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Quadratic fields with infinite Hilbert 2-class field towers

by

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1. Introduction. Let K be a finite extension field of the rational number field $\mathbb{Q}$, and let C_K be the 2-*class group* of K (i.e., the Sylow 2-subgroup of the ideal class group of K) in the usual sense. Let K_1 be the *Hilbert* 2-*class field* of K (i.e., the maximal abelian unramified extension of K whose Galois group is a 2-group), and let K_i be the Hilbert 2-class field of K_{i-1} for $i \geq 2$. Then

$$K \subset K_1 \subset \ldots \subset K_i \subset \ldots$$

is the *Hilbert* 2-*class field tower* of K. If $K_i \neq K_{i-1}$ for all i, then the Hilbert 2-class field tower is said to be *infinite*.

Next we define the 2-class rank and the 4-class rank of K. Let $C_K^i = \{a^i : a \in C_K\}$. We define the 2-*class rank* r_K by

$$r_K = \operatorname{rank} C_K = \dim_{\mathbb{F}_2}(C_K/C_K^2) \tag{1}$$

where $\mathbb{F}_2$ is the finite field with two elements, and we are viewing the elementary abelian 2-group C_K/C_K^2 as a vector space over $\mathbb{F}_2$. We define the 4-*class rank* s_K by

$$s_K = \operatorname{rank} C_K^2 = \dim_{\mathbb{F}_2}(C_K^2/C_K^4). \tag{2}$$

We note that $0 \leq s_K \leq r_K$.

Now suppose K is an imaginary quadratic extension of $\mathbb{Q}$. It is known (cf. [1, p. 233]) that the Hilbert 2-class field tower of K is infinite if $r_K \geq 5$. We shall prove some results for the cases where $r_K = 3$ or 4. For nonnegative integers r and s, square-free positive integers m, and positive real numbers x, we define

$$\begin{aligned}
V_r &= \{K = \mathbb{Q}(\sqrt{-m}\,) : \text{the 2-class rank } r_K = r\},\\
V_{r;x} &= \{K = \mathbb{Q}(\sqrt{-m}\,) \in V_r : m \leq x\},\\
V_{r,s;x} &= \{K \in V_{r;x} : \text{the 4-class rank } s_K = s\},\\
V^*_{r,s;x} &= \{K \in V_{r,s;x} : \text{the Hilbert 2-class field tower of } K \text{ is infinite}\}.
\end{aligned}$$

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Then we define a density

$$\delta^*_{r,s} = \liminf_{x\to\infty} \frac{|V^*_{r,s;x}|}{|V_{r;x}|} \tag{3}$$

where $|V|$ denotes the cardinality of a finite set V. We shall prove the following theorem in Section 2 of this paper.

THEOREM 1. *For imaginary quadratic fields let* $\delta^*_{r,s}$ *be defined by* (3)*. Then* $\delta^*_{3,s} > 0$ *for* $1 \le s \le 3$ *and* $\delta^*_{4,s} > 0$ *for* $0 \le s \le 4$*.*

REMARK. Thus a positive proportion of the imaginary quadratic fields with 2-class rank equal to 3 have infinite Hilbert 2-class field towers and 4-class rank equal to s for each value $s = 1$, 2, and 3. Similarly, a positive proportion of the imaginary quadratic fields with 2-class rank equal to 4 have infinite Hilbert 2-class field towers and 4-class rank equal to s for each value $s = 0$, 1, 2, 3, and 4.

REMARK. An essential part of the proof of Theorem 1 depends on Corollary 3 in a paper of Hajir [6]. In fact, Theorem 1 can be viewed as a generalization of ideas introduced in [6].

