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On interaction of Hecke–Shimura rings: symplectic versus orthogonal

by

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Dedicated to Professor Andrzej Schinzel
on the occasion of his 75th birthday

Introduction. There is a hope that further progress in number theory is closely related to study of interaction of various representations of Hecke–Shimura rings (*HS-rings*) of arithmetical discrete subgroups of Lie groups on suitable spaces of automorphic forms. Important examples of, say, "vertical" interaction are given by lifts of automorphic structures to similar groups of higher orders (see, e.g., [An01]). Not less, if not more, interesting are cases of "horizontal" interaction arising from consideration of HS-rings of different Lie groups, say, symplectic and orthogonal (see, e.g., [An06]).

In general, an *automorphic structure* on a Lie group is a HS-ring of an arithmetical discrete subgroup of the group together with a linear representation of the ring on a space of automorphic forms by Hecke operators. An *interaction* mapping from one automorphic structure to an automorphic structure on another group is a mapping of HS-rings of the discrete subgroups compatible with the action of the corresponding Hecke operators on suitable spaces of automorphic forms.

In this paper we construct interaction mappings for HS-rings of certain subgroups of integral symplectic groups and groups of units of integral nonsingular quadratic forms in an even number of variables. For the action of Hecke operators on theta-functions see, e.g., [An90], [An92], and [An09].

Notation. We fix the letters $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ for the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively.

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If $\mathbb{A}$ is a set, $\mathbb{A}_n^m$ denotes the set of all $m \times n$-matrices with elements in $\mathbb{A}$. If $\mathbb{A}$ is a ring with the identity element, 1_n denotes the identity element of the ring $\mathbb{A}_n^n$ and 0_n is the zero element of that ring.

The transpose of a matrix M is denoted by ${}^tM$. For two matrices Q and N of appropriate sizes we set $Q[N] = {}^tNQN$. For a complex square matrix A we write $\mathbf{e}\{A\} = \exp(\pi\sqrt{-1}\,\mathrm{Tr}(A))$, where $\mathrm{Tr}(A)$ is the sum of the diagonal entries of A.

1. Theta-functions, symplectic HS-rings, Hecke operators. For details of forthcoming definitions, formulas, and corresponding references see [An09]. In particular, the references [An86], [An87], and [An95] can be useful.

Let

$$\mathbf{q}(X) = \frac{1}{2}\,{}^tXQX = \frac{1}{2}Q[X] \qquad (X = {}^t(x_1, \ldots, x_m))$$

be a fixed integral nonsingular quadratic form in m variables with matrix $Q = {}^tQ$. The matrix of an integral form is an *even matrix*, i.e., an integral symmetric matrix with even diagonal entries. The least positive integer q such that the matrix qQ^{-1} is even is called the *level of the form* $\mathbf{q}$. On integral quadratic forms see [Ca78]. Speaking about quadratic forms, we mainly use the equivalent matrix language.

The *majorant space* $\mathbb{M}(\mathbf{q}) = \mathbb{M}(Q)$ *of the nonsingular form* $\mathbf{q}$ is defined by

$$\mathbb{M}(Q) = \{H \in \mathbb{R}_m^m \mid H = {}^tH,\ H > 0,\ HQ^{-1}H = Q\}.$$

The set $\mathbb{M}(Q)$ is a homogeneous space of the real orthogonal group of the form $\mathbf{q}$,

$$O(\mathbf{q}) = O(Q) = O_{\mathbb{R}}(Q) = \{U \in \mathbb{R}_m^m \mid {}^tUQU = Q\}$$

operating on $\mathbb{M}(Q)$ by the rule $O(Q) \ni U : H \mapsto {}^tUHU$. If the form $\mathbf{q}$ is positive definite, then the set $\mathbb{M}(Q)$ reduces to the single matrix Q.

