell,\ell}(q) = 0 \quad \text{if } k \text{ is odd.} \tag{2.15}$$

Applying the binomial theorem to $(r\sqrt{\ell} + s\sqrt{-m})^{2k}$, and then interchanging the order of summation in (2.13), we obtain

$$\Phi_{k,\ell,m}(q) = \sum_{j=0}^{k} (-1)^j \binom{2k}{2j} \ell^{k-j} m^j \varphi_{k-j}(q^\ell)\varphi_j(q^m) \tag{2.16}$$

as $\sum_{n=-\infty}^{\infty} n^{2k-1} q^{n^2} = 0$ for $k \in \mathbb{N}$. In anticipation of evaluating $\Phi_{k,\ell,m}(q)$ for $k = 1$ and 2, we define for $\ell, m \in \mathbb{N}$ the quantities

$$\begin{cases} A_{\ell,m}(q) := \ell\xi_1(-q^\ell) - m\xi_1(-q^m), \\ B_{\ell,m}(q) := \ell^2\xi_3(-q^\ell) + m^2\xi_3(-q^m). \end{cases} \tag{2.17}$$

Clearly

$$A_{\ell,m}(q) = -A_{m,\ell}(q), \qquad B_{\ell,m}(q) = B_{m,\ell}(q). \tag{2.18}$$

From (2.17) and (2.18), we have

$$A_{\ell,\ell}(q) = 0, \qquad B_{\ell,\ell}(q) = 2\ell^2\xi_3(-q^\ell). \tag{2.19}$$

THEOREM 2.2. *For $\ell, m \in \mathbb{N}$ we have*

(i) $\Phi_{1,\ell,m}(q) = -2A_{\ell,m}(q)\varphi(q^\ell)\varphi(q^m)$,
(ii) $\Phi_{2,\ell,m}(q) = 2(6A^2_{\ell,m}(q) - B_{\ell,m}(q))\varphi(q^\ell)\varphi(q^m)$.

Proof. (i) Taking $k = 1$ in (2.16), we have

$$\Phi_{1,\ell,m}(q) = \ell\varphi_1(q^\ell)\varphi(q^m) - m\varphi(q^\ell)\varphi_1(q^m).$$

Appealing to Theorem 2.1(i) and (2.17), we deduce

$$\begin{aligned}\Phi_{1,\ell,m}(q) &= -2\ell\varphi(q^\ell)\varphi(q^m)\xi_1(-q^\ell) + 2m\varphi(q^\ell)\varphi(q^m)\xi_1(-q^m) \\ &= -2\varphi(q^\ell)\varphi(q^m)(\ell\xi_1(-q^\ell) - m\xi_1(-q^m)) = -2A_{\ell,m}(q)\varphi(q^\ell)\varphi(q^m).\end{aligned}$$

(ii) Taking $k = 2$ in (2.16), we have

$$\Phi_{2,\ell,m}(q) = \ell^2\varphi_2(q^\ell)\varphi(q^m) - 6\ell m\varphi_1(q^\ell)\varphi_1(q^m) + m^2\varphi(q^\ell)\varphi_2(q^m).$$

Appealing to Theorem 2.1(i), (ii) for the values of $\varphi_1(q)$ and $\varphi_2(q)$, we deduce

$$\begin{aligned}&\Phi_{2,\ell,m}(q) \\ &\quad = 2(6(\ell\xi_1(-q^\ell) - m\xi_1(-q^m))^2 - (\ell^2\xi_3(-q^\ell) + m^2\xi_3(-q^m)))\varphi(q^\ell)\varphi(q^m).\end{aligned}$$

Then, appealing to (2.17), we obtain

$$\Phi_{2,\ell,m}(q) = 2(6A^2_{\ell,m}(q) - B_{\ell,m}(q))\varphi(q^\ell)\varphi(q^m). \ \blacksquare$$

By (2.4) we have

$$\varphi(q^\ell)\varphi(q^m) = \frac{E_{2\ell}^5 E_{2m}^5}{E_\ell^2 E_m^2 E_{4\ell}^2 E_{4m}^2}. \tag{2.20}$$

Thus, by Theorem 2.2(i) and (2.20), we see that $\Phi_{1,\ell,m}(q)$ can be expressed as an infinite product if $A_{\ell,m}(q)$ can be expressed as a product of finitely many E_r ($r \in \mathbb{N}$). Similarly, by Theorem 2.2(ii), (2.19) and (2.20), $\Phi_{2,\ell,\ell}(q)$ can be expressed as an infinite product if $B_{\ell,\ell}(q)$ can be expressed as a product of finitely many E_r ($r \in \mathbb{N}$). To this end we prove the following result.

THEOREM 2.3. *For $q \in \mathbb{C}$ with $|q| < 1$ we have*

(i) $A_{1,2}(q) = -q\dfrac{E_1^4 E_8^4}{E_2^2 E_4^2}$,

(ii) $A_{1,3}(q) = -q\dfrac{E_1^2 E_3^2 E_4^2 E_{12}^2}{E_2^2 E_6^2}$,

(iii) $A_{1,4}(q) = -q\dfrac{E_1^2 E_4^{10} E_{16}^2}{E_2^5 E_8^5}$,

(iv) $B_{1,1}(q) = -2q\dfrac{E_1^8 E_4^8}{E_2^8}$.

