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Effective polynomial upper bounds to perigees and numbers of $(3x + d)$-cycles of a given oddlength

by

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1. Introduction. Let d be a positive odd integer not divisible by 3, and let T_d be the function defined on the set of positive integers, as follows: for all $m \in \mathbb{N}$,

$$T_d(m) = \begin{cases} (3m + d)/2 & \text{if } m \text{ is odd,} \\ m/2, & \text{otherwise.} \end{cases} \tag{1.1}$$

Repeated iterations of the function T_d generate $(3x+d)$- (or T_d-) trajectories

$$\tau_d(m) = \{m, T_d(m), T_d^2(m), \ldots\} \tag{1.2}$$

for all $d \in \mathbf{D} = \{1, 5, 7, 11, 13, \ldots\}$ and $m \in \mathbb{N}$. By definition, a trajectory $\tau_d(m)$ is a *cycle of length* l, $\mathbf{C} = \mathbf{C}(m, d) = \tau_d(m)$, $\mathrm{length}(\mathbf{C}) = l$, if $T_d^l(m) = m$ and, for any $j \in [1, l-1]$, $T_d^j(m) \neq m$ (note that $l > 1$, since the mapping T_d has no fixed points). The minimal member of a T_d-cycle $\mathbf{C}$ is odd, and is called its *perigee*, $n_0 = \mathrm{prg}(\mathbf{C})$. Thus, the number k of odd members of a T_d-cycle, called here its *oddlength*, is a positive integer, $k \geq 1$. The length and oddlength of a cycle are related by the inequality $l \geq \lceil k \log_2 3 \rceil$ [Belaga, Mignotte 1998] (see Theorem 3.2(1) below). Note also that no member of a T_d-trajectory (1.2), excluding possibly the first one, is divisible by 3, and thus, all odd members of a T_d-cycle belong to $\mathbf{D}$.

It has been conjectured that the dynamical system $\mathcal{D}_d = \{\mathbb{N}, T_d\}$ has no divergent T_d-trajectories (1.2), and that the number $\varsigma(d)$ of cyclic T_d-trajectories is finite [Lagarias 1990], [Belaga, Mignotte 1998]. In the particular case $d = 1$, the well-known $3x+1$ conjecture [Lagarias 1985], [Wirsching 1998] is even more specific: any trajectory $\tau_1(m)$ enters ultimately the (only) $3x + 1$ cycle $\{1 \to 2 \to 1\}$.

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The present paper is concerned with the cyclic part of the above $3x+d$ conjecture, and more generally, with quantitative (and when available, numerical) characteristics of the cyclic structure of systems $\mathcal{D}_d$. Let $\mathcal{C}(d)$ and $\mathcal{C}_k(d)$ be the sets of all T_d-cycles and, respectively, of all such cycles with k odd members, or, in our terminology, of oddlength $k \geq 1$.

Technically, our main result is the following general upper bound on the perigee of a T_d-cycle of length l and oddlength k: for all $d \in \mathbf{D}$ and $\mathbf{C} \in \mathcal{C}(d)$,

$$\left.\begin{cases} \text{length}(\mathbf{C}) = l \\ \text{oddlength}(\mathbf{C}) = k \end{cases}\right\} \Rightarrow\ n_0 = \text{prg}(\mathbf{C}) \leq \frac{d}{2^{l/k} - 3}. \tag{1.3}$$

The inequality (1.3) has *four* important implications.

The *first* one is an upper bound on the ratio of the length of a T_d-cycle to its oddlength, which, together with the well-known lower bound (2.5), Theorem 2.1(1), confines this ratio to the interval:

$$\log_2 3 < \varrho_d(\mathbf{C}) = \frac{\text{length}(\mathbf{C})}{\text{oddlength}(\mathbf{C})} \leq \log_2(d+3). \tag{1.4}$$

The upper bound is sharp, and so is, in all probability, the lower bound; but the considerations leading to the corresponding conclusions are quite different in nature.

Consider first the case of the upper bound. For any $r \geq 2$, the T_{2^r-3}-cycle $\mathbf{C}^0_{2^r-3}$ of length $r+1$, starting at (the odd number) 1, has no other odd members:

$$\mathbf{C}^0_{2^r-3} = \{1, 2^{r-1}, 2^{r-2}, \ldots, 2\};$$
$$\text{oddlength}(\mathbf{C}^0_{2^r-3}) = 1; \qquad \text{length}(\mathbf{C}^0_{2^r-3}) = r = \log_2((2^r-3)+3).$$

As to the lower bound, the calculations carried out in [Belaga, Mignotte 2000] (e.g., there exists a T_{233}-cycle starting at 919, of length 13 and oddlength 8, $1.584 < \log_2 3 < 1.585 < 1.625 = 13/8$) show the high plausibility of the following conjecture:

CONJECTURE 1.1. *For any* $\varepsilon > 0$, *there exist a triplet of positive integers*, $d \in \mathbf{D}, (k,l) \in \mathbb{N}^2$, $\log_2 3 < l/k < \log_2 3 + \varepsilon$, *and a* T_d*-cycle of length* l *and oddlength* k.