Let Q be a nonsingular even matrix of order m. For $n = 1, 2, \ldots,$ we define the *theta-function of* Q *of genus* n *with translations* $V = (V_1, V_2) \in \mathbb{C}_n^m \times \mathbb{C}_n^m$ as a function of $V \in \mathbb{C}_{2n}^m$, of

$$Z \in \mathbb{H}^n = \{Z = X + \sqrt{-1}\,Y \in \mathbb{C}_n^n \mid {}^tZ = Z,\ Y > 0\}$$

(the *Siegel upper half-plane of genus* n), and of $H \in \mathbb{M}(Q)$, given by the series

$$\begin{aligned}(1.1)\qquad \Theta(V, Z; H, Q) &= \Theta^n(V, Z; H, Q)\\ &= \sum_{N \in \mathbb{Z}_n^m} \mathbf{e}\{Q[N - V_2]X + \sqrt{-1}\,H[N - V_2]Y + 2 \cdot {}^tV_1QN - {}^tV_1QV_2\}.\end{aligned}$$

The series converges absolutely and uniformly on the product of any compact subset of $\mathbb{C}_{2n}^m$, the set $\{Z = X + \sqrt{-1}\,Y \in \mathbb{H}^n \mid Y \geq \varepsilon 1_n\}$ for any $\varepsilon > 0$, and any compact subset of $\mathbb{M}(Q)$. Therefore, the series defines a real-analytic function on $\mathbb{C}^m_{2n} \times \mathbb{H}^n \times \mathbb{M}(Q)$.

According to [An95, Theorems 4.1–4.3], theta-functions have the following automorphic properties: if Q is an even nonsingular matrix of even order $m = 2k$ and signature (s, l), then, for each matrix M in the group

$$\Gamma^n_0(q) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z}) \;\middle|\; C \equiv 0 \pmod{q} \right\},$$

the theta-function (1.1) of Q of genus $n \geq 1$ satisfies the functional equation

$$\Theta(V \cdot {}^tM, M\langle Z\rangle; H, Q) = j_Q(M,\, Z)\Theta(V, Z; H, Q), \tag{1.2}$$

where $M\langle Z\rangle = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\langle Z\rangle = (AZ+B)(CZ+D)^{-1}$,

$$j_Q(M,\, Z) = \chi_Q(\det D)(\det(CZ+D))^{(s-l)/2}|\det(CZ+D)|^l$$

is the automorphic factor, and χ_Q is the character of the quadratic form with matrix Q.

On the other hand, the above theta-function has symmetries corresponding to the action of the *group* $E(Q) = \{M \in \mathrm{GL}_m(\mathbb{Z}) \mid Q[M] = Q\}$ *of units of the matrix* Q. Namely, it is an easy consequence of the definition that

$$\Theta^n(MV, Z; H[M^{-1}], Q) = \Theta^n(V, Z; H, Q) \quad \text{for } M \in E(Q). \tag{1.3}$$

Further, for $q \geq 1$, let us introduce the multiplicative semigroup

$$\Sigma^n_0(q) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{Z}^{2n}_{2n} \;\middle|\; {}^tM \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} M = \mu(M) \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}, \right.$$
$$\left. C \equiv 0_n \pmod{q},\, \mu(M) > 0,\, \gcd(\mu(M),\, q) = 1 \right\}.$$

The group $\Gamma^n_0(q)$ can be considered as a subgroup of $\Sigma^n_0(q)$ consisting of the matrices $M \in \Sigma^n_0(q)$ with *multiplier* $\mu(M) = 1$. Let

$$\mathcal{H}^n_0(q) = \mathcal{H}(\Gamma^n_0(q),\, \Sigma^n_0(q))$$

be the Hecke–Shimura ring (over $\mathbb{C}$) of the semigroup $\Sigma^n_0(q)$ relative to the subgroup $\Gamma^n_0(q)$. Here we only note that this ring consists of all those finite formal linear combinations T with complex coefficients of symbols $(\Gamma^n_0(q)M)$, corresponding in one-to-one way to different left cosets $\Gamma^n_0(q)M \subset \Sigma^n_0(q)$, which are invariant with respect to right multiplication by all elements of $\Gamma^n_0(q)$:

$$\begin{aligned} T &= \sum_\alpha c_\alpha(\Gamma^n_0(q)\, M_\alpha), \\ T\gamma &= \sum_\alpha c_\alpha(\Gamma^n_0(q)\, M_\alpha\gamma) = T \qquad (\forall \gamma \in \Gamma^n_0(q)). \end{aligned} \tag{1.4}$$

