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Zero-cycles and rational points on some surfaces over a global function field

by

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1. Introduction. Study of the case of curves (Cassels, Tate) and of the case of rational surfaces (Colliot-Thélène et Sansuc [CT/S81], where a more precise conjecture is made for rational surfaces) has led to the following conjecture for *zero-cycles* on arbitrary varieties over global fields (Kato and Saito [K/S86], Saito [S89], Colliot-Thélène [CT95], [CT99]).

Conjecture 1.1. *Let X be a smooth, projective, geometrically integral variety over a global field k. If there exists a family $\{z_v\}_{v\in\Omega}$ of local zero-cycles of degree* 1 (*here v runs through the set Ω of places of k*) *such that for all $A \in \mathrm{Br}(X)$,*

$$\sum_{v\in\Omega} \mathrm{inv}_v(A(z_v)) = 0 \in \mathbb{Q}/\mathbb{Z},$$

then there exists a zero-cycle of degree 1 *on X. In other words, the Brauer–Manin obstruction to the existence of a zero-cycle of degree* 1 *on X is the only obstruction.*

Over number fields, this conjecture has been established in special cases in work of (alphabetical order, and various combinations) Colliot-Thélène, Frossard, Salberger, Sansuc, Skorobogatov, Swinnerton-Dyer, Wittenberg (see the introduction of [W10]). None of these results applies to smooth surfaces of degree d at least 3 in 3-dimensional projective space—for $d \geq 5$ these surfaces are of general type. In Section 2, we establish the conjecture in the special case of a global field $k = \mathbb{F}(t)$ purely transcendental over a finite field $\mathbb{F}$ and of smooth surfaces $X \subset \mathbb{P}^3_k$ defined by an equation

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$f + tg = 0$, where f and g are two forms of arbitrary degree d over the field $\mathbb{F}$.

According to a conjecture of Colliot-Thélène and Sansuc [CT/S80], the Brauer–Manin obstruction to the existence of a *rational point* on a smooth, geometrically rational surface defined over a global field should be the only obstruction. Such should in particular be the case for smooth cubic surfaces in 3-dimensional projective space $\mathbb{P}^3_k$. In Section 3, we establish the conjecture in the special case of a global field $k = \mathbb{F}(t)$ purely transcendental over a finite field $\mathbb{F}$ and of smooth cubic surfaces $X \subset \mathbb{P}^3_k$ defined by an equation $f + tg = 0$, where f and g are two cubic forms over the field $\mathbb{F}$. Simple though they be, such surfaces may fail to obey the Hasse principle.

2. Zero-cycles of degree 1 on surfaces of arbitrary degre. The following theorem is due to S. Saito [S89]. It says that if a strong integral form of the Tate conjecture on 1-dimensional cycles is true, then the above conjecture holds, at least if we stay away from the characteristic of the field. For an alternative proof of Theorem 2.1, see [CT99, Prop. 3.2].

THEOREM 2.1 (Saito). *Let $\mathbb{F}$ be a finite field and $C/\mathbb{F}$ a smooth, projective, geometrically integral curve over $\mathbb{F}$. Let $k = \mathbb{F}(C)$ be its function field. Let $\mathcal{X}$ be a smooth, projective, geometrically integral $\mathbb{F}$-variety of dimension n and $f : \mathcal{X} \to C$ a faithfully flat map whose generic fibre X/k is smooth and geometrically integral. Assume:*

(1) *For each prime $l \neq \mathrm{char}(\mathbb{F})$, the cycle map*

$$T_X : \mathrm{CH}^{n-1}(\mathcal{X}) \otimes \mathbb{Z}_l \to H^{2n-2}_{\text{ét}}(\mathcal{X}, \mathbb{Z}_l(n-1))$$

from the Chow group of dimension 1 cycles on $\mathcal{X}$ to étale cohomology is onto.

(2) *There exists a family $\{z_v\}_{v \in \Omega}$ of local zero-cycles of degree 1 (here v runs through the set Ω of places of k) such that for all $A \in \mathrm{Br}(X)$,*

$$\sum_{v \in \Omega} \mathrm{inv}_v(A(z_v)) = 0 \in \mathbb{Q}/\mathbb{Z}.$$

Then there exists a zero-cycle on X of degree a power of $\mathrm{char}(\mathbb{F})$.