Proof. (i) By (2.17) we have $A_{1,2}(q) = \xi_1(-q) - 2\xi(-q^2)$. From (2.2) we have

$$\xi_1(-q) = \sum_{n=1}^{\infty} \frac{n(-q)^n}{1-q^{2n}}, \qquad \xi_1(-q^2) = \sum_{n=1}^{\infty} \frac{n(-q^2)^n}{1-q^{4n}}.$$

Hence

$$A_{1,2}(q) = \sum_{n=1}^{\infty} n(-1)^n \left(\frac{q^n}{1-q^{2n}} - \frac{2q^{2n}}{1-q^{4n}} \right).$$

Now

$$\frac{q^n}{1-q^{2n}} = \frac{q^n}{1-(-1)^n q^n} - (-1)^n \frac{q^{2n}}{1-q^{2n}}$$

and

$$\frac{2q^{2n}}{1-q^{4n}} = \frac{q^{2n}}{1-(-1)^n q^{2n}} - (1+(-1)^n)\frac{q^{4n}}{1-q^{4n}} + \frac{q^{2n}}{1-q^{2n}}.$$

Thus

$$\begin{aligned} A_{1,2}(q) &= \sum_{n=1}^{\infty} n(-1)^n \left(\frac{q^n}{1-(-1)^n q^n} - \frac{q^{2n}}{1-(-1)^n q^{2n}} \right) \\ &\quad - \sum_{n=1}^{\infty} n(-1)^n (1+(-1)^n) \left(\frac{q^{2n}}{1-q^{2n}} - \frac{q^{4n}}{1-q^{4n}} \right). \end{aligned}$$

Define

$$F(q) := \sum_{n=1}^{\infty} n(-1)^n \left(\frac{q^n}{1-(-1)^n q^n} - \frac{q^{2n}}{1-(-1)^n q^{2n}} \right),$$
$$G(q) := \sum_{n=1}^{\infty} n \left(\frac{q^{4n}}{1-q^{4n}} - \frac{q^{8n}}{1-q^{8n}} \right).$$

Then

$$A_{1,2}(q) = F(q) - 4G(q).$$

It is well-known that

$$\varphi^4(q) = 1 + 8 \sum_{\substack{n=1 \\ 4\nmid n}}^{\infty} \frac{nq^n}{1-q^n}$$

(see for example Berndt [3, p. 61]). Thus

$$\begin{aligned}
\tfrac{1}{8}(\varphi^4(-q) - \varphi^4(-q^2)) &= \sum_{\substack{n=1 \\ 4\nmid n}}^{\infty} n(-1)^n \left(\frac{q^n}{1-(-1)^n q^n} - \frac{q^{2n}}{1-(-1)^n q^{2n}} \right) \\
&= \sum_{n=1}^{\infty} n(-1)^n \left(\frac{q^n}{1-(-1)^n q^n} - \frac{q^{2n}}{1-(-1)^n q^{2n}} \right) \\
&\quad - 4 \sum_{n=1}^{\infty} n \left(\frac{q^{4n}}{1-q^{4n}} - \frac{q^{8n}}{1-q^{8n}} \right) \\
&= F(q) - 4G(q).
\end{aligned}$$

Hence, by (2.7), (2.12), (2.5) and (2.6), we have

$$\begin{aligned}
A_{1,2}(q) &= \tfrac{1}{8}(\varphi^4(-q) - \varphi^4(-q^2)) = \tfrac{1}{8}(\varphi^4(-q) - \varphi^2(q)\varphi^2(-q)) \\
&= -\tfrac{1}{8}\varphi^2(-q)(\varphi^2(q) - \varphi^2(-q)) = -q\varphi^2(-q)\psi^2(q^4) \\
&= -q\left(\frac{E_1^2}{E_2}\right)^2 \left(\frac{E_8^2}{E_4}\right)^2 = -q\frac{E_1^4 E_8^4}{E_2^2 E_4^2}.
\end{aligned}$$

(ii) We recall (see e.g. [2, p. 223]) the identity

$$\sum_{\substack{n=1 \\ 3\nmid n}}^{\infty} \frac{nq^n}{1-q^n} = q\psi^2(q)\psi^2(q^3).$$

By (2.2) and (2.17) the left hand side is $\xi_1(q) - 3\xi_1(q^3) = A_{1,3}(-q)$. By (2.6) the right hand side is $q\frac{E_2^4 E_6^4}{E_1^2 E_3^2}$. Thus

$$A_{1,3}(-q) = q\frac{E_2^4 E_6^4}{E_1^2 E_3^2}.$$

Changing q to $-q$, and appealing to (1.2), we obtain

$$A_{1,3}(q) = -q\frac{E_1^2E_3^2E_4^2E_{12}^2}{E_2^2E_6^2}.$$

(iii) We note that in the course of the proof of part (i), we showed that

$$A_{1,2}(q) = -q\varphi^2(-q)\psi^2(q^4). \tag{2.21}$$

Appealing to (2.17), (2.21), (2.8), (2.7), (2.9), (2.10), (2.11), (2.5), (2.4) and (2.6), we obtain

$$\begin{aligned}A_{1,4}(q) &= \xi_1(-q) - 4\xi_1(-q^4) = (\xi_1(-q) - 2\xi_1(-q^2)) + 2(\xi_1(-q^2) - 2\xi_1(-q^4))\\ &= A_{1,2}(q) + 2A_{1,2}(q^2) = -q\varphi^2(-q)\psi^2(q^4) - 2q^2\varphi^2(-q^2)\psi^2(q^8)\\ &= -q\varphi(-q)\psi(q^8)(\varphi(-q)\varphi(q^4) + 2q\varphi(q)\psi(q^8))\\ &= -\tfrac{1}{2}q\varphi(-q)\psi(q^8)(\varphi(-q)(\varphi(q)+\varphi(-q)) + \varphi(q)(\varphi(q)-\varphi(-q)))\\ &= -\tfrac{1}{2}q\varphi(-q)\psi(q^8)(\varphi^2(q)+\varphi^2(-q)) = -q\varphi(-q)\varphi^2(q^2)\psi(q^8)\\ &= -q\left(\frac{E_1^2}{E_2}\right)\left(\frac{E_4^5}{E_2^2E_8^2}\right)^2\left(\frac{E_{16}^2}{E_8}\right) = -q\frac{E_1^2E_4^{10}E_{16}^2}{E_2^5E_8^5}.\end{aligned}$$