The semigroup $\Sigma_0^n(q)$ operates on the space $\mathfrak{F}^{m,n}$ of all complex-valued real-analytic functions $F = F(V, Z) : \mathbb{C}_{2n}^m \times \mathbb{H}^n \to \mathbb{C}$ by the *Petersson operators*

$$\Sigma_0^n(q) \ni M : F = F(V, Z) \mapsto F|M = F|_{\mathbf{j}}M = j_Q(M, Z)^{-1}F(V\,{}^tM, M\langle Z\rangle), \tag{1.5}$$

where $j_Q(M, Z)$ is the automorphic factor. Petersson operators map the space $\mathfrak{F}^{n,m}$ into itself and satisfy the rule $F|M|M' = F|MM'$. This allows us to define the standard representation $T \mapsto |T = |_{\mathbf{j}}T$ of the HS-ring $\mathcal{H}_0^n(q)$ on the subspace

$$\mathfrak{F}_\iota^{n,m} = \{F \in \mathfrak{F}^{n,m} \mid F|\gamma = F \text{ for all } \gamma \in \Gamma_0^n(q)\} \tag{1.6}$$

of all $\Gamma_0^n(q)$-invariant functions of $\mathfrak{F}^{n,m}$ by *Hecke operators*: the Hecke operator $|T = |_{\mathbf{j}}T$ corresponding to an element of $\mathcal{H}_0^n(q)$ of the form (1.4) is defined by

$$F|T = \sum_\alpha c_\alpha F|M_\alpha \qquad (F = F(V, Z) \in \mathfrak{F}_\iota^{n,m}), \tag{1.7}$$

where $|M_\alpha$ are the Petersson operators (1.5). The Hecke operators are independent of the choice of the representatives $M_\alpha \in \Gamma_0^n(q)M_\alpha$ and map $\mathfrak{F}_\iota^{n,m}$ into itself. The map $T \mapsto |T$ is a linear representation of the ring $\mathcal{H}_0^n(q)$ on $\mathfrak{F}_\iota^{n,m}$.

Theta-functions of different genera n of a fixed quadratic form are related by *Siegel operators* $\Phi^{n,r} : \mathfrak{F}^{n,m} \to \mathfrak{F}^{r,m}$, where $0 \le r \le n$, whereas the actions of Hecke operators on these spaces are related by the *Zharkovskaya homomorphisms* $\Psi_Q^{n,r} : \mathcal{H}_0^n(q) \to \mathcal{H}_0^r(q)$ of HS-rings. For definition and properties of these mappings see [An09, §4]. Here we only note that the Zharkovskaya homomorphism $\Psi_Q^{n,r}$ is not always epimorphic, but it is epimorphic if $n, r \ge m/2$ [An96, Proposition 3.3].

The functional equations (1.2) show that the theta-functions (1.1), viewed as functions of V and Z, belong to the space $\mathfrak{F}_\iota^{n,m}$. Explicit formulas for the action of Hecke operators on theta-functions show that images of this action can often be written as finite linear combinations with constant coefficients of similar theta-functions. According to [An86, Theorem 1] and [An09, Propositions 5.1, 5.2(2)], for each homogeneous element $T \in \mathcal{H}_0^n(q)$ with multiplier μ (i.e., a linear combination of left cosets $(\Gamma_0^n(q)M_\alpha)$ with a fixed multiplier $\mu(\Gamma_0^n(q)M_\alpha) = \mu(M_\alpha) = \mu$), which in the case $n < m$ belongs to the image of the ring $\mathcal{H}_0^m(q)$ under the Zharkovskaya map $\Psi_Q^{m,n} : \mathcal{H}_0^m(q) \to \mathcal{H}_0^n(q)$, the image of the theta-function (1.1) under the Hecke operator $|T$ can be written as a linear combination with constant coefficients of similar theta-functions in the form

$$(1.8)\quad \Theta^n(V,Z;H,Q)|T = \begin{cases} \sum_{D\in A(Q,\mu)/\mathrm{GL}_m(\mathbb{Z})} I(D,Q,\Psi_Q^{n,m}T)\Theta^n(\mu D^{-1}V,Z;\mu^{-1}H[D],\mu^{-1}Q[D]), \\ 0, \end{cases}$$

depending on whether the set

$$(1.9)\qquad A(Q,\mu)=\{D\in\mathbb{Z}_m^m \mid \mu^{-1}Q[D]\in\mathbb{E}^m,\ \det\mu^{-1}Q[D]=\det Q\},$$

of all *automorphs of Q with multiplier μ*, is not empty or empty, where $T'=\Psi_Q^{n,m}T\in\mathcal{H}_0^m(q)$ is the image of T under the Zharkovskaya map if $n\ge m$ and an inverse image if $n<m$, and where for $T'=\sum_\alpha c_\alpha(\Gamma_0^m(q)N_\alpha)$ written with "triangular" representatives $N_\alpha=\left(\begin{smallmatrix}A_\alpha & B_\alpha\\ 0_m & D_\alpha\end{smallmatrix}\right)$, the coefficients on the right are the *interaction sums*

$$(1.10)\quad I(D,Q,T') = \sum_{\alpha,\ D\,{}^tD_\alpha\equiv 0 \pmod{\mu}} c_\alpha\chi_Q(|\det D_\alpha|)|\det D_\alpha|^{-m/2}\mathbf{e}\{\mu^{-2}Q[D]\cdot {}^tD_\alpha B_\alpha\}.$$

