ACTA ARITHMETICA
155.1 (2012)

On the Hausdorff dimension of the expressible set of certain sequences

by

Jaroslav Hančl (Ostrava), Radhakrishnan Nair (Liverpool),
Lukáš Novotný (Ostrava) and Jan Šustek (Ostrava)

Dedicated to Professor Andrzej Schinzel
on the occasion of his seventy-fifth birthday

1. Introduction. A long standing issue in number theory is to find conditions on series to decide if their sums are rational or not. Very occasionally, spectacular special results like R. Apéry's proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ come along [1]. General methods are however very rare. Motivated by investigations in this vein Erdős [3] called a sequence $\{a_n\}_{n=1}^{\infty}$ *irrational* if the set

$$X\{a_n\}_{n=1}^{\infty} = \left\{ \sum_{n=1}^{\infty} \frac{1}{a_n c_n} : c_n \in \mathbb{N} \right\},$$

which we refer to henceforth as its *expressible set*, contains no rational numbers. Sequences that are not irrational are called *rational*. In [3] it is shown that if $\liminf_{n\to\infty} a_n^{1/2^n} > 1$, $\limsup_{n\to\infty} a_n^{1/2^n} = \infty$ and $a_n \in \mathbb{N}$ then $\sum_{n=1}^{\infty} 1/a_n$ is an irrational number. From this we can deduce that the sequence $\{2^{2^n}\}$ is irrational. In [5] it is shown that if for given $\varepsilon > 0$ we have $a_n < 2^{(2-\varepsilon)^n}$ and $a_n \in \mathbb{R}^+$ for all $n \in \mathbb{N}$ then the sequence $\{a_n\}_{n=1}^{\infty}$ is rational and in fact that $X\{a_n\}_{n=1}^{\infty}$ contains an interval. It appears to be the case that, in general, evaluating the Lebesgue measure of the set $X\{a_n\}_{n=1}^{\infty}$ is not easy. This has led to a number of studies under particular hypotheses. For instance Hančl, Schinzel and Šustek [7] studied the case of geometric sequences. Also Hančl and Šustek [8] studied boundedly expressible sets. In [6] we give conditions on $\{a_n\}_{n=1}^{\infty}$ to ensure that the Lebesgue measure of the set $X\{a_n\}_{n=1}^{\infty}$ is zero. In particular the following is shown. Let α, δ

2010 *Mathematics Subject Classification*: Primary 11K55; Secondary 11J72.
Key words and phrases: expressible set, Hausdorff dimension.

DOI: 10.4064/aa155-1-8

and ε be positive real numbers with $0 < \alpha < 1$ and let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of positive integers with $\{a_n\}_{n=1}^{\infty}$ nondecreasing. Suppose also that $\limsup_{n\to\infty} a_n^{1/3^n} = \infty$, that $a_n \geq n^{1+\varepsilon}$ and that $b_n \leq 2^{\log_2^\alpha a_n}$ for every sufficiently large n. Then the expressible set of the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ has Lebesgue measure zero.

For a set $E \subseteq \mathbb{R}$ we call $(U_i)_{i=1}^{\infty}$ a *δ-cover* of E if $E \subseteq \bigcup_{i=1}^{\infty} U_i$ and $\operatorname{diam} U_i = \sup_{x,y\in U_i} |x-y| < \delta$. We define an outer measure by

$$\mathcal{H}_\delta^s(E) = \sup \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s,$$

where the supremum is taken over all δ-covers. We also define $\mathcal{H}^s(E) = \lim_{s\to 0} \mathcal{H}_\delta^s(E)$. There is a nonnegative real number s_0 such that if $s > s_0$ then $\mathcal{H}^s(E) = 0$ and if $s < s_0$ then $\mathcal{H}^s(E) = \infty$. We call s_0 the *Hausdorff dimension* of E. We denote s_0 by $\dim_{\mathrm{H}}(E)$. See [4] for a more systematic discussion of the relevant ideas.

In this paper we prove the following refinement of our theorem from [6].

Theorem 1.1. *Let α, δ and ε be positive real numbers with $0 < \alpha < 1$ and let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of positive integers with $\{a_n\}_{n=1}^{\infty}$ nondecreasing. Also suppose that*

$$\limsup_{n\to\infty} a_n^{1/(3+\delta)^n} = \infty$$

and that

$$a_n \geq n^{1+\varepsilon}, \qquad b_n \leq 2^{\log_2^\alpha a_n}$$

for every sufficiently large n. Then

$$\dim_{\mathrm{H}}\left(X\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}\right) \leq \frac{2}{2+\delta}.$$

We hope to return to the issue of lower bounds on another occasion.

