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A symmetric diophantine system concerning fifth powers

by

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This paper is concerned with the diophantine system

$$x_1^k + x_2^k + x_3^k + x_4^k = y_1^k + y_2^k + y_3^k + y_4^k \quad \text{for } k = 1, 2, 4, 5. \tag{1}$$

The analogous diophantine system with $k = 1, 2, 4, 6$ was in effect solved as long back as in 1937 by Chernick [1, p. 633] who gave a parametric solution of the system of equations $\sum_{i=1}^4 x_i^k = \sum_{i=1}^4 y_i^k$ for $k = 2, 4, 6$. With appropriate changes of sign, Chernick's solution is also valid for $k = 1$. Till now, however, no integer solutions of the diophantine system (1) have been published. We obtain a parametric solution of this system and show how infinitely many parametric solutions may be generated. We also obtain several numerical solutions which are not generated by these parametric solutions.

To solve the system (1), we may make the substitution

$$x_i = a_i m + b_i n, \quad y_i = a_i m - b_i n \quad \text{for } i = 1, 2, 3, 4, \tag{2}$$

upon which these four equations reduce, on transposing to the left-hand side and removing trivial factors such as n and mn, to the following four equations respectively:

$$b_1 + b_2 + b_3 + b_4 = 0, \tag{3}$$

$$a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 = 0, \tag{4}$$

$$(a_1^3 b_1 + a_2^3 b_2 + a_3^3 b_3 + a_4^3 b_4) m^2 + (a_1 b_1^3 + a_2 b_2^3 + a_3 b_3^3 + a_4 b_4^3) n^2 = 0, \tag{5}$$

$$\begin{aligned} &5(a_1^4 b_1 + a_2^4 b_2 + a_3^4 b_3 + a_4^4 b_4) m^4 \\ &\quad + 10(a_1^2 b_1^3 + a_2^2 b_2^3 + a_3^2 b_3^3 + a_4^2 b_4^3) m^2 n^2 + (b_1^5 + b_2^5 + b_3^5 + b_4^5) n^4 = 0. \end{aligned} \tag{6}$$

We may solve equations (3) and (4) for b_3 and b_4, substitute these values in equations (5) and (6), and eliminate m from the resulting equations when we get a condition that may be written as

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$$(a_1+a_2-a_3-a_4)(a_1-a_2+a_3-a_4)(a_1-a_2-a_3+a_4) \times (a_3-a_4)b_1b_2(a_1b_1+a_2b_2-a_3b_1-a_3b_2) \times (a_1b_1+a_2b_2-a_4b_1-a_4b_2)\phi(a_i,b_i)=0, \tag{7}$$

where $\phi(a_i,b_i)$ is a polynomial of degree 8 in the variables a_1,a_2,a_3,a_4,b_1,b_2. Equating any factor on the left-hand side of (7) (except $\phi(a_i,b_i)$) to 0 leads to triviality, and it is difficult to obtain a parametric solution of $\phi(a_i,b_i)=0$ that leads to a nontrivial solution of our diophantine system.

As the aforementioned direct attempt fails, computer trials were performed to generate integer solutions of our diophantine system. A total of 204 integer solutions were obtained, and on analysing the data, it was observed that 44 of these solutions also simultaneously satisfied the additional condition

$$(x_1-x_2)(x_3-x_4)=(y_1-y_2)(y_3-y_4). \tag{8}$$

Accordingly we will solve the system (1) together with the auxiliary condition (8). Now on making the substitution (2), we get equations (3)–(6) as before, while (8) reduces to

$$(a_3-a_4)b_1-(a_3-a_4)b_2+(a_1-a_2)b_3-(a_1-a_2)b_4=0. \tag{9}$$

We now solve the three equations (3), (4) and (9) for b_i and get

$$\begin{aligned} b_1 &= 2a_1a_2-a_1a_3-a_1a_4-2a_2^2+a_2a_3+a_2a_4+a_3^2-2a_3a_4+a_4^2,\\ b_2 &= -2a_1^2+2a_1a_2+a_1a_3+a_1a_4-a_2a_3-a_2a_4+a_3^2-2a_3a_4+a_4^2,\\ b_3 &= a_1^2-2a_1a_2-a_1a_3+a_1a_4+a_2^2-a_2a_3+a_2a_4+2a_3a_4-2a_4^2,\\ b_4 &= a_1^2-2a_1a_2+a_1a_3-a_1a_4+a_2^2+a_2a_3-a_2a_4-2a_3^2+2a_3a_4. \end{aligned} \tag{10}$$