Now suppose K is a real quadratic extension of $\mathbb{Q}$. From [1, p. 233] the Hilbert 2-class field tower of K is infinite if $r_K \ge 6$. We shall prove some results for the cases where $r_K = 4$ or 5. For nonnegative integers r and s, square-free integers $m > 1$, and positive real numbers x, we define

$$\begin{aligned} W_r &= \{K = \mathbb{Q}(\sqrt{m}\,) : \text{the 2-class rank } r_K = r\},\\ W_{r;x} &= \{K = \mathbb{Q}(\sqrt{m}\,) \in W_r : m \le x\},\\ W_{r,s;x} &= \{K \in W_{r;x} : \text{the 4-class rank } s_K = s\},\\ W^*_{r,s;x} &= \{K \in W_{r,s;x} : \text{the Hilbert 2-class field tower of } K \text{ is infinite}\}, \end{aligned}$$

and

$$\varepsilon^*_{r,s} = \liminf_{x\to\infty} \frac{|W^*_{r,s;x}|}{|W_{r;x}|}\ . \tag{4}$$

We shall prove the following theorem in Section 3 of this paper.

THEOREM 2. *For real quadratic fields let* $\varepsilon^*_{r,s}$ *be defined by* (4)*. Then* $\varepsilon^*_{4,s} > 0$ *for* $0 \le s \le 4$ *and* $\varepsilon^*_{5,s} > 0$ *for* $0 \le s \le 5$*.*

REMARK. So a positive proportion of the real quadratic fields with 2-class rank equal to 4 have infinite Hilbert 2-class field towers and 4-class rank equal to s for each value $s = 0$, 1, 2, 3, and 4. Similarly, a positive proportion of the real quadratic fields with 2-class rank equal to 5 have infinite Hilbert 2-class field towers and 4-class rank equal to s for each value $s = 0$, 1, 2, 3, 4, and 5.

2. Proof of Theorem 1. Let K be an imaginary quadratic field in which exactly t finite primes are ramified, where t is a positive integer. From genus theory we know that the 2-class rank r_K equals $t-1$. With $t = r+1$, we see that the set $V_{r;x}$ in this paper is the same as the set $A_{t;x}$ in [4]. So from equation (2.5) in [4],

$$|V_{r;x}| \sim \frac{1}{2}\cdot\frac{1}{r!}\cdot\frac{x(\log\log x)^r}{\log x} \quad (\text{as } x\to\infty). \tag{5}$$

Now suppose $K = \mathbb{Q}(\sqrt{-p_1\ldots p_t}\,)$, where $p_1 < \ldots < p_t$ are primes with $p_i \equiv 1 \pmod 4$ for $1 \le i \le t-1$ and $p_t \equiv 3 \pmod 4$. From equations (2.6) and (2.7) in [4], the 4-class rank s_K satisfies

$$s_K = t - 1 - \operatorname{rank} M_K \tag{6}$$

where M_K is a $t\times t$ matrix over $\mathbb{F}_2$ whose entries a_{ij} are defined by Legendre symbols as follows:

$$(-1)^{a_{ij}} = \begin{cases} \left(\dfrac{P_j}{p_i}\right) & \text{if } i \ne j, \\[2ex] \left(\dfrac{\overline{P}_j}{p_i}\right) & \text{if } i = j, \end{cases} \tag{7}$$

with $P_j = p_j$ if $p_j \equiv 1 \pmod 4$, $P_j = -p_j$ if $p_j \equiv 3 \pmod 4$, and $\overline{P}_j = -p_1\ldots p_t/P_j$. From quadratic reciprocity and properties of Legendre symbols, the matrix M_K is completely determined by the set of values $\bigl\{\bigl(\frac{p_j}{p_i}\bigr)$ for $1 \le i < j \le t\bigr\}$. For positive real numbers x, let

$$\begin{aligned} S(K,t;x) = \biggl\{ &\mathbb{Q}(\sqrt{-p_1'\ldots p_t'}\,) \text{ with primes } p_1' < \ldots < p_t', \\ &p_i' \equiv p_i \pmod 4 \text{ for } 1 \le i \le t, \\ &\left(\frac{p_j'}{p_i'}\right) = \left(\frac{p_j}{p_i}\right) \text{ for } 1 \le i < j \le t, \text{ and } p_1'\ldots p_t' \le x \biggr\}. \end{aligned}$$

From equation (2.12) in [4],

$$|S(K,t;x)| \sim 2^{-(t^2+t)/2}\cdot\frac{1}{(t-1)!}\cdot\frac{x(\log\log x)^{t-1}}{\log x} \quad (\text{as } x\to\infty). \tag{8}$$

The proof of this formula depends on character sum estimates similar to those used in Section 4 of [3] and Section 5 of [5]. Alternatively, one can use the analytic machinery developed in [2]. Note that from (5) and with $t = r+1$ in (8), we get

$$\lim_{x\to\infty}\frac{|S(K,r+1;x)|}{|V_{r;x}|} = 2^{-(r^2+3r)/2} > 0. \tag{9}$$

The fact that this limit is positive will be a key part of the proof of Theorem 1. However, first we need the following lemma, which follows from Corollary 3 in [6].