2. Our main tool. Theorem 1.1 is an immediate consequence of the following more general result.

Theorem 2.1. *Let L be a positive integer and let α, β and ε be real numbers with $0 < \alpha < 1$, $0 < \varepsilon$ and $0 \leq \beta < \varepsilon/(1+\varepsilon)$. Assume that $x_1, \ldots, x_L, M_1, \ldots, M_L$ are real numbers such that $x_i \neq 0$ and $M_i \geq 1$ for every $i = 1, \ldots, L$. Let $\{a_n\}_{n=1}^{\infty}$ be a nondecreasing sequence of positive integers. Suppose that $\{b_{i,n}\}_{n=1}^{\infty}$, $i = 1, \ldots, L$, are sequences of integers such that for every $i = 1, \ldots, L$ and every sufficiently large n,*

$$|b_{i,n}| \leq 2^{\log_2^\alpha a_n} a_n^{M_i}. \tag{2.1}$$

Assume that for every n,

$$b_n = \sum_{i=1}^{L} b_{i,n} x_i \neq 0. \tag{2.2}$$

For every sufficiently large n, let

$$|b_n| \leq 2^{\log_2^{\alpha} a_n} a_n^{\beta} \tag{2.3}$$

and let

$$a_n \geq n^{1+\varepsilon}. \tag{2.4}$$

Suppose that

$$P = \sup\Big\{ M : \limsup_{n\to\infty} a_n^{1/(\frac{1+M}{1-\beta}+1)^n} = \infty \Big\} > \sum_{i=1}^{L} M_i. \tag{2.5}$$

Then

$$\dim_{\mathrm{H}}\bigg(X\bigg\{ \frac{a_n}{b_n} \bigg\}_{n=1}^{\infty} \bigg) \leq \frac{1 + \sum_{i=1}^{L} M_i}{1+P}. \tag{2.6}$$

3. Proof. For the proof of Theorem 2.1 we need the following two lemmas.

LEMMA 3.1. *Assume that sequences* $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ *and* $\{b_{i,n}\}_{n=1}^{\infty}$ *satisfy the assumptions* (2.1)–(2.4) *of Theorem* 2.1. *Let* $M \geq 1$ *be a real number such that*

$$\limsup_{n\to\infty} a_n^{1/(\frac{1+M}{1-\beta}+1)^n} = \infty. \tag{3.1}$$

Then for every sequence $\{c_n\}_{n=1}^{\infty}$ *of positive integers the inequality*

$$\bigg| \sum_{n=1}^{\infty} \frac{b_n}{a_n c_n} - \frac{\sum_{i=1}^{L} p_i x_i}{q} \bigg| < \frac{1}{(q \log_2^2 q) 2^{L \log_2^{(1+2\alpha)/3} q} q^M}$$

has infinitely many solutions $p_1, \ldots, p_L \in \mathbb{Z}$, $q \in \mathbb{N}$ *with*

$$p_i = O(2^{\log_2^{(1+2\alpha)/3} q} q^{M_i}) \quad \textit{for every } i = 1, \ldots, L.$$

LEMMA 3.2. *Let* $\{H_j : j \in \mathbb{N}\}$ *be a family of hypercubes. Suppose* $S \subset \mathbb{R}^n$ *is a set such that*

$$S \subseteq \{ t \in \mathbb{R}^n : t \in H_j \textit{ for infinitely many } j \in \mathbb{N} \}.$$

If

$$\sum_{j=1}^{\infty} (\operatorname{diam} H_j)^m < \infty \tag{3.2}$$

for some $m > 0$, *then* $\dim_{\mathrm{H}}(S) \leq m$.

Lemma 3.1 is the same as Lemma 6 in [6]. Lemma 3.2 is known as the Hausdorff–Cantelli lemma and its proof can be found in [2].

Proof of Theorem 2.1. From assumption (2.5) we obtain (3.1) for every $M \in (\sum_{i=1}^{L} M_i, P)$. Hence we can use Lemma 3.1. For brevity let

$$h_i(q) := 2^{\log_2^{(1+2\alpha)/3} q} q^{M_i}.$$

From Lemma 3.1 we deduce that for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the inequality

$$\left|\sum_{n=1}^{\infty} \frac{b_n}{a_n c_n} - \frac{\sum_{i=1}^{L} p_i x_i}{q}\right| < \frac{1}{q^{M+1} 2^{L \log_2^{(1+2\alpha)/3} q}}$$

has infinitely many solutions $p_1, \dots, p_L \in \mathbb{Z}$, $q \in \mathbb{N}$ with

$$p_i = O(h_i(q))$$

for every $i = 1, \dots, L$. This implies that the expressible set of the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ is a subset of the set

$$S := \bigcap_{N=1}^{\infty} \bigcup_{q=N}^{\infty} \bigcup_{p_1=-\lfloor Eh_1(q)\rfloor}^{\lfloor Eh_1(q)\rfloor} \cdots \bigcup_{p_L=-\lfloor Eh_L(q)\rfloor}^{\lfloor Eh_L(q)\rfloor} J_{p_1,\dots,p_L,q},$$

where E is a positive real constant not depending on q and

$$J_{p_1,\dots,p_L,q} := \left(\frac{\sum_{i=1}^{L} p_i x_i}{q} - \frac{1}{q^{M+1} 2^{L \log_2^{(1+2\alpha)/3} q}}, \frac{\sum_{i=1}^{L} p_i x_i}{q} + \frac{1}{q^{M+1} 2^{L \log_2^{(1+2\alpha)/3} q}}\right).$$