Note that the interaction sums satisfy the relations

$$(1.11)\quad I(\lambda D\gamma,Q,T')=I(D,Q[\lambda],T')\qquad (\forall T'\in\mathcal{H}_0^m(q),\ \lambda,\gamma\in\mathrm{GL}_m(\mathbb{Z})).$$

2. Orthogonal HS-rings. In this section we briefly recall the definition of orthogonal Hecke–Shimura rings. For details see [An90].

For an even nonsingular matrix Q of even order m and level q, we fix a system of representatives

$$(2.1)\qquad \langle Q\rangle=\{Q_1,\dots,Q_h\}$$

of all different classes with respect to integral equivalence of even matrices of the same order, signature, divisor, level, and determinant as the matrix Q. Given such a system, we define the groups $\mathbf{E}_i=E(Q_i)=\{M\in\mathrm{GL}_m(\mathbb{Z})\mid Q_i[M]=Q_i\}$ of *units of the matrix Q_i* and the sets

$$(2.2)\quad \mathbf{A}_{ij}=\{D\in\mathbb{Z}_m^m\mid Q_i[D]=\mu(D)Q_j,\ \mu(D)>0,\ \gcd(\mu(D),q)=1\}$$

of (*regular*) *automorphs of Q_i to Q_j*. It can be verified that the groups $\mathbf{E}_i$ and sets $\mathbf{A}_{ij}$ satisfy the following three conditions: $\mathbf{A}_{ij}\mathbf{A}_{jk}\subset\mathbf{A}_{ik}$, $\mathbf{E}_i\subset\mathbf{A}_{ii}$, and each double coset $\mathbf{E}_iD\mathbf{E}_j$ with $D\in\mathbf{A}_{ij}$ is a finite union of left cosets modulo $\mathbf{E}_i$. Let us denote by $\mathbb{D}_{ij}$ the set of those finite formal linear combinations with integral coefficients of symbols $(\mathbf{E}_iM)$, corresponding in one-to-one way to different left cosets $\mathbf{E}_iM$ contained in $\mathbf{A}_{ij}$, which are invariant with respect to right multiplication by all elements of $\mathbf{E}_j$:

$$(2.3)\qquad \mathbb{D}_{ij}\ni t=\sum_\alpha a_\alpha(\mathbf{E}_i\,M_\alpha),\quad t\lambda=\sum_\alpha a_\alpha(\mathbf{E}_i\,M_\alpha\lambda)=t\quad(\forall\lambda\in\mathbf{E}_j).$$

Finally, denote by

$$\mathbf{D} = D(\mathbf{E}_1, \dots, \mathbf{E}_h;\, \mathbf{A}_{11}, \mathbf{A}_{12}, \dots, \mathbf{A}_{hh}) = D((\mathbf{E}_i); (\mathbf{A}_{ij})) \tag{2.4}$$

the set of all $h \times h$-matrices $\mathbf{t} = (t_{ij})$ with ij-entries $t_{ij} \in \mathbb{D}_{ij}$. With respect to standard addition and multiplication of matrices, where the product of the linear combinations

$$t_{ij} = \sum_{\alpha} a_\alpha(\mathbf{E}_i M_\alpha) \in \mathbb{D}_{ij} \quad \text{and} \quad t_{jk} = \sum_{\beta} b_\beta(\mathbf{E}_j N_\beta) \in \mathbb{D}_{jk}$$

is defined by

$$t_{ij} \cdot t_{jk} = \sum_{\alpha,\,\beta} a_\alpha b_\beta(\mathbf{E}_i\, M_\alpha N_\beta) \in \mathbb{D}_{ik},$$

the set $\mathbf{D}$ is an associative ring, called the (*regular*) *Hecke–Shimura ring of the system* $(Q_1, \dots, Q_h)$.