Substituting these values of b_i in (5) and (6), and eliminating m from the resulting two equations, we get the condition

$$(a_1+a_2-a_3-a_4)(a_1-a_2+a_3-a_4)(a_1-a_2-a_3+a_4) \times b_1b_2b_3b_4\phi_1(a_i)\phi_2(a_i)=0, \tag{11}$$

where the values of b_i are given by (10), and

$$\begin{aligned} \phi_1(a_i) &= a_1^3-a_1^2a_2-a_1a_2^2+a_2^3-a_3^3+a_3^2a_4+a_3a_4^2-a_4^3,\\ \phi_2(a_i) &= (2a_2-a_3-a_4)a_1^5-(8a_2^2+a_2a_3+a_2a_4-2a_3^2-2a_4^2)a_1^4\\ &\quad+2(6a_2^2+a_2a_3+a_2a_4-a_3^2+6a_3a_4-a_4^2)a_1^3a_2\\ &\quad+(-8a_2^4+2a_2^3a_3+2a_2^3a_4+8a_2^2a_3^2-40a_2^2a_3a_4+8a_2^2a_4^2\\ &\quad-2a_3^4+2a_3^3a_4-8a_3^2a_4^2+2a_3a_4^3-2a_4^4)a_1^2\\ &\quad+(2a_2^5-a_2^4a_3-a_2^4a_4-2a_2^3a_3^2+12a_2^3a_3a_4-2a_2^3a_4^2\\ &\quad-12a_2a_3^3a_4+40a_2a_3^2a_4^2-12a_2a_3a_4^3+a_3^5 \end{aligned} \tag{12}$$

$$+ a_3^4a_4 - 2a_3^3a_4^2 - 2a_3^2a_4^3 + a_3a_4^4 + a_4^5)a_1$$
$$- (a_2a_3 + a_2a_4 - a_3^2 + 2a_3a_4 - a_4^2)(a_2^4 - a_2^3a_3 - a_2^3a_4$$
$$- a_2^2a_3^2 + 2a_2^2a_3a_4 - a_2^2a_4^2 + a_2a_3^3 + a_2a_3^2a_4$$
$$+ a_2a_3a_4^2 + a_2a_4^3 - 2a_3^3a_4 + 4a_3^2a_4^2 - 2a_3a_4^3).$$

Equating any factor on the left-hand side of (11) (except $\phi_1(a_i)$ and $\phi_2(a_i)$) to 0 leads to triviality, and it is difficult to obtain a solution of $\phi_2(a_i) = 0$. We can, however, solve $\phi_1(a_i) = 0$ by writing $a_4 = a_3 + t(a_1 - a_2)$, in which case we get $(a_1 - a_2)^2\{(t^3 - 1)a_1 - (t^3 + 1)a_2 + 2a_3t^2\} = 0$, and so we readily obtain the following solution:

$$\begin{aligned} a_3 &= \{(-t^3 + 1)a_1 + (t^3 + 1)a_2\}/(2t^2), \\ a_4 &= \{(t^3 + 1)a_1 - (t^3 - 1)a_2\}/(2t^2). \end{aligned} \tag{13}$$

With these values of a_3 and a_4, equation (5) reduces to

$$\begin{aligned} &\{-(a_1 - a_2)^2t^6 + 4a_1a_2t^4 + (a_1 + a_2)^2(4t^2 + 1)\}m^2 \\ &\quad - 4(t^2 - 1)^2(a_1 - a_2)^2\{2(a_1 - a_2)^2t^4 + (a_1 + a_2)^2(t^2 + 1)\}n^2 = 0, \end{aligned} \tag{14}$$

and when this equation is satisfied, it is easily verified that (6) is also satisfied. Now on substituting

$$\begin{aligned} a_1 &= a_2X, \\ m &= \frac{2(t^2 - 1)(a_1 - a_2)a_2^2nY}{-(a_1 - a_2)^2t^6 + 4a_1a_2t^4 + (a_1 + a_2)^2(4t^2 + 1)}, \end{aligned} \tag{15}$$

equation (14) reduces to

$$\begin{aligned} Y^2 = &- (t^6 - 4t^2 - 1)(2t^4 + t^2 + 1)X^4 \\ &+ 4(2t^6 - 2t^4 + 3t^2 + 1)(t^2 + 1)^2X^3 \\ &- (12t^{10} + 14t^8 + 6t^6 - 28t^4 - 30t^2 - 6)X^2 \\ &+ 4(2t^6 - 2t^4 + 3t^2 + 1)(t^2 + 1)^2X \\ &- (t^6 - 4t^2 - 1)(2t^4 + t^2 + 1). \end{aligned} \tag{16}$$