LEMMA 1. *Suppose* $K = \mathbb{Q}(\sqrt{-p_1 \ldots p_t})$ *with* $p_1, \ldots, p_t$ *distinct primes and* $t \geq 4$. *Also suppose* $p_1 \equiv p_2 \equiv 1 \pmod 4$ *and* $\left(\frac{p_j}{p_i}\right) = 1$ *for* $i = 1, 2$ *and* $j = 3, 4$. *Then the Hilbert* 2*-class field tower of* K *is infinite.*

Proof. Let $F = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2})$. Since $p_1 \equiv p_2 \equiv 1 \pmod 4$ and $\left(\frac{p_j}{p_i}\right) = 1$ for $i = 1, 2$ and $j = 3, 4$, p_3 and p_4 split completely in the totally real degree 4 extension F of $\mathbb{Q}$. Then by Corollary 3 in [6], $E = FK$ has infinite Hilbert 2-class field tower. Since E is contained in the Hilbert 2-class field of K, the Hilbert 2-class field tower of K is infinite.

Now consider $K = \mathbb{Q}(\sqrt{-p_1 p_2 p_3 p_4})$ with primes $p_1 < p_2 < p_3 < p_4$ such that $p_i \equiv 1 \pmod 4$ for $1 \leq i \leq 3$ and $p_4 \equiv 3 \pmod 4$. Then the 2-class rank r_K equals 3.

CASE $s = 1$: Suppose $\left(\frac{p_j}{p_i}\right) = 1$ for $i = 1, 2$ and $j = 3, 4$; $\left(\frac{p_2}{p_1}\right) = -1$; $\left(\frac{p_4}{p_3}\right) = -1$. From Lemma 1, K has an infinite Hilbert 2-class field tower, and from (7) one can check that $\operatorname{rank} M_K = 2$. Then the 4-class rank s_K is 1 from (6). Furthermore every field in the set $S(K, 4; x)$ has an infinite Hilbert 2-class field tower and 4-class rank equal to 1. So $S(K, 4; x) \subset V^*_{3,1;x}$, and then equations (3) and (9) imply $\delta^*_{3,1} > 0$.

CASE $s = 2$: Suppose $\left(\frac{p_j}{p_i}\right) = 1$ for $i = 1, 2$ and $j = 3, 4$; $\left(\frac{p_2}{p_1}\right) = -1$; $\left(\frac{p_4}{p_3}\right) = 1$. Then a similar analysis shows $S(K, 4; x) \subset V^*_{3,2;x}$, and then $\delta^*_{3,2} > 0$.

CASE $s = 3$: Suppose $\left(\frac{p_j}{p_i}\right) = 1$ for $1 \leq i < j \leq 4$. Then $S(K, 4; x) \subset V^*_{3,3;x}$, and $\delta^*_{3,3} > 0$.

Now consider $K = \mathbb{Q}(\sqrt{-p_1 p_2 p_3 p_4 p_5})$ with primes $p_1 < p_2 < p_3 < p_4 < p_5$ such that $p_i \equiv 1 \pmod 4$ for $1 \leq i \leq 4$ and $p_5 \equiv 3 \pmod 4$. Then the 2-class rank r_K equals 4.

CASE $s = 0$: Suppose $\left(\frac{p_j}{p_i}\right) = 1$ for $1 \leq i < j \leq 4$ and $\left(\frac{p_5}{p_i}\right) = -1$ for $1 \leq i \leq 4$. Then $S(K, 5; x) \subset V^*_{4,0;x}$, and $\delta^*_{4,0} > 0$.