For $1 \le i \le h$, let us define the space $\mathcal{G}_i^{n,m}$ of all (real-) analytic functions $G_i = G_i(V,\, H) : \mathbb{C}_{2n}^m \times \mathbb{M}(Q_i) \to \mathbb{C}$ satisfying $G_i(MV,\, H[M^{-1}]) = G_i(V,\, H)$ for each $M \in \mathbf{E}_i$. Then it is easy to see that for $t_{ij} = \sum_\alpha a_\alpha(E_i\, M_\alpha) \in \mathbb{D}_{ij}$ we have

$$G_i \circ t_{ij} = \sum_{\alpha} a_\alpha G_i(M_\alpha V,\, \mu(M_\alpha) H[M_\alpha^{-1}]) \in \mathcal{G}_j^{n,m}, \tag{2.5}$$

and $(G_i \circ t_{ij}) \circ t_{jk} = G_i \circ (t_{ij} \cdot t_{jk})$.

Let $\mathcal{G}^{n,m}$ be the linear space of all rows $\mathbf{G} = (G_1, \dots, G_h)$ with components $G_i \in \mathcal{G}_i^{n,m}$. For a row $\mathbf{G} = (G_i) \in \mathcal{G}^{n,m}$ and $\mathbf{t} = (t_{ij}) \in \mathbf{D}$, we set

$$\mathbf{G} \circ \mathbf{t} = \Big(\sum_{1 \le i \le h} G_i \circ t_{i1}, \dots, \sum_{1 \le i \le h} G_i \circ t_{ih} \Big). \tag{2.6}$$

It is easy to see that the operators $\circ\, \mathbf{t}$ map the space $\mathcal{G}^{n,m}$ into itself and satisfy the rule $\circ\, \mathbf{t} \circ \mathbf{t}' = \circ(\mathbf{t} \cdot \mathbf{t}')$ for $\mathbf{t}, \mathbf{t}' \in \mathbf{D}$. Thus, the operators define a linear representation of the ring $\mathbf{D}$ on the space $\mathcal{G}^{n,m}$.

3. Interaction mappings. Let $T \in \mathcal{H}_0^n(q)$ be a homogeneous element with multiplier $\mu(T) = \mu$. Assume that the set $A(Q_j,\, \mu)$ of the form (1.9) is not empty for a matrix Q_j of the system (2.1). Then, for each $D \in A(Q_j,\, \mu)$, the matrix $\mu^{-1} Q_j[D]$ is integrally equivalent to one of the matrices from the system (2.1), say Q_i. By choosing an appropriate representative in the coset $D \cdot \mathrm{GL}_m(\mathbb{Z})$, one can assume that $\mu^{-1} Q_j[D] = Q_i$, i.e., $Q_j[D] = \mu Q_i$, and the coset $D \cdot \mathrm{GL}_m(\mathbb{Z})$ for such D reduces to the coset $D \cdot \mathbf{E}_i$ of the group $\mathbf{E}_i = E(Q_i)$ of units of Q_i. This shows that one can take

$$A(Q_j, \mu) = \bigcup_{1 \le i \le h} A_{ji}(\mu), \qquad A(Q_j, \mu)/\mathrm{GL}_m(\mathbb{Z}) = \bigcup_{1 \le i \le h} A_{ji}(\mu)/\mathbf{E}_i, \tag{3.1}$$

where

$$A_{ji}(\mu) = \{D \in \mathbf{A}_{ji} \mid \mu(D) = \mu\} = \{D \in \mathbb{Z}^m_m \mid Q_j[D] = \mu Q_i\}.$$

Then the relation (1.8) for $Q = Q_j$ and $H = H_j \in \mathbb{M}(Q_j)$ takes the form

$$(\Theta^n|T)(V, Z; H_j, Q_j)$$
$$= \begin{cases} \sum_{1\le i\le h} \sum_{D\in A_{ji}(\mu)/\mathbf{E}_i} I(D, Q_j, \Psi^{n,m}T)\Theta^n(\mu D^{-1}V, Z; \mu^{-1}H_j[D], \mu^{-1}Q_j[D]), \\ 0, \end{cases}$$

depending on whether $A(Q_j,\, \mu)$ is not empty or empty, where $\Psi^{n,m} = \Psi^{n,m}_Q$ is the Zharkovskaya mapping for a matrix $Q \in \langle Q\rangle$. Since μ is prime to the level q of the matrices Q_j, the condition $D \in A_{ji}(\mu)/\mathbf{E}_i$ is equivalent to $\mu D^{-1} \in \mathbf{E}_i\backslash A_{ij}(\mu)$. Therefore, by replacing $D \mapsto \mu D^{-1}$, the last relations can be rewritten in the form

$$(3.2)\qquad (\Theta^n|T)(V, Z; H_j, Q_j)$$
$$= \begin{cases} \sum_{1\le i\le h} \sum_{D\in \mathbf{E}_i\backslash A_{ij}(\mu)} I(\mu D^{-1}, Q_j, \Psi^{n,m}T)\Theta^n(DV, Z; \mu H_j[D^{-1}], \mu Q_j[D^{-1}]), \\ 0, \end{cases}$$

depending on whether $A(Q_j, \mu)$ is not empty or empty.