In this statement, $A(z_v)$ is the element of the Brauer group of the local field k_v obtained by evaluation of A on the zero-cycle z_v. The map inv_v : $\mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$ is the local invariant of class field theory.

Here is one case where assumption (1) in the previous theorem is fulfilled.

THEOREM 2.2. *Let $\mathbb{F}$ be a finite field and l a prime, $l \neq \mathrm{char}(\mathbb{F})$. For a smooth, projective, geometrically integral threefold $\mathcal{X}$ over $\mathbb{F}$ which is birational to $\mathbb{P}^3_F$, the cycle map $T_{\mathcal{X}} : \mathrm{CH}^2(\mathcal{X}) \otimes \mathbb{Z}_l \to H^4_{\text{ét}}(\mathcal{X}, \mathbb{Z}_l(2))$ is onto.*

Proof. If $\mathcal{X} = \mathbb{P}^3_{\mathbb{F}}$, then $\mathrm{CH}^2(\mathcal{X}) = \mathbb{Z}$ and one easily checks that the cycle map

$$T_{\mathcal{X}} : \mathrm{CH}^2(\mathcal{X}) \otimes \mathbb{Z}_l \to H^4_{\text{ét}}(\mathcal{X}, \mathbb{Z}_l(2))$$

is simply the identity map $\mathbb{Z}_l \to \mathbb{Z}_l$. Using the standard formulas for the computation of Chow groups and of cohomology for a blow-up along a smooth projective subvariety, as well as the vanishing of Brauer groups of smooth projective curves over a finite field, one shows: For $\mathcal{X}$ a smooth projective threefold, the cokernel of the above cycle map $T_{\mathcal{X}}$ is invariant under blow-up of smooth projective subvarieties on $\mathcal{X}$.

By a result of Abhyankar [Abh66, Thm. 9.1.6], there exists a smooth projective variety $\mathcal{X}'$ which is obtained from $\mathbb{P}^3_{\mathbb{F}}$ by a sequence of blow-ups along smooth projective $\mathbb{F}$-subvarieties, and which is equipped with a birational $\mathbb{F}$-morphism $p : \mathcal{X}' \to \mathcal{X}$.

There are push-forward maps π_* and pull-back maps π^* both for Chow groups and for étale cohomology, and for the birational map π we have $\pi_* \circ \pi^* = \mathrm{id}$. Moreover these maps are compatible with the cycle class map. Thus the cokernel of $T_{\mathcal{X}}$ is a subgroup of the cokernel of $T_{\mathcal{X}'}$, hence is zero. ∎

Combining Theorems 2.1 and 2.2, we get:

THEOREM 2.3. *Let $\mathbb{F}$ be a finite field and $C/\mathbb{F}$ a smooth, projective, geometrically integral curve over $\mathbb{F}$. Let $k = \mathbb{F}(C)$ be its function field. Let $\mathcal{X}$ be a smooth, projective, geometrically integral $\mathbb{F}$-variety of dimension n and $f : \mathcal{X} \to C$ a faithfully flat map whose generic fibre X/k is smooth and geometrically integral. Assume:*

(1) $\dim \mathcal{X} = 3$ *and $\mathcal{X}$ is $\mathbb{F}$-rational.*
(2) *There exists a family $\{z_v\}_{v\in\Omega}$ of local zero-cycles of degree* 1 (*here v runs through the set Ω of places of k*) *such that for all $A \in \mathrm{Br}(X)$,*

$$\sum_{v\in\Omega} \mathrm{inv}_v(A(z_v)) = 0 \in \mathbb{Q}/\mathbb{Z}.$$

Then there exists a zero-cycle on X of degree a power of $\mathrm{char}(\mathbb{F})$.

We may now prove the main result of this section.