(iv) The following identity is well-known:

$$q\psi^8(q) = \sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^{2n}}$$

(see for example Cooper [8, eq. (3.71), p. 136]). Hence, by (2.2), we have $\xi_3(q) = q\psi^8(q)$. Appealing to (2.6), we deduce

$$\xi_3(q) = q\frac{E_2^{16}}{E_1^8}.$$

Changing q to $-q$, and appealing to (1.2), we obtain

$$\xi_3(-q) = -q\frac{E_1^8E_4^8}{E_2^8}.$$

Then, by (2.19), we have $B_{1,1}(q) = 2\xi_3(-q) = -2q\frac{E_1^8E_4^8}{E_2^8}$. ∎

We are now ready to evaluate $\Phi_{1,1,2}(q)$, $\Phi_{1,1,3}(q)$, $\Phi_{1,1,4}(q)$ and $\Phi_{2,1,1}(q)$.

THEOREM 2.4. *For $q \in \mathbb{C}$ with $|q| < 1$ we have*

(i) $\Phi_{1,1,2}(q) = \sum_{r,s=-\infty}^{\infty}(r+s\sqrt{-2})^2q^{r^2+2s^2} = 2qE_1^2E_2E_4E_8^2$,

(ii) $\Phi_{1,1,3}(q) = \sum_{r,s=-\infty}^{\infty}(r+s\sqrt{-3})^2q^{r^2+3s^2} = 2qE_2^3E_6^3$,

(iii) $\Phi_{1,1,4}(q) = \sum_{r,s=-\infty}^{\infty} (r + s\sqrt{-4})^2 q^{r^2+4s^2} = 2qE_4^6$,

(iv) $\Phi_{2,1,1}(q) = \sum_{r,s=-\infty}^{\infty} (r + s\sqrt{-1})^4 q^{r^2+s^2} = 4qE_1^4 E_2^2 E_4^4$.

Proof. (i) By Theorem 2.2(i), Theorem 2.3(i) and (2.20) we have

$$\begin{aligned}\Phi_{1,1,2}(q) &= -2A_{1,2}(q)\varphi(q)\varphi(q^2) = -2\left(-q\frac{E_1^4 E_8^4}{E_2^2 E_4^2}\right)\left(\frac{E_2^3 E_4^3}{E_1^2 E_8^2}\right)\\ &= 2qE_1^2 E_2 E_4 E_8^2.\end{aligned}$$

(ii) By Theorem 2.2(i), Theorem 2.3(ii) and (2.20) we have

$$\begin{aligned}\Phi_{1,1,3}(q) &= -2A_{1,3}(q)\varphi(q)\varphi(q^3) = -2\left(-q\frac{E_1^2 E_3^2 E_4^2 E_{12}^2}{E_2^2 E_6^2}\right)\left(\frac{E_2^5 E_6^5}{E_1^2 E_3^2 E_4^2 E_{12}^2}\right)\\ &= 2qE_2^3 E_6^3.\end{aligned}$$

(iii) By Theorem 2.2(i), Theorem 2.3(iii) and (2.20) we have

$$\begin{aligned}\Phi_{1,1,4}(q) &= -2A_{1,4}(q)\varphi(q)\varphi(q^4) = -2\left(-q\frac{E_1^2 E_4^{10} E_{16}^2}{E_2^5 E_8^5}\right)\left(\frac{E_2^5 E_8^5}{E_1^2 E_4^4 E_{16}^2}\right)\\ &= 2qE_4^6.\end{aligned}$$

(iv) By Theorem 2.2(ii), (2.19), Theorem 2.3(iv) and (2.4) we have

$$\begin{aligned}\Phi_{2,1,1}(q) &= 2(6A_{1,1}^2(q) - B_{1,1}(q))\varphi^2(q) = -2B_{1,1}(q)\varphi^2(q)\\ &= -2\left(-2q\frac{E_1^8 E_4^8}{E_2^8}\right)\left(\frac{E_2^{10}}{E_1^4 E_4^4}\right) = 4qE_1^4 E_2^2 E_4^4. \ ∎\end{aligned}$$

As $\sum_{r=-\infty}^{\infty} r^{2k-1} q^{r^2} = 0$ for $k \in \mathbb{N}$, we have

$$(2.22)\qquad \sum_{r,s=-\infty}^{\infty} (r + s\sqrt{-m})^2 q^{r^2+ms^2} = \sum_{r,s=-\infty}^{\infty} (r^2 - ms^2) q^{r^2+ms^2}, \qquad m \in \mathbb{N},$$

and

$$\begin{aligned}\sum_{r,s=-\infty}^{\infty} (r + s\sqrt{-1})^4 q^{r^2+s^2} &= \sum_{r,s=-\infty}^{\infty} (r^4 - 6r^2 s^2 + s^4) q^{r^2+s^2}\\ &= \sum_{r,s=-\infty}^{\infty} ((r^4 - 3r^2 s^2) q^{r^2+s^2} + (s^4 - 3s^2 r^2) q^{s^2+r^2})\end{aligned}$$

so that

$$(2.23)\qquad \sum_{r,s=-\infty}^{\infty} (r + s\sqrt{-1})^4 q^{r^2+s^2} = 2\sum_{r,s=-\infty}^{\infty} (r^4 - 3r^2 s^2) q^{r^2+s^2}.$$