For $n \ge 1$ and $i,\, j = 1, \dots, h$ set

$$(3.3)\qquad \tau^n_{ij}(T) = \begin{cases} \sum_{D\in \mathbf{E}_i\backslash A_{ij}(\mu)} I(\mu D^{-1}, Q_j, \Psi^{n,m}T)(\mathbf{E}_i D) & \text{if } A(Q_j, \mu) \ne \emptyset, \\ 0 & \text{if } A(Q_j, \mu) = \emptyset. \end{cases}$$

It follows from (1.11) that for each $\gamma \in \mathbf{E}_j$ the linear combinations (3.3) satisfy

$$\tau^n_{ij}(T)\gamma = \sum_{D\in \mathbf{E}_i\backslash A_{ij}(\mu)} I(\mu D^{-1}, Q_j[\gamma^{-1}], \Psi^{n,m}T)(\mathbf{E}_i D) = \tau^n_{ij}(T).$$

Thus, $\tau^n_{ij}(T) \in \mathbb{D}_{ij}$, and so the matrix $\tau^n(T) = \tau^n_{\langle Q\rangle}(T) = (\tau^n_{ij}(T))$ belongs to the orthogonal HS-ring (2.4). Extending the mapping by linearity to arbitrary $T \in \mathcal{H}^n_0(q)$, we obtain a linear mapping of HS-rings

$$(3.4)\qquad \mathcal{H}^n_0(q) \ni T \mapsto \tau^n(T) = (\tau^n_{ij}(T)) \in \mathbf{D} = D((\mathbf{E}_i); (\mathbf{A}_{ij})).$$

Since the theta-functions (1.1) satisfy not only the functional equations (1.2) but also (1.3), it follows that each theta-function $\Theta^n_i(V_i,\, H_i) = \Theta^n(V_i, Z; H_i, Q_i)$ considered as a function in $V_i \in \mathbb{C}^m_{2n}$ and $H_i \in \mathbb{M}(Q_i)$ belongs to the space $\mathcal{G}^{n,m}_i$, and each *theta-vector* (Θ^n_i) *of genus* n *corresponding to the system* $\langle Q\rangle$, with components $\Theta^n_i = \Theta^n_i(V_i,\, H_i) = \Theta^n(V_i, Z; H_i, Q_i)$, is an element of $\mathcal{G}^{n,m}$. The following theorem expresses images of the theta-vectors under componentwise action of symplectic Hecke operators in terms of the action of orthogonal Hecke operators by means of the interaction mapping (3.4).

Theorem 1. *Let Q be an even nonsingular matrix of even order m, and $\langle Q\rangle$ a system of representatives* (2.1). *Then for each $n \geq m/2$ the componentwise action of a Hecke operator $|T$ with $T \in \mathcal{H}_0^n(q)$ on the theta-vector $(\Theta_i^n(V_i, H_i))$ considered as an element of the space $\mathcal{G}^{n,m}$ can be written in terms of the action of the operator $\circ\,\tau^n(T)$ defined by* (2.5) *and* (2.6) *in the form*

$$(\Theta_1^n|T,\dots,\Theta_h^n|T) = (\Theta_1^n,\dots,\Theta_h^n)\circ\tau^n(T) = \Big(\sum_{1\le i\le h}\Theta_i^n\circ\tau_{i1}^n(T),\dots,\sum_{1\le i\le h}\Theta_i^n\circ\tau_{ih}^n(T)\Big).$$