THEOREM 2.4. *Let $\mathbb{F}$ be a finite field, let f, g be two nonproportional homogeneous forms in* 4 *variables, of degree d prime to the characteristic of $\mathbb{F}$. Let $k = \mathbb{F}(t)$. Suppose the k-surface $X \subset \mathbb{P}^3_k$ defined by $f + tg = 0$ is smooth. If there is no Brauer–Manin obstruction to the Hasse principle for zero-cycles of degree* 1 *on X, then:*

(i) *There exists a zero-cycle of degree* 1 *on the k-surface X.*
(ii) *There exists a zero-cycle of degree* 1 *on the $\mathbb{F}$-curve Γ defined by $f = g = 0$ in $\mathbb{P}^3_{\mathbb{F}}$.*

Proof. Let $\mathcal{X}_1 \subset \mathbb{P}^3_{\mathbb{F}} \times_F \mathbb{P}^1_{\mathbb{F}}$ be the schematic closure of $X \subset \mathbb{P}^3_{\mathbb{F}(t)}$. The $\mathbb{F}$-variety $\mathcal{X}_1$ has an affine birational model with equation

$$\phi(x, y, z) + t\psi(x, y, z) = 0,$$

hence t is determined by x, y, z, thus $\mathcal{X}$ is $\mathbb{F}$-birational to $\mathbb{P}^3_{\mathbb{F}}$. Since $\mathcal{X}_1$ admits a smooth projective model over $\mathbb{F}$, a result of Cossart [Co92, Théorème, p. 115] shows that there exists a smooth projective threefold $\mathcal{X}/\mathbb{F}$ and an $\mathbb{F}$-birational morphism $\mathcal{X} \to \mathcal{X}_1$ which is an isomorphism over the smooth locus of $\mathcal{X}_1$, hence in particular which induces an isomorphism over $\operatorname{Spec} \mathbb{F}(t) \subset \mathbb{P}^1_{\mathbb{F}}$. That is, the generic fibre of $\mathcal{X} \to \mathbb{P}^1_{\mathbb{F}}$ is k-isomorphic to X/k.

Statement (i) then follows from Thm. 2.3. Statement (ii) follows from (i) as a special application of a result of Colliot-Thélène and Levine [CT/L10, Théorème 1, p. 217]. ∎

Remark 2.5. Theorem 2.4 is of interest only in the case where the $\mathbb{F}$-curve Γ does not contain a geometrically integral component. Otherwise the two statements immediately follow from the Weil estimates for the number of points on geometrically integral curves. These estimates actually provide more: they show that if there exists such a component, then on any field extension $\mathbb{F}'$ of $\mathbb{F}$ of high enough degree, there exists an $\mathbb{F}'$-point on Γ, hence for any such field there exists an $\mathbb{F}'(t)$-point on the $\mathbb{F}(t)$-surface X.

Remark 2.6. One could try to circumvent the cohomological machinery, i.e. Theorems 2.1 and 2.2. For this, in each of the special cases where there are zero-cycles of degree 1 everywhere locally on X but there is no zero-cycle of degree 1 on the curve Γ, one should:

(i) Check that the Brauer group is not trivial, find generators.
(ii) Check that there is a Brauer–Manin obstruction.

Already when the common degree of f and g is 3, which we shall now more particularly examine, this seems no easy enterprise.

3. Rational points on cubic surfaces. The proof of the following theorem is independent of the previous results.

Theorem 3.1. *Let $\mathbb{F}$ be a finite field, let f, g be two nonproportional cubic forms over $\mathbb{F}$ in 4 variables. Assume the characteristic of $\mathbb{F}$ is not 3. Let $k = \mathbb{F}(t)$. Suppose the k-surface $X \subset \mathbb{P}^3_k$ defined by $f + tg = 0$ is smooth. Let $\Gamma \subset \mathbb{P}^3_{\mathbb{F}}$ be the complete intersection curve defined by $f = g = 0$. The following conditions are equivalent:*

(i) *There exists a k-rational point on the k-variety X.*
(ii) *There exists a zero-cycle of degree 1 on the k-variety X.*
(iii) *There exists a zero-cycle of degree 1 on the $\mathbb{F}$-curve Γ.*

(iv) *There exists a closed point of degree prime to* 3 *on the* $\mathbb{F}$*-curve* Γ.

(v) *There exists a closed point of degree a power of* 2 *on the* $\mathbb{F}$*-curve* Γ.