3. Proof of Theorem 1.1. Let $k \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$. Let $r, s, t, u \in \mathbb{N}_0$ be such that

$$r + s + t + u = k. \tag{3.1}$$

Let $v, w, x, y \in \mathbb{N}_0$ be such that

$$v + w + x + y = \ell. \tag{3.2}$$

Set

$$m = k + 2\ell \tag{3.3}$$

so that $m \in \mathbb{N}$ and $m \geq 2$. We consider the product

$$\begin{aligned} \Pi(q) &:= \varphi(q)^{r+v+w+x+2y}\varphi(q^2)^{s+v}\varphi(q^3)^{t+w}\varphi(q^4)^{u+x} \\ &\quad \times A_{1,2}(q)^v A_{1,3}(q)^w A_{1,4}(q)^x B_{1,1}(q)^y. \end{aligned} \tag{3.4}$$

Using the infinite product representations of $\varphi(q)$, $\varphi(q^2)$, $\varphi(q^3)$ and $\varphi(q^4)$, which follow from (2.4), as well as the values of $A_{1,2}(q)$, $A_{1,3}(q)$, $A_{1,4}(q)$ and $B_{1,1}(q)$ given in Theorem 2.3, (3.4) becomes

$$\begin{aligned} \Pi(q) &= (-1)^\ell 2^y q^\ell E_1^{-2r+2v+4y} E_2^{5r-2s+v+3w+2y} E_3^{-2t} \\ &\quad \times E_4^{-2r+5s-2u+v+6x+4y} E_6^{5t+3w} E_8^{-2s+5u+2v} E_{12}^{-2t} E_{16}^{-2u}. \end{aligned} \tag{3.5}$$

On the other hand, from Theorem 2.2(i), we have

$$A_{1,2}(q) = \frac{\Phi_{1,1,2}(q)}{-2\varphi(q)\varphi(q^2)}, \quad A_{1,3}(q) = \frac{\Phi_{1,1,3}(q)}{-2\varphi(q)\varphi(q^3)}, \quad A_{1,4}(q) = \frac{\Phi_{1,1,4}(q)}{-2\varphi(q)\varphi(q^4)},$$

and, from (2.19) and Theorem 2.2(ii),

$$B_{1,1}(q) = \frac{\Phi_{2,1,1}(q)}{-2\varphi^2(q)}.$$

Then we deduce from (3.2) and (3.4) that

$$\begin{aligned} \Pi(q) &:= \frac{(-1)^\ell}{2^\ell}\varphi(q)^r\varphi(q^2)^s\varphi(q^3)^t\varphi(q^4)^u \\ &\quad \times \Phi_{1,1,2}(q)^v\Phi_{1,1,3}(q)^w\Phi_{1,1,4}(q)^x\Phi_{2,1,1}(q)^y. \end{aligned} \tag{3.6}$$

Hence, by (2.4), (2.13), (2.22) and (2.23), we obtain

$$\begin{aligned} \Pi(q) &= \frac{(-1)^\ell}{2^\ell}\Big(\sum_{i\in\mathbb{Z}} q^{i^2}\Big)^r\Big(\sum_{i\in\mathbb{Z}} q^{2i^2}\Big)^s\Big(\sum_{i\in\mathbb{Z}} q^{3i^2}\Big)^t\Big(\sum_{i\in\mathbb{Z}} q^{4i^2}\Big)^u \\ &\quad \times \Big(\sum_{(i,j)\in\mathbb{Z}^2} (i^2-2j^2)q^{i^2+2j^2}\Big)^v\Big(\sum_{(i,j)\in\mathbb{Z}^2} (i^2-3j^2)q^{i^2+3j^2}\Big)^w \\ &\quad \times \Big(\sum_{(i,j)\in\mathbb{Z}^2} (i^2-4j^2)q^{i^2+4j^2}\Big)^x\Big(2\sum_{(i,j)\in\mathbb{Z}^2} (i^4-3i^2j^2)q^{i^2+j^2}\Big)^y. \end{aligned} \tag{3.7}$$

Next we express the factors in the product (3.7) in the following way:

$$\Big(\sum_{i\in\mathbb{Z}} q^{i^2}\Big)^r = \sum_{(x_1,\dots,x_r)\in\mathbb{Z}^r} q^{x_1^2+\cdots+x_r^2},$$

$$\Big(\sum_{i\in\mathbb{Z}} q^{2i^2}\Big)^s = \sum_{(x_{r+\ell+y+1},\dots,x_{r+s+\ell+y})\in\mathbb{Z}^s} q^{2x_{r+\ell+y+1}^2+\cdots+2x_{r+s+\ell+y}^2},$$

$$\Big(\sum_{i\in\mathbb{Z}} q^{3i^2}\Big)^t = \sum_{(x_{r+s+\ell+v+y+1},\dots,x_{r+s+t+\ell+v+y})\in\mathbb{Z}^t} q^{3x_{r+s+\ell+v+y+1}^2+\cdots+3x_{r+s+t+\ell+v+y}^2},$$

$$\Big(\sum_{i\in\mathbb{Z}} q^{4i^2}\Big)^u = \sum_{(x_{r+s+t+\ell+v+w+y+1},\dots,x_{k+\ell+v+w+y})\in\mathbb{Z}^u} q^{4x_{r+s+t+\ell+v+y+1}^2+\cdots+4x_{k+\ell+v+w+y}^2};$$

$$\Big(\sum_{(i,j)\in\mathbb{Z}^2} (i^2-2j^2)q^{i^2+2j^2}\Big)^v$$

$$= \sum \prod_{g=r+1}^{r+v} (x_g^2 - 2x_{g+s+\ell+y}^2) q^{x_{r+1}^2+\cdots+x_{r+v}^2+2x_{r+s+\ell+y+1}^2+\cdots+2x_{r+s+\ell+v+y}^2},$$