Proof. Suppose first that $T \in \mathcal{H}_0^n(q)$ is a homogeneous element with multiplier $\mu(T) = \mu$, and that the set $A(Q_j, \mu)$ of the form (1.9) is not empty for a matrix Q_j of the system (2.1). Then, according to the definition of the operators $\circ\, t_{ij}^n$, the action of $\circ\,\tau_{ij}^n(T)$ on components of the theta-vector can be written in the form

$$(\Theta_i^n\circ\tau_{ij}^n(T))(V_i,H_i) = \sum_{D\in\mathbf{E}_i\backslash A_{ij}(\mu)} I(\mu D^{-1},Q_j,\Psi^{n,m}T)\Theta_i^n(DV_i,\mu H_i[D^{-1}]).$$

Hence the formulas (3.2) can be rewritten as

$$\Theta_j^n|T = \sum_{1\le i\le h}\Theta_i^n\circ\tau_{ij}^n(T) \qquad (j=1,\dots,h).$$

Thus, under componentwise action of the symplectic Hecke operator $|T$ on the theta-vector we obtain

$$(\Theta_i^n)|T = (\Theta_1^n|T,\dots,\Theta_h^n|T) = \Big(\sum_{1\le i\le h}\Theta_i^n\circ\tau_{i1}^n(T),\dots,\sum_{1\le i\le h}\Theta_i^n\circ\tau_{ih}^n(T)\Big)$$
$$= (\Theta_1^n,\dots,\Theta_h^n)\circ\tau^n(T) = (\Theta_i^n)\circ\tau^n(T).$$

This formula together with (1.8) is true for all homogeneous elements $T \in \mathcal{H}_0^n(q)$ with multiplier μ and such that the set $A(Q, \mu)$ of the form (1.9) is not empty for a matrix $Q \in \langle Q\rangle$. By linearity, the formulas remain true for all $T \in \mathcal{H}_0^n(q)$. If all of the sets $A(Q_j, \mu)$ are empty, then $(\Theta_i^n)|T = \mathbf{0}$, and the formula remains true with the convention $\tau^n(T) = \mathbf{0}_m^m$. The theorem is proved. ∎

Note that when $m/2 \leq n < m$, the inverse image $\Psi^{n,m}T \in \mathcal{H}_0^m(q)$ is not unique, which causes an indeterminacy in the definition of the mapping (3.4), but in view of the theorem this does not affect the action of the operators $\circ\,\tau^n(T)$ on theta-vectors. We call the mapping $T \mapsto \tau^n(T)$ the *interaction mapping of HS-rings*.

THEOREM 2. *Let Q be an even nonsingular matrix of even order m, and $\langle Q\rangle$ a system of representatives of the form* (2.1)*. Then for every $n \geq m$ the mapping* (3.4) *is a linear ring homomorphism of Hecke–Shimura rings.*

Proof. The mapping (3.4) is linear by definition. We consider first the case $n = m$. By linearity it is sufficient to prove in this case that $\tau^m(T\,T') = \tau^m(T)\tau^m(T')$ for any homogeneous $T, T' \in \mathcal{H}_0^m(q)$. This matrix relation is equivalent to the system of relations

$$\tau_{ij}^m(TT') = \sum_{k=1}^{h} \tau_{ik}^m(T)\tau_{kj}^m(T') \qquad (i, j = 1, \ldots, h). \tag{3.5}$$

If $\mu(T) = \mu$ and $\mu(T') = \mu'$, then $\mu(TT') = \mu\mu'$ and by (3.3) we have

$$\tau_{ij}^m(TT') = \begin{cases} \displaystyle\sum_{D''\in \mathbf{E}_i\backslash A_{ij}(\mu\mu')} I(\mu\mu'(D'')^{-1}, Q_j, TT')(\mathbf{E}_i D'') & \text{if } A(Q_j, \mu\mu') \neq \emptyset, \\ 0 & \text{if } A(Q_j, \mu\mu') = \emptyset. \end{cases}$$

By definitions we can write

$$\tau_{ik}^m(T)\tau_{kj}^m(T') = \sum_{D\in \mathbf{E}_i\backslash A_{ik}(\mu)}\ \sum_{D'\in \mathbf{E}_k\backslash A_{kj}(\mu')} I(\mu D^{-1}, Q_k, T) I(\mu'(D')^{-1}, Q_j, T')(\mathbf{E}_i DD')$$

if $A(Q_k,\, \mu) \neq \emptyset$ and $A(Q_j,\, \mu') \neq \emptyset$. Otherwise, $\tau_{ik}^m(T)\tau_{kj}^m(T') = 0$. Therefore, in order to prove (3.5) it is sufficient to show that

$$I(\mu\mu'(D'')^{-1}, Q_j, TT') = \sum_{k=1}^{h} \sum_{\substack{(D,D')\in(\mathbf{E}_i\backslash A_{ik}(\mu), \mathbf{E}_k\backslash A_{kj}(\mu')) \\ DD'\in \mathbf{E}_i D''}} I(\mu D^{-1}, Q_k, T) I(\mu'(D')^{-1}, Q_j, T') \tag{3.6}$$

for each $D'' \in A_{ij}(\mu\mu')$, unless both sides are zero.