Proof. That (i) implies (ii) is trivial. That (ii) implies (iii) is a special case of [CT/L10]. Statements (iii) and (iv) are equivalent, since Γ is a curve of degree 9. If (v) holds, then Γ has a point in a tower of quadratic extensions of $\mathbb{F}$, hence the cubic surface X has a point in a tower of quadratic extensions of k. An extremely well known argument shows that if a cubic surface over a field has a point in a separable quadratic extension of that field, then it has a rational point: the line joining two conjugate points is defined over the ground field, and either it is entirely contained in the cubic surface or it meets it in a third, rational point. Iterating this remark, we see that X has a rational point, i.e. (i) holds.

Let us prove that (iii) implies (v). To this end, one may replace $\mathbb{F}$ by its maximal pro-2-extension F, which we now do. For an odd integer n, we let F_n/F be the unique, cyclic, field extension of F of degree n.

For Z/L a variety over a field L, the index $\mathrm{ind}(Z) = \mathrm{ind}(Z/L)$ is the gcd of the L-degrees of closed points on Z. The index of an L-variety is equal to the index of its reduced L-subvariety. The index of an L-variety which is a finite union of L-varieties is the gcd of the indices of each of them. The assumption made in (iii) is precisely that the index of the curve Γ is 1.

Since F has no quadratic or quartic extension, an effective zero-cycle of degree $1, 2, 4$ contains an F-rational point, and an effective zero-cycle of degree $3, 6, 9$ either contains an F-point or has index a multiple of 3.

If Γ contains a geometrically integral component, then $\Gamma(F) \neq \emptyset$ (Weil estimates, see Remark 2.5).

Suppose Γ does not contain a geometrically integral component. One then easily checks that the degree 9 curve $\overline{\Gamma}$ can break up only in one of the following ways:

$$\begin{aligned}
9 &= 3(1+1+1),\\
9 &= 2(1+1+1) + (1+1+1),\\
9 &= (2+2+2) + (1+1+1),\\
9 &= (1+1+1) + (1+1+1) + (1+1+1),\\
9 &= (1+\cdots+1) \quad \text{(9 times)},\\
9 &= (3+3+3).
\end{aligned}$$

Here $m(a+a+a)$ means the sum of three conjugate integral curves of degree a over $\overline{F}$ with multiplicity m.

An integral curve of degree 2 over $\overline{F}$ is a smooth plane conic, contained in a well-defined plane. An integral curve of degree 3 over $\overline{F}$ is either a plane cubic or a smooth twisted cubic.

Let the integral curve $C \subset \mathbb{P}^3_F$ break up as $(1+1+1)$. The singular set consists of at most three points. Then either $C(F) \neq \emptyset$ or 3 divides $\mathrm{ind}(C)$.

Let the integral curve $C \subset \mathbb{P}^3_F$ break up as $(2+2+2)$. Each conic is defined over F_3. Two distinct smooth conics on $f = 0$ define two distinct planes, hence they intersect in at most two geometric points. Such points must already be in F_3. Thus any closed point in the singular locus of C has degree 1 or 3. One concludes that either $C(F) \neq \emptyset$ or 3 divides $\mathrm{ind}(C)$.

Let the integral curve $\Gamma \subset \mathbb{P}^3_F$ break up as $(1+ \cdots +1)$ (9 times). The nine lines are defined over F_9, the degree 9 extension of F. So are their intersection points. This implies that any singular closed point on Γ has degree a power of 3. Thus $\Gamma(F) \neq \emptyset$ or 3 divides $\mathrm{ind}(\Gamma)$.

Let the integral curve $\Gamma \subset \mathbb{P}^3_F$ break up as $(3+3+3)$, and assume that this corresponds to a decomposition as three conjugate plane cubics. Each of these is defined over F_3. The intersection number of two of these cubics is 3. The points of intersection of two such curves are thus defined over F_9. We conclude that the singular locus of Γ splits over F_9. This implies that the degree of any closed point in that locus is a power of 3. Thus either $\Gamma(F) \neq \emptyset$ or 3 divides $\mathrm{ind}(\Gamma)$.