Every element $\xi \in S$ lies in infinitely many sets $J_{p_1,\dots,p_L,q}$. The family

$$\mathcal{J} := \{J_{p_1,\dots,p_L,q} : p_1, \dots, p_L \in \mathbb{Z},\ q \in \mathbb{N},\ |p_i| \le \lfloor Eh_i(q)\rfloor\}$$

is a cover of the set S. The diameter of every set $J \in \mathcal{J}$ is

$$J = \frac{2}{q^{M+1} 2^{L \log_2^{(1+2\alpha)/3} q}}.$$

Then, for every m with

$$\frac{1 + \sum_{i=1}^{L} M_i}{1 + M} < m < 1 \tag{3.3}$$

we have

$$\sum_{J \in \mathcal{J}} (\operatorname{diam} J)^m = \sum_{q=1}^{\infty} \sum_{p_1=-\lfloor Eh_1(q)\rfloor}^{\lfloor Eh_1(q)\rfloor} \cdots \sum_{p_L=-\lfloor Eh_L(q)\rfloor}^{\lfloor Eh_L(q)\rfloor} \frac{2^m}{q^{(M+1)m} 2^{Lm \log_2^{(1+2\alpha)/3} q}}$$

$$\le D \sum_{q=1}^{\infty} q^{\sum_{i=1}^{L} M_i - (M+1)m} 2^{L(1-m) \log_2^{(1+2\alpha)/3} q} < \infty.$$

Here D is a suitable positive constant not depending on q, since

$$\sum_{i=1}^{L} M_i - (M+1)m < -1.$$

From Lemma 3.2 it follows that $\dim_{\mathrm{H}}(S) \leq m$. This fact holds for every m satisfying (3.3). Hence

$$\dim_{\mathrm{H}}\left(X\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}\right) \leq \dim_{\mathrm{H}}(S) \leq \frac{1+\sum_{i=1}^{L} M_i}{1+M}.$$

The fact that this inequality holds for every $M \in (\sum_{i=1}^{L} M_i, P)$ gives

$$\dim_{\mathrm{H}}\left(X\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}\right) \leq \frac{1+\sum_{i=1}^{L} M_i}{1+P}$$

if $P < \infty$ and

$$\dim_{\mathrm{H}}\left(X\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}\right) = 0$$

if $P = \infty$, and concludes the proof of Theorem 2.1. ■

Acknowledgements. This paper has been developed in the framework of the IT4Innovations Centre of Excellence project, reg. no. CZ.1.05/1.1.00/02.0070, supported by Operational Programme 'Research and Development for Innovations', funded by Structural Funds of the European Union, by the Moravian-Silesian Region and state budget of the Czech Republic and by grants ME09017, P201/12/2351, MSM 6198898701 and 01798/2011/RRC.

References

[1] R. Apéry, *Irrationalité de $\zeta(2)$ et $\zeta(3)$*, Astérisque 61 (1979), 11–13.

[2] V. I. Bernik and M. M. Dodson, *Metric Diophantine Approximation on Manifolds*, Cambridge Univ. Press, Cambridge, 1999.

[3] P. Erdős, *Some problems and results on the irrationality of the sum of infinite series*, J. Math. Sci. 10 (1975), 1–7.

[4] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge Univ. Press, Cambridge, 1985.

[5] J. Hančl, *Expression of real numbers with the help of infinite series*, Acta Arith. 59 (1991), 97–104.

[6] J. Hančl, R. Nair, and J. Šustek, *On the Lebesgue measure of the expressible set of certain sequences*, Indag. Math. (N.S.) 17 (2006), 567–581.

[7] J. Hančl, A. Schinzel and J. Šustek, *On expressible sets of geometric sequences*, Funct. Approx. Comment. Math. 39 (2008), 71–95.

[8] J. Hančl and J. Šustek, *Boundedly expressible sets*, Czechoslovak Math. J. 59 (2009), 649–654.

Jaroslav Hančl
Department of Mathematics
University of Ostrava
and
Centre of Excellence IT4Innovations
– Division UO – IRAFM
30. dubna 22
701 03 Ostrava 1, Czech Republic
E-mail: jaroslav.hancl@osu.cz

Radhakrishnan Nair
Mathematical Sciences
University of Liverpool
Peach Street
Liverpool L69 7ZL, United Kingdom
E-mail: nair@liverpool.ac.uk

Lukáš Novotný, Jan Šustek
Department of Mathematics
University of Ostrava
30. dubna 22
701 03 Ostrava 1, Czech Republic
E-mail: lukas.novotny@osu.cz
jan.sustek@osu.cz

Received on 14.5.2011
and in revised form on 16.11.2011 (6699)