On the other hand, by [An96, Proposition 3.8], for every $\tilde{D} \in A(Q_j, \mu\mu')$ the interaction sums satisfy

$$I(\tilde{D}, Q_j, TT') = \sum_{\substack{(D_1,D_2)\in(A(Q_j,\mu)/\Lambda, A(\mu^{-1}Q_j[D_1],\mu')/\Lambda) \\ D_1 D_2\in \tilde{D}\Lambda}} I(D_1, Q_j, T) I(D_2, \mu^{-1}Q_j[D_1], T'),$$

where $\Lambda = \mathrm{GL}_m(\mathbb{Z})$. By (3.1), the inclusion $D_1 \in A(Q_j, \mu)/\Lambda$ means that $D_1 \in A_{jk}(\mu)/\mathbf{E}_k$ for some k with $1 \leq k \leq h$ and vice versa. If $D_1 \in A_{jk}(\mu)/\mathbf{E}_k$, then $\mu^{-1}Q_j[D_1] = Q_k$. Hence, again by (3.1), the condition $D_2 \in A(\mu^{-1}Q_j[D_1], \mu')/\Lambda$ means that $D_2 \in A_{ki}(\mu')/\mathbf{E}_i$ for some i, $1 \leq i \leq h$.

Then $D_1D_2 \in \tilde{D}\mathbf{E}_i$, and the last relation turns into

$$I(\tilde{D}, Q_j, TT') = \sum_{k=1}^{h} \sum_{\substack{(D_1,D_2)\in(A_{jk}(\mu)/\mathbf{E}_k, A_{ki}(\mu')/\mathbf{E}_i) \\ D_1D_2\in\tilde{D}\mathbf{E}_i}} I(D_1, Q_j, T)I(D_2, Q_k, T').$$

Since μ and μ' are prime to the level q of all matrices of $\langle Q\rangle$, the conditions $D_1 \in A_{jk}(\mu)/\mathbf{E}_k$ and $D_2 \in A_{ki}(\mu')/\mathbf{E}_i$ are equivalent to $\mu D_1^{-1} \in \mathbf{E}_k\backslash A_{kj}(\mu)$ and $\mu' D_2^{-1} \in \mathbf{E}_i\backslash A_{ik}(\mu')$, respectively. Therefore, after the substitution $\mu D_1^{-1} = D$, $\mu' D_2^{-1} = D'$ and $\mu\mu'\tilde{D}^{-1} = D''$, i.e. $D_1 = \mu D^{-1}$, $D_2 = \mu'(D')^{-1}$ and $\tilde{D} = \mu\mu'(D'')^{-1}$, we come to the relation

$$\begin{aligned} I(\mu\mu'(D'')^{-1},\, Q_j,\, TT') \\ &= \sum_{k=1}^{h} \sum_{\substack{(D,D')\in(\mathbf{E}_k\backslash A_{kj}(\mu),\mathbf{E}_i\backslash A_{ik}(\mu')) \\ D'D\in\mathbf{E}_iD''}} I(\mu D^{-1}, Q_j, T)I(\mu'(D')^{-1}, Q_k, T') \\ &= \sum_{k=1}^{h} \sum_{\substack{(D',D)\in(\mathbf{E}_i\backslash A_{ik}(\mu'),\,\mathbf{E}_k\backslash A_{kj}(\mu)) \\ D'D\in\mathbf{E}_iD''}} I(\mu'(D')^{-1}, Q_k, T')I(\mu D^{-1}, Q_j, T), \end{aligned}$$

which is actually (3.6) for $T', T \in \mathcal{H}^m(q)$, since $TT' = T'T$, by commutativity of the ring $\mathcal{H}^m(q)$.

The case $n \geq m$ follows from the case $n = m$, since by definition the mapping $T \mapsto \tau^n(T) = \tau^m(\Psi^{n,m}(T))$ is the composition of the Zharkovskaya homomorphism $\Psi^{n,m} = \Psi_Q^{n,m}$ and the homomorphism τ^m. ∎

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