Let the curve $\Gamma \subset \mathbb{P}^3_F$ break up as $(3+3+3)$, and assume that Γ breaks up as the sum of three conjugate twisted cubics. The curve Γ lies on the smooth cubic surface X over $F(t)$ defined by $f + tg = 0$. Each twisted curve is defined over F_3. Let σ be a generator of $\mathrm{Gal}(F_3(t)/F(t))$. Write $\Gamma = C + \sigma(C) + \sigma^2(C)$ on $X_{F_3(t)}$. Using intersection theory on the smooth surface $X_{F_3(t)}$, which is invariant under the action of $\mathrm{Gal}(F_3(t)/F(t))$, and letting H be the class of a plane section, we find

$$27 = (3H.3H) = (\Gamma.\Gamma) = 3(C.C) + 6(C.\sigma(C)).$$

The curve C is a twisted cubic, hence a smooth curve of genus 0 on the smooth cubic surface X, whose canonical bundle K is given by $-H$. The formula for the arithmetic genus of a curve on a surface, namely

$$2(p_a(C) - 1) = (C.C) + (C.K),$$

gives $(C.C) = 1$. This implies $(C.\sigma(C)) = 4$, hence $(\sigma(C).\sigma^2(C)) = 4$ and $(\sigma^2(C).C) = 4$. Since each of these twisted cubics is defined over F_3 and since F_3 has no field extension of degree 2 or 4, this implies that the points of intersection of any two of these twisted cubics are defined over F_9. We conclude that the singular locus of Γ splits over F_9. This implies that the degree of any closed point in that locus is a power of 3. Thus either $\Gamma(F) \neq \emptyset$ or 3 divides $\mathrm{ind}(\Gamma)$.

In all cases we have proved: Either $\Gamma(F) \neq \emptyset$ or 3 divides $\operatorname{ind}(\Gamma)$. The assumption $\operatorname{ind}(\Gamma) = 1$ now implies $\Gamma(F) \neq \emptyset$. ∎

REMARK 3.2. If the order q of the finite field $\mathbb{F}$ is large enough and $f + tg = 0$ is solvable in $\mathbb{F}(t)$, a variant of the proof for the equivalence of (iv) and (v) shows that $f + tg = 0$ has a solution in polynomials of degree at most 5. This raises the interesting general question whether there are integers $N(d)$ with the following property: Suppose that $G(X_0, \dots, X_4, t)$ is a polynomial defined over $\mathbb{F}$, homogeneous of degree 3 in the X_i and of degree d in t; if $G = 0$ is solvable in $\mathbb{F}(t)$, then it has a solution in polynomials of degree at most $N(d)$.

We may now prove:

THEOREM 3.3. *Let $\mathbb{F}$ be a finite field, let f, g be two nonproportional cubic forms in 4 variables. Assume the characteristic of $\mathbb{F}$ is not 3. Let $k = \mathbb{F}(t)$. Suppose the cubic surface $X \subset \mathbb{P}^3_k$ over k defined by $f + tg = 0$ is smooth. If there is no Brauer–Manin obstruction to the Hasse principle for rational points on X, then there exists a k-rational point on X.*

Proof. Combine Theorems 2.4 and 3.1. ∎

REMARK 3.4. Again, it would be nice to avoid the cohomological machinery, i.e. Theorems 2.1 and 2.2. When X has no rational points over $\mathbb{F}(t)$ but points in all the completions of $\mathbb{F}(t)$, one should exhibit an explicit Brauer–Manin obstruction for X. For this purpose, it would probably be helpful to use [SD93]. Down to earth computations, which we shall not insert here, have led to the following result. If a smooth cubic surface X given by $f + tg = 0$ is a counterexample to the Hasse principle over $\mathbb{F}(t)$, then, after replacing $\mathbb{F}$ by its maximal pro-2-extension F, the following holds: When going over to the algebraic closure of F, the curve Γ in the proof of Theorem 3.1 breaks up as a sum of nine conjugate lines, or a sum of three twisted cubics, or a sum of three conjugate conics plus a sum of three coplanar conjugate lines; when using the word "conjugate" we mean that the Galois action is transitive. Only in these three cases may we expect a Brauer–Manin obstruction.

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