CASES $1 \leq s \leq 3$: If $s = 1$, 2, or 3, suppose $\left(\frac{p_j}{p_i}\right) = 1$ for $1 \leq i < j \leq 4$; $\left(\frac{p_5}{p_i}\right) = 1$ for $1 \leq i \leq s$; $\left(\frac{p_5}{p_i}\right) = -1$ for $s+1 \leq i \leq 4$. Then $S(K, 5; x) \subset V^*_{4,s;x}$ and $\delta^*_{4,s} > 0$.

CASE $s = 4$: Suppose $\left(\frac{p_j}{p_i}\right) = 1$ for $1 \leq i < j \leq 5$. Then $S(K, 5; x) \subset V^*_{4,4;x}$, and $\delta^*_{4,4} > 0$.

Thus the proof of Theorem 1 is complete.

REMARK. It is known that $V^*_{3,3;x} = V_{3,3;x}$ and $V^*_{4,s;x} = V_{4,s;x}$ for $s = 3, 4$ (see [6] and [7]). One could use these facts to give an alternative proof that $\delta^*_{3,3} > 0$, $\delta^*_{4,3} > 0$, and $\delta^*_{4,4} > 0$.

3. Proof of Theorem 2. Let $K = \mathbb{Q}(\sqrt{m}\,)$, where $m > 1$ is a square-free integer. Let r_K be the 2-class rank of K, and let t be the number of primes that ramify in $K/\mathbb{Q}$. It is well known that

$$r_K = \begin{cases} t-1 & \text{if no prime dividing } m \\ & \text{is congruent to 3 (mod 4)}, \\ t-2 & \text{if at least one prime dividing } m \\ & \text{is congruent to 3 (mod 4)}. \end{cases} \tag{10}$$

For nonnegative integers r and positive real numbers x, we let

$$\begin{aligned} Y_{r;x} = \{k = \mathbb{Q}(\sqrt{m}\,) : m = p_1 \ldots p_{r+2} \le x \\ \text{with odd primes } p_1 < \ldots < p_{r+2} \\ \text{and with a positive even number of } p_i \equiv 3 \pmod 4\}. \end{aligned} \tag{11}$$

If N_x is the number of square-free positive integers $m \le x$ with $r+2$ prime factors, then

$$N_x \sim \frac{1}{(r+1)!} \cdot \frac{x(\log\log x)^{r+1}}{\log x} \qquad (\text{as } x \to \infty)$$

(see [8, Theorem 437]). If $N_{\mathrm{e},x}$ is the number of square-free positive integers $m \le x$ with $r+2$ prime factors and an even number of these primes congruent to 3 (mod 4), then $N_{\mathrm{e},x} \sim \frac{1}{2} N_x$. In $Y_{r;x}$, we are excluding the set

$$\{m = p_1 \ldots p_{r+2} \le x \text{ with each } p_i \equiv 1 \pmod 4\},$$

which has cardinality asymptotic to $2^{-(r+2)} N_x$. We are also excluding the set

$$\{m = p_1 \ldots p_{r+2} \le x \text{ with } 2 \mid m\},$$

which has cardinality $o(N_x)$. So

$$|Y_{r;x}| \sim \left(\frac{1}{2} - \frac{1}{2^{r+2}}\right) \cdot \frac{1}{(r+1)!} \cdot \frac{x(\log\log x)^{r+1}}{\log x} \qquad (\text{as } x \to \infty). \tag{12}$$

Now recall that $W_{r;x} = \{K = \mathbb{Q}(\sqrt{m}\,) :$ 2-class rank r_K equals r and $m \le x\}$. We note that

$$|W_{r;x}| \sim |Y_{r;x}| \qquad (\text{as } x \to \infty)$$

since $Y_{r;x} \subset W_{r;x}$ and the set of elements of $W_{r;x}$ that are not in $Y_{r;x}$ has cardinality $o(|Y_{r;x}|)$. Then for nonnegative integers s, we define

$$\begin{aligned} Y^*_{r,s;x} = \{K \in Y_{r;x} : \text{the 4-class rank } s_K = s, \text{ and the} \\ \text{Hilbert 2-class field tower of } K \text{ is infinite}\}. \end{aligned}$$