It is readily verified that a solution of (16) is given by

$$X = (t + 1)/(t - 1), \quad Y = 4t^2(3t^2 + 1)/(t - 1)^2. \tag{17}$$

Using these values of (X, Y), we obtain the values of a_1 and m from (15), then the values of a_3, a_4 and b_i from (13) and (10), and finally we get a solution of our diophantine system from (2). This, however, turns out to be a trivial solution. Now using the solution (17) and following a standard procedure described by Dickson [2, p. 639], we obtain another solution of (16):

$$(18)\qquad \begin{aligned} X &= \frac{(t-1)(t^6+2t^5+13t^4+12t^3+15t^2+2t+3)}{(t+1)(t^6-2t^5+13t^4-12t^3+15t^2-2t+3)},\\ Y &= \frac{4t^2(t^4+4t^2-1)(t^6+3t^4+11t^2+1)(3t^2+1)^2}{(t+1)^2(t^6-2t^5+13t^4-12t^3+15t^2-2t+3)^2} \end{aligned}$$

Using this solution and working backwards as before, we obtain a one-parameter nontrivial solution of our diophantine system. On writing

$$(19)\qquad \begin{aligned} f_1(t) &= t^{14}+4t^{13}+17t^{12}+12t^{11}+93t^{10}-56t^9+285t^8\\ &\quad -232t^7+99t^6-204t^5+19t^4-36t^3-t^2-1,\\ f_2(t) &= t^{14}+2t^{13}+11t^{12}+8t^{11}+t^{10}-46t^9-21t^8\\ &\quad -128t^7-5t^6+102t^5+9t^4+56t^3+3t^2+6t+1, \end{aligned}$$

our parametric solution may be written as follows:

$$(20)\qquad \begin{aligned} &x_1 = f_1(t), && x_2 = f_2(t), && x_3 = t^{14}f_1(-1/t), && x_4 = t^{14}f_2(-1/t),\\ &y_1 = f_2(-t), && y_2 = f_1(-t), && y_3 = t^{14}f_1(1/t), && y_4 = t^{14}f_2(1/t). \end{aligned}$$

Using the solution (18) and following the same procedure, we may obtain another parametric solution of (16), and this process may be continued indefinitely. We thus obtain infinitely many parametric solutions of (16), and these solutions may be used as indicated to obtain infinitely many parametric solutions of our diophantine system. All of these solutions would still not give the complete solution of our diophantine system as these solutions necessarily satisfy the condition (8), and we have already noted the existence of solutions that do not satisfy this condition.

Taking $t = 3$ in the solution given by (20), we get, after removing common divisors, the following numerical solution:

$$\begin{aligned} &x_1 = 27437, && x_2 = 13666, && x_3 = 7408, && x_4 = -14006,\\ &y_1 = 6934, && y_2 = 14072, && y_3 = -13907, && y_4 = 27406. \end{aligned}$$

This is not the smallest numerical solution of the diophantine system (1). The

Table I. Solutions of the system (1)

x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4
469	238	224	−290	424	394	85	−262
1009	817	−19	−437	953	907	−131	−359
1033	701	343	−635	973	853	215	−599
1192	729	186	−582	1113	966	−266	−288
1657	902	364	−830	1612	1145	22	−686
1673	931	277	−791	1621	1187	−167	−551
2127	661	579	−897	2119	951	213	−813

seven smallest numerical solutions found during a selective computer search are shown in Table I. The search was exhaustive for $\max(|x_i|, |y_i|) \leq 1200$, hence the first four solutions are indeed the four smallest existing solutions. We also note that the first six solutions do not satisfy condition (8).

The existence of relatively small numerical solutions suggests the possibility of a parametric solution of lower degree. We could not, however, find such a solution and it remains an open question whether there exists a solution of the diophantine system (1) of degree lower than 14.

References

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