where the sum is over $(x_{r+1},\dots,x_{r+v},x_{r+s+\ell+y+1},\dots,x_{r+s+\ell+v+y})\in\mathbb{Z}^{2v}$;

$$\Big(\sum_{(i,j)\in\mathbb{Z}^2} (i^2-3j^2)q^{i^2+3j^2}\Big)^w = \sum \prod_{g=r+v+1}^{r+v+w} (x_g^2 - 3x_{g+s+t+\ell+y}^2)$$
$$\times\, q^{x_{r+v+1}^2+\cdots+x_{r+v+w}^2+3x_{r+s+t+\ell+v+y+1}^2+\cdots+3x_{r+s+t+\ell+v+w+y}^2},$$

where the sum is over

$$(x_{r+v+1},\dots,x_{r+v+w},x_{r+s+t+\ell+v+y+1},\dots,x_{r+s+t+\ell+v+w+y})\in\mathbb{Z}^{2w};$$

$$\Big(\sum_{(i,j)\in\mathbb{Z}^2} (i^2-4j^2)q^{i^2+4j^2}\Big)^x = \sum \prod_{g=r+v+w+1}^{r+v+w+x} (x_g^2 - 4x_{g+s+t+\ell+y+u}^2)$$
$$\times\, q^{x_{r+v+w+1}^2+\cdots+x_{r+v+w+x}^2+4x_{r+s+t+\ell+v+y+w+u+1}^2+\cdots+4x_m^2},$$

where the sum is over

$$(x_{r+v+w+1},\dots,x_{r+v+w+x},x_{r+s+t+\ell+v+y+w+u+1},\dots,x_m)\in\mathbb{Z}^{2x};$$

and

$$\Big(\sum_{(i,j)\in\mathbb{Z}^2} (i^4-3i^2j^2)q^{i^2+j^2}\Big)^y$$

$$= \sum \prod_{g=r+v+w+x+1}^{r+\ell} (x_g^4 - 3x_g^2x_{g+y}^2) q^{x_{r+v+w+x+1}^2+\cdots+x_{r+\ell}^2+x_{r+\ell+1}^2+\cdots+x_{r+\ell+y}^2},$$

where the sum is over $(x_{r+v+w+x+1},\dots,x_{r+\ell},x_{r+\ell+1},\dots,x_{r+\ell+y})\in\mathbb{Z}^{2y}$.

Using these in (3.7) we obtain

$$\Pi(q) = (-1)^{\ell} 2^{y} \sum_{(x_1,\ldots,x_m)\in\mathbb{Z}^m} P(x_1,\ldots,x_m) q^{Q(x_1,\ldots,x_m)},$$

that is,

$$\Pi(q) = (-1)^{\ell} 2^{y} \sum_{n=0}^{\infty} \sum_{\substack{(x_1,\ldots,x_m)\in\mathbb{Z}^m \\ Q(x_1,\ldots,x_m)=n}} P(x_1,\ldots,x_m). \tag{3.8}$$

Equating the two expressions for $\Pi(q)$ given in (3.5) and (3.8), we deduce

$$\begin{aligned} q^{\ell} E_1^{-2r+2v+4y} E_2^{5r-2s+v+3w+2y} E_3^{-2t} E_4^{-2r+5s-2u+v+6x+4y} \\ \times E_6^{5t+3w} E_8^{-2s+5u+2v} E_{12}^{-2t} E_{16}^{-2u} \\ = \sum_{n=0}^{\infty} \sum_{\substack{(x_1,\ldots,x_m)\in\mathbb{Z}^m \\ Q(x_1,\ldots,x_m)=n}} P(x_1,\ldots,x_m) q^n. \end{aligned} \tag{3.9}$$

Equating coefficients of q^n in (3.9) for $n \geq \ell$, we obtain (1.12). ∎

Incidentially, equating coefficients of q^n for $0 \leq n \leq \ell - 1$, we deduce

$$\sum_{\substack{(x_1,\ldots,x_m)\in\mathbb{Z}^m \\ Q(x_1,\ldots,x_m)=n}} P(x_1,\ldots,x_m) = 0, \qquad n = 0, 1, \ldots, \ell - 1.$$

We close this section by illustrating Theorem 1.1 with four examples.

Example 3.1. We choose

$$v = 1, \qquad r = s = t = u = w = x = y = 0,$$

so that $k = 0$, $\ell = 1$, $m = 2$. Then

$$P(x_1, x_2) = \tfrac{1}{2}(x_1^2 - 2x_2^2), \qquad Q(x_1, x_2) = x_1^2 + 2x_2^2,$$

and Theorem 1.1 gives the following result.

Theorem 3.1. *Let $n \in \mathbb{N}$. Then*

$$[qE_1^2 E_2 E_4 E_8^2]_n = \sum_{\substack{(x_1,x_2)\in\mathbb{Z}^2 \\ x_1^2+2x_2^2=n}} \tfrac{1}{2}(x_1^2 - 2x_2^2).$$

Example 3.2. We choose

$$r = v = 1, \qquad s = t = u = w = x = y = 0,$$

so that $k = \ell = 1$, $m = 3$. Then

$$P(x_1, x_2, x_3) = \tfrac{1}{2}(x_2^2 - 2x_3^2), \qquad Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2x_3^2,$$

and Theorem 1.1 gives the following result.