Then from (4) and the above discussion, we get

$$\varepsilon^*_{r,s} = \liminf_{x\to\infty} \frac{|Y^*_{r,s;x}|}{|Y_{r;x}|}. \tag{13}$$

Now suppose $K = \mathbb{Q}(\sqrt{p_1 \dots p_t})$, where $p_1 < \dots < p_t$ are primes with $p_i \equiv 1 \pmod 4$ for $1 \le i \le t-2$ and $p_{t-1} \equiv p_t \equiv 3 \pmod 4$. Then the 2-class rank r_K is $t-2$. For the 4-class rank, we shall use results from Section 5 of [4]. However, first we remark that the 2-class groups considered in Section 5 of [4] are the narrow 2-class groups. For the field K that we are considering, the narrow 2-class rank is $t-1$ rather than $t-2$, but the narrow 4-class rank and the usual 4-class rank are the same. Hence from equations (5.5) and (5.6) in [4], the 4-class rank s_K satisfies

$$s_K = t - 1 - \operatorname{rank} M_K \tag{14}$$

where M_K is the $t \times t$ matrix over $\mathbb{F}_2$ whose entries a_{ij} satisfy

$$(-1)^{a_{ij}} = \begin{cases} \left(\dfrac{P_j}{p_i}\right) & \text{if } i \ne j, \\ \left(\dfrac{\overline{P}_j}{p_i}\right) & \text{if } i = j, \end{cases} \tag{15}$$

with $P_j = p_j$ if $p_j \equiv 1 \pmod 4$, $P_j = -p_j$ if $p_j \equiv 3 \pmod 4$, and $\overline{P}_j = p_1 \dots p_t / P_j$. Note that $\operatorname{rank} M_K \ge 1$ since either $\left(\frac{P_{t-1}}{p_t}\right) = -1$ or $\left(\frac{P_t}{p_{t-1}}\right) = -1$ by quadratic reciprocity since $p_{t-1} \equiv p_t \equiv 3 \pmod 4$. So $s_K \le t-2$. Furthermore, from quadratic reciprocity and properties of Legendre symbols, the matrix M_K is completely determined by the set of values $\left\{\left(\frac{p_j}{p_i}\right)\right.$ for $\left.1 \le i < j \le t\right\}$. For positive real numbers x, let

$$\begin{aligned} S'(K,t;x) = \Big\{ &\mathbb{Q}(\sqrt{p'_1 \dots p'_t}) \text{ with primes } p'_1 < \dots < p'_t, \\ &p'_i \equiv p_i \pmod 4 \text{ for } 1 \le i \le t, \\ &\left(\frac{p'_j}{p'_i}\right) = \left(\frac{p_j}{p_i}\right) \text{ for } 1 \le i < j \le t, \text{ and } p'_1 \dots p'_t \le x \Big\}. \end{aligned}$$

Then analogously to (8) we have

$$|S'(K,t;x)| \sim 2^{-(t^2+t)/2} \cdot \frac{1}{(t-1)!} \cdot \frac{x(\log\log x)^{t-1}}{\log x} \quad (\text{as } x \to \infty). \tag{16}$$

From (12) and (16) with $t = r+2$, we get

$$\lim_{x\to\infty} \frac{|S'(K,r+2;x)|}{|Y_{r;x}|} = 2^{-(r+1)(r+2)/2} \cdot (2^{r+1}-1)^{-1} > 0. \tag{17}$$

Now we prove a lemma analogous to Lemma 1.

LEMMA 2. *Suppose $K = \mathbb{Q}(\sqrt{p_1 \dots p_t})$ with distinct primes $p_1, \dots, p_t$ and $t \geq 6$. Also suppose $p_1 \equiv p_2 \equiv 1 \pmod 4$ and $\left(\frac{p_j}{p_i}\right) = 1$ for $i = 1, 2$ and $j = 3, 4, 5$. Then the Hilbert 2-class field tower of K is infinite.*