Cf. also the inequalities (1.12) below.

Second, the inequality (1.3) implies the following general and uniform upper bound on the perigees of T_d-cycles of oddlength $k \geq 1$:

$$n_0 = \text{prg}(\mathbf{C}) \leq \mathbf{U}_{d,k} = \frac{d}{2^{\lceil k \log_2 3 \rceil / k} - 3}. \tag{1.5}$$

The bound (1.5) has an effective polynomial numerical equivalent (see the estimate (1.9) below). It is also sharp in the following natural sense (Theorem 3.2, (3.11)(1)): the average value of an odd member of a T_d-cycle of the oddlength $k \geq 1$ is bigger than $\mathbf{U}_{d,k}$. Thus, for example, the T_5-cycle $\mathbf{C} = \{23 \to 37 \to 58 \to 29 \to 46\}$ has 3 odd members, $n_0 = \operatorname{prg}(\mathbf{C}) = 23 < \mathbf{U}_{5,3} \approx 28.6038 < 29 < 37$.

Third, since no two T_d-cycles have a common member, any such cycle is fully determined by its perigee. Thus, the upper bound (1.5) not only implies that the set $\mathcal{C}_k(d)$ of T_d-cycles of oddlength $k \geq 1$ is finite, but supplies us with an effective general upper bound on the number $\varsigma_k(d) = \#\mathcal{C}_k(d)$ of T_d-cycles of oddlength k:

$$(1.6)\quad \varsigma_k(d) \leq \frac{1}{3}\mathbf{U}_{d,k} = \frac{1}{3} \cdot \frac{d}{2^{\lceil k \log_2 3\rceil/k} - 3} = \frac{1}{9} \cdot \frac{d}{2^{(\lceil k \log_2 3\rceil - k\log_2 3)/k} - 1}$$

(the factor 1/3 is due to the aforementioned inclusion $n_0 \in \mathbf{D}$).

Any numerical evaluation of the expression $\mathbf{U}_{d,k}$ depends on our knowledge of effective *lower bounds* for diophantine approximations of linear combinations of logarithms $\log 2$ and $\log 3$ (cf. the left inequality in (1.4)),

$$(1.7)\qquad \varepsilon_k = \lceil k \log_2 3\rceil - k \log_2 3 = \frac{1}{\log 2} \min_{l > k \log_2 3} (l \log 2 - k \log 3).$$

According to [Baker, Wüstholz 1993], for some effectively calculable constant $C_1 > 0$, we have:

$$(1.8)\qquad \forall k, l \in \mathbb{Z},\ k < l, \quad |l \log 2 - k \log 3| > k^{-C_1}.$$

One easily deduces from (1.8) the existence of an effectively calculable constant $C_2 > 0$ such that for all $d \in \mathbf{D}$ and $k > 2$,

$$(1.9)\qquad \mathbf{U}_{d,k} \leq d k^{C_2}.$$

The original bound [Baker, Wüstholz 1993] on the constant C_1 (and thus, of the closely related C_2) has been enormous. Using less general but more appropriate techniques (linear combination of only two logarithms) of [Laurent *et al.* 1995, Corollary 2], one can easily reduce the value C_2 to a two-digit number, $C_2 < 32$.

Fourth, as is clear from the right side expressions of the upper bounds (1.3), (1.5), (1.6), the values of pairs (k, l) corresponding to potentially "rich" or "numerous" families of d-cycles do not actually depend on d (which enters all three expressions as a linear factor) but only on how close to zero the value $|l \log 2 - k \log 3|$ is.

Thus, any result concerning (non-)existence of d-cycles, for a specific value of d, of oddlength k and length l would probably imply, or at least strongly hint at, the (non-)existence of d'-cycles, for all $d' \in \mathbf{D}$, as well.

Historical remarks. The present author is not aware of any previous effective (and in any sense sharp) upper bound on the minimal odd member of a T_d-cycle. The following general exponential upper bound on the number $\varsigma_k(d)$ of T_d-cycles of oddlength $k \geq 1$ was actually (implicitly) proved in

[Belaga, Mignotte 1998] and refined in [Belaga, Mignotte 2000]: for all $d \in \mathbf{D}$ and $k \in \mathbb{N}$,

$$\varsigma_k(d) < d\left(\frac{3}{2}\right)^k \frac{2^{\varepsilon_k}}{2^{\varepsilon_k} - 1}, \qquad \varepsilon_k = \lceil k \log_2 3 \rceil - k \log_2 3. \tag{1.10}$$

The bound (1.10) was derived from an identical upper bound on the *maximal odd member* of a cycle, the corresponding numerical upper bound being based on the aforementioned estimate of [Baker, Wüstholz 1993]: for all $d \in \mathbf{D}$ and $k \in \mathbb{N}$,

$$\varsigma_k(d) < dk^C \left(\frac{3}{2}\right)^k. \tag{1.11}$$

Comments and future prospects