THEOREM 3.2. *Let $n \in \mathbb{N}$. Then*

$$\left[q\frac{E_2^6E_8^2}{E_4}\right]_n = \sum_{\substack{(x_1,x_2,x_3)\in\mathbb{Z}^3\\ x_1^2+x_2^2+2x_3^2=n}} \tfrac{1}{2}(x_2^2-2x_3^2).$$

EXAMPLE 3.3. We choose

$$w=y=1, \qquad r=s=t=u=v=x=0,$$

so that $k=0$, $\ell=2$, $m=4$. Then

$$P(x_1,x_2,x_3,x_4)=\tfrac{1}{4}(x_1^2-3x_4^2)(x_2^4-3x_2^2x_3^2),$$
$$Q(x_1,x_2,x_3,x_4)=x_1^2+x_2^2+x_3^2+3x_4^2.$$

Theorem 1.1 gives

$$[q^2E_1^4E_2^5E_4^4E_6^3]_n = \sum_{\substack{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4\\ x_1^2+x_2^2+x_3^2+3x_4^2=n}} \tfrac{1}{4}(x_1^2-3x_4^2)(x_2^4-3x_2^2x_3^2), \qquad n\geq 2.$$

Mapping $x_1\mapsto x_3$, $x_2\mapsto x_1$, $x_3\mapsto x_2$ in this sum, we obtain the following result.

THEOREM 3.3. *Let $n\in\mathbb{N}$ satisfy $n\geq 2$. Then*

$$[q^2E_1^4E_2^5E_4^4E_6^3]_n = \frac{1}{4}\sum_{\substack{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4\\ x_1^2+x_2^2+x_3^2+3x_4^2=n}} x_1^2(x_1^2-3x_2^2)(x_3^2-3x_4^2).$$

EXAMPLE 3.4. We choose

$$r=v=x=1, \qquad s=t=u=w=y=0,$$

so that $k=1$, $\ell=2$, $m=5$. Then

$$P(x_1,x_2,x_3,x_4,x_5)=\tfrac{1}{4}(x_2^2-2x_4^2)(x_3^2-4x_5^2),$$
$$Q(x_1,x_2,x_3,x_4,x_5)=x_1^2+x_2^2+x_3^2+2x_4^2+4x_5^2.$$

By Theorem 1.1 we have

$$[q^2E_2^6E_4^5E_8^2]_n = \frac{1}{4}\sum_{\substack{(x_1,x_2,x_3,x_4,x_5)\in\mathbb{Z}^5\\ x_1^2+x_2^2+x_3^2+2x_4^2+4x_5^2=n}} (x_2^2-2x_4^2)(x_3^2-4x_5^2), \qquad n\geq 2.$$

Clearly, for n odd we have $[q^2E_2^6E_4^5E_8^2]_n=0$ so

$$[q^2E_2^6E_4^5E_8^2]_{2n} = \frac{1}{4}\sum_{\substack{(x_1,x_2,x_3,x_4,x_5)\in\mathbb{Z}^5\\ x_1^2+x_2^2+x_3^2+2x_4^2+4x_5^2=2n}} (x_2^2-2x_4^2)(x_3^2-4x_5^2), \qquad n\geq 1.$$

Replacing q by q^2 we obtain the following theorem.

THEOREM 3.4. *Let $n \in \mathbb{N}$. Then*

$$[qE_1^6E_2^5E_4^2]_n = \frac{1}{4} \sum_{\substack{(x_1,x_2,x_3,x_4,x_5)\in\mathbb{Z}^5 \\ x_1^2+x_2^2+x_3^2+2x_4^2+4x_5^2=2n}} (x_2^2-2x_4^2)(x_3^2-4x_5^2).$$

4. Applications of Theorem 1.1. We give two applications of Theorem 1.1.

First application: Sums of 10, 12 and 14 squares. Let N be an integer with $N \geq 2$. We choose

$$r = N-2, \quad s=0, \quad t=0, \quad u=0, \quad v=0, \quad w=0, \quad x=0, \quad y=1,$$

so that $k = N-2$, $\ell = 1$, $m = N$. Then

$$P(x_1,\dots,x_N) = \tfrac{1}{2}(x_{N-1}^4 - 3x_{N-1}^2x_N^2), \quad Q(x_1,\dots,x_N) = x_1^2+x_2^2+\dots+x_N^2.$$

Theorem 1.1 gives, for all $n \in \mathbb{N}$,

$$[qE_1^{8-2N}E_2^{5N-8}E_4^{8-2N}]_n = \sum_{\substack{(x_1,\dots,x_N)\in\mathbb{Z}^N \\ x_1^2+\dots+x_N^2=n}} \tfrac{1}{2}(x_{N-1}^4 - 3x_{N-1}^2x_N^2). \tag{4.1}$$

Thus, relabelling x_1 as x_{N-1}, x_2 as x_N, x_{N-1} as x_1 and x_N as x_2, we obtain

$$\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{(x_1,\dots,x_N)\in\mathbb{Z}^N \\ x_1^2+\dots+x_N^2=n}} (x_1^4-3x_1^2x_2^2)\Bigg)q^n = 2qE_1^{8-2N}E_2^{5N-8}E_4^{8-2N}. \tag{4.2}$$

Taking $N=2$ in (4.2) we obtain

$$\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{(x_1,x_2)\in\mathbb{Z}^2 \\ x_1^2+x_2^2=n}} (x_1^4-3x_1^2x_2^2)\Bigg)q^n = 2qE_1^4E_2^2E_4^4. \tag{4.3}$$

The number $r_{10}(n)$ of representations of $n \in \mathbb{N}$ as a sum of 10 squares is given by

$$r_{10}(n) = \frac{4}{5}\sum_{\substack{d\in\mathbb{N} \\ d\mid n}}\left(\frac{-4}{d}\right)d^4 + \frac{64}{5}\sum_{\substack{d\in\mathbb{N} \\ d\mid n}}\left(\frac{-4}{n/d}\right)d^4 + \frac{32}{5}a(n)$$