Proof. Since $\left(\frac{p_j}{p_i}\right) = 1$ for $i = 1, 2$ and $j = 3, 4, 5$, we see that p_3, p_4, and p_5 split completely in $F = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2})$. Also p_6 splits in at least one of the subfields $\mathbb{Q}(\sqrt{p_1})$, $\mathbb{Q}(\sqrt{p_2})$, $\mathbb{Q}(\sqrt{p_1 p_2})$ of F. Let $L = F(\sqrt{p_3 \dots p_t})$. Then there are at least 14 ramified primes in L/F. Let E_F be the group of units in the ring of algebraic integers in F, and let $E_F^2 = \{u^2 : u \in E_F\}$. Since $\dim_{\mathbb{F}_2}(E_F/E_F^2) = 4$, from genus theory the 2-class rank r_L satisfies

$$r_L \geq 14 - 1 - 4 = 9.$$

From [1, p. 233], L has an infinite Hilbert 2-class field tower if $r_L \geq 2 + 2\sqrt{\gamma_L + 1}$, where γ_L is the number of infinite primes of L. Since $\gamma_L = 8$, we get $r_L \geq 9 > 2 + 2\sqrt{8+1}$, and thus L does have an infinite Hilbert 2-class field tower. Since L is contained in the Hilbert 2-class field of K, we conclude that K has an infinite Hilbert 2-class field tower.

Now consider $K = \mathbb{Q}(\sqrt{p_1 \dots p_6})$ with primes $p_1 < \dots < p_6$ such that $p_i \equiv 1 \pmod 4$ for $1 \leq i \leq 4$ and $p_5 \equiv p_6 \equiv 3 \pmod 4$. Then the 2-class rank r_K equals 4. Analogously to the procedure we used in proving Theorem 1, we list conditions on Legendre symbols that imply that the 4-class rank s_K equals s for a given value of s (by (14) and (15)) and so that K has an infinite Hilbert 2-class field tower (by Lemma 2). Then we get $S'(K, 6; x) \subset Y^*_{4,s;x}$, and using (13) and (17), we get $\varepsilon^*_{4,s} > 0$.

CASE $s = 0$: Suppose $\left(\frac{p_j}{p_i}\right) = 1$ for $1 \leq i < j \leq 5$; $\left(\frac{p_6}{p_i}\right) = -1$ for $1 \leq i \leq 5$.

CASES $1 \leq s \leq 4$: For $s = 1$, 2, 3, or 4, suppose $\left(\frac{p_j}{p_i}\right) = 1$ for $1 \leq i < j \leq 5$; $\left(\frac{p_6}{p_i}\right) = 1$ for $1 \leq i \leq s$; $\left(\frac{p_6}{p_i}\right) = -1$ for $s + 1 \leq i \leq 5$.

Next consider $K = \mathbb{Q}(\sqrt{p_1 \dots p_7})$ with primes $p_1 < \dots < p_7$ such that $p_i \equiv 1 \pmod 4$ for $1 \leq i \leq 5$ and $p_6 \equiv p_7 \equiv 3 \pmod 4$. Then the 2-class rank r_K equals 5. We will get $S'(K, 7; x) \subset Y^*_{5,s;x}$ and $\varepsilon^*_{5,s} > 0$ if we choose the primes as follows:

CASE $s = 0$: Suppose $\left(\frac{p_j}{p_i}\right) = 1$ for $1 \leq i < j \leq 6$; $\left(\frac{p_7}{p_i}\right) = -1$ for $1 \leq i \leq 6$.

CASES $1 \leq s \leq 5$: If $s = 1$, 2, 3, 4, or 5, suppose $\left(\frac{p_j}{p_i}\right) = 1$ for $1 \leq i < j \leq 6$; $\left(\frac{p_7}{p_i}\right) = 1$ for $1 \leq i \leq s$; $\left(\frac{p_7}{p_i}\right) = -1$ for $s + 1 \leq i \leq 6$.

Then the proof of Theorem 2 is complete.

REMARK. It is known that $W^*_{4,4;x} = W_{4,4;x}$ and $W^*_{5,s;x} = W_{5,s;x}$ for $s = 4, 5$ (see [9]). One could use these facts to give an alternative proof that $\varepsilon^*_{4,4} > 0$, $\varepsilon^*_{5,4} > 0$, and $\varepsilon^*_{5,5} > 0$.

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