(see for example [1, p. 1429]), where

$$\sum_{n=1}^{\infty} a(n)q^n = qE_1^4E_2^2E_4^4.$$

Thus

$$a(n) = \tfrac{1}{2}\sum_{\substack{(x_1,x_2)\in\mathbb{Z}^2 \\ x_1^2+x_2^2=n}} (x_1^4-3x_1^2x_2^2)$$

and so we have

$$r_{10}(n) = \frac{4}{5}\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right)d^4 + \frac{64}{5}\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{n/d}\right)d^4 + \frac{16}{5}\sum_{\substack{(x_1,x_2)\in\mathbb{Z}^2\\ x_1^2+x_2^2=n}}(x_1^4 - 3x_1^2x_2^2),$$

which is a formula first given by Liouville [17] in 1865 in a slightly different form. When $n \equiv 3 \pmod 4$ there are no $(x_1, x_2) \in \mathbb{Z}^2$ such that $x_1^2 + x_2^2 = n$, so

$$r_{10}(n) = \frac{4}{5}\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right)d^4 + \frac{64}{5}\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{n/d}\right)d^4, \qquad n \equiv 3 \pmod 4,$$

a formula first given by Eisenstein [10, p. 135; Werke I, p. 501] (see also Glaisher [15, p. 482]).

Taking $N = 3$ in (4.1) we obtain, for $n \in \mathbb{N}$,

$$[qE_1^2E_2^7E_4^2]_n = \sum_{\substack{(x_1,x_2,x_3)\in\mathbb{Z}^3\\ x_1^2+x_2^2+x_3^2=n}} \tfrac{1}{2}(x_2^4 - 3x_2^2x_3^2). \tag{4.4}$$

When $n \equiv 7 \pmod 8$ there are no integers x_1, x_2, x_3 satisfying $x_1^2 + x_2^2 + x_3^2 = n$, so

$$[qE_1^2E_2^7E_4^2]_n = 0, \qquad n \equiv 7 \pmod 8. \tag{4.5}$$

Taking $N = 4$ in (4.1) we obtain, for $n \in \mathbb{N}$,

$$[qE_2^{12}]_n = \sum_{\substack{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4\\ x_1^2+x_2^2+x_3^2+x_4^2=n}} \tfrac{1}{2}(x_3^4 - 3x_3^2x_4^2), \tag{4.6}$$

so

$$\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4\\ x_1^2+x_2^2+x_3^2+x_4^2=n}} (x_1^4 - 3x_1^2x_2^2)\Bigg)q^n = 2qE_2^{12}. \tag{4.7}$$

Now the number $r_{12}(n)$ of representations of $n \in \mathbb{N}$ as the sum of 12 squares is given by

$$r_{12}(n) = 8\sigma_5(n) - 512\sigma_5(n/4) + 16b(n)$$

(see for example [22, p. 241]), where

$$\sum_{n=1}^{\infty} b(n)q^n = qE_2^{12}.$$

Thus

$$b(n) = \tfrac{1}{2}\sum_{\substack{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^2\\ x_1^2+x_2^2+x_1^2+x_2^2=n}} (x_1^4 - 3x_1^2x_2^2)$$

and so

$$r_{12}(n) = 8\sigma_5(n) - 512\sigma_5(n/4) + 8 \sum_{\substack{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^2 \\ x_1^2+x_2^2+x_1^2+x_2^2=n}} (x_1^4 - 3x_1^2x_2^2),$$

which is a formula of Bulygin [4] (see also for example Carlitz [5, p. 411], Glaisher [15, p. 484] and Lomadze [18, p. 9]).

Taking $N = 6$ in (4.2) we obtain, for $n \in \mathbb{N}$,

$$\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{(x_1,\dots,x_6)\in\mathbb{Z}^6 \\ x_1^2+\dots+x_6^2=n}} (x_1^4 - 3x_1^2x_2^2)\Bigg)q^n = 2q\frac{E_2^{22}}{E_1^4E_4^4}. \tag{4.8}$$

Taking $m = 3$ in Cooper [8, Theorem 3.3, p. 131] we have

$$\varphi^{14}(q) = 1 + \frac{4}{61}\sum_{j=1}^{\infty}\frac{(-1)^j(2j-1)^6q^{2j-1}}{1-q^{2j-1}} + \frac{256}{61}\sum_{j=1}^{\infty}\frac{j^6q^j}{1+q^{2j}} + \frac{1456}{61}qE_1^4(-q)E_1^{10}(q^2).$$

Now

$$\sum_{j=1}^{\infty}\frac{(-1)^j(2j-1)^6q^{2j-1}}{1-q^{2j-1}} = -\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{d\in\mathbb{N} \\ d\mid n}}\left(\frac{-4}{d}\right)d^6\Bigg)q^n,$$

$$\sum_{j=1}^{\infty}\frac{j^6q^j}{1+q^{2j}} = \sum_{n=1}^{\infty}\Bigg(\sum_{\substack{d\in\mathbb{N} \\ d\mid n}}\left(\frac{-4}{n/d}\right)d^6\Bigg)q^n,$$

and by (1.2) and (4.8),

$$qE_1^4(-q)E_1^{10}(q^2) = q\frac{E_2^{22}}{E_1^4E_4^4} = \tfrac{1}{2}\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{(x_1,\dots,x_6)\in\mathbb{Z}^6 \\ x_1^2+\dots+x_6^2=n}} (x_1^4 - 3x_1^2x_2^2)\Bigg)q^n.$$

Hence

$$\sum_{n=0}^{\infty} r_{14}(n)q^n = \varphi^{14}(q) = 1 + \frac{256}{61}\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{d\in\mathbb{N} \\ d\mid n}}\left(\frac{-4}{n/d}\right)d^6\Bigg)q^n - \frac{4}{61}\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{d\in\mathbb{N} \\ d\mid n}}\left(\frac{-4}{d}\right)d^6\Bigg)q^n + \frac{728}{61}\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{(x_1,\dots,x_6)\in\mathbb{Z}^6 \\ x_1^2+\dots+x_6^2=n}} (x_1^4 - 3x_1^2x_2^2)\Bigg)q^n.$$

Equating coefficients of q^n ($n \in \mathbb{N}$), we obtain

$$r_{14}(n) = \frac{256}{61}\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{n/d}\right)d^6 - \frac{4}{61}\sum_{\substack{d\in\mathbb{N}\\ d\mid n}}\left(\frac{-4}{d}\right)d^6 + \frac{728}{61}\sum_{\substack{(x_1,\dots,x_6)\in\mathbb{Z}^6\\ x_1^2+\dots+x_6^2=n}}(x_1^4 - 3x_1^2x_2^2).$$

This formula can be found in Bulygin [4], Glaisher [15, p. 480] and Lomadze [18, p. 9].

Second application: Ramanujan's tau function. We recall that Ramanujan's tau function $\tau(n)$ is defined for $n \in \mathbb{N}$ by

$$q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n.$$

A number of explicit formulae for $\tau(n)$ have appeared in the literature: see for example Chan, Cooper and Toh [7], Dyson [9], Ewell [11, 12, 13], Gallardo [14] and Niebur [20]. We just mention Dyson's formula,

$$\tau(n) = \sum_{\substack{(x_1,\dots,x_5)\in\mathbb{Z}^5\\ (x_1,x_2,x_3,x_4,x_5)\equiv(1,2,3,4,0) \pmod 5\\ x_1+\dots+x_5=0\\ x_1^2+\dots+x_5^2=10n}} F(x_1,\dots,x_5),$$

where

$$F(x_1,\dots,x_5) := \frac{1}{1!2!3!4!}\prod_{1\le r<s\le 5}(x_r - x_s),$$

as well as the formula of Chan, Cooper and Toh,

$$\tau(n) = -\frac{1}{4320\sqrt{3}}\sum_{\substack{(x_1,\dots,x_4)\in\mathbb{Z}^4\\ x_1^2+x_2^2+x_3^2+3x_4^2=12n\\ x_1\equiv 1 \pmod 6\\ x_2\equiv 4 \pmod 6\\ x_3\equiv 2 \pmod 6\\ x_4\equiv 1 \pmod 4}}(-1)^{(x_3-2)/6}\,\mathrm{Im}((x_1+ix_2)^4)\,\mathrm{Im}((x_3+ix_4\sqrt{3})^6).$$

We use Theorem 1.1 to prove the following new formula for $\tau(n)$.

THEOREM 4.1. *For $n \in \mathbb{N}$ we have*

$$\tau(n) = \tfrac{1}{4}\sum_{\substack{(x_1,\dots,x_8)\in\mathbb{Z}^8\\ x_1^2+\dots+x_8^2=2n}} x_1^2x_2^2(x_1^2-3x_3^2)(x_2^2-3x_4^2).$$

Proof. We choose

$$r=4,\quad s=0,\quad t=0,\quad u=0,\quad v=0,\quad w=0,\quad x=0,\quad y=2,$$

so that $k=4$, $\ell=2$, $m=8$. Then

$$P(x_1,\dots,x_8) = \frac{1}{2^2}(x_5^4-3x_5^2x_7^2)(x_6^4-3x_6^2x_8^2),\quad Q(x_1,\dots,x_8) = x_1^2+\dots+x_8^2.$$

With this choice Theorem 1.1 gives

$$[q^2E_2^{24}]_n = \frac{1}{4} \sum_{\substack{(x_1,\ldots,x_8)\in\mathbb{Z}^8\\ x_1^2+\cdots+x_8^2=n}} (x_5^4 - 3x_5^2x_7^2)(x_6^4 - 3x_6^2x_8^2), \qquad n \geq 2.$$

Hence

$$[q^2E_2^{24}]_n = \frac{1}{4} \sum_{\substack{(x_1,\ldots,x_8)\in\mathbb{Z}^8\\ x_1^2+\cdots+x_8^2=n}} x_1^2x_2^2(x_1^2 - 3x_3^2)(x_2^2 - 3x_4^2), \qquad n \geq 2.$$

Clearly, for n odd we have

$$[q^2E_2^{24}]_n = 0.$$

Thus

$$[q^2E_2^{24}]_{2n} = \frac{1}{4} \sum_{\substack{(x_1,\ldots,x_8)\in\mathbb{Z}^8\\ x_1^2+\cdots+x_8^2=2n}} x_1^2x_2^2(x_1^2 - 3x_3^2)(x_2^2 - 3x_4^2), \qquad n \geq 1.$$

Now

$$[q^2E_2^{24}]_{2n} = [qE_1^{24}]_n = \tau(n)$$

and the asserted formula follows. ■

Many other applications of Theorem 1.1 are possible.

5. Final comments. An important ingredient in the proof of Theorem 1.1 is the fact that each of $A_{1,2}(q)$, $A_{1,3}(q)$, $A_{1,4}(q)$ and $B_{1,1}(q)$ is expressible as a single infinite product consisting of a product of certain of the E_k ($k \in \mathbb{N}$) (Theorem 2.3). Any other $A_{\ell,m}(q)$ or $6A_{\ell,m}^2(q) - B_{\ell,m}(q)$ expressible in this manner would permit an extension of Theorem 1.1 via Theorems 2.2 and 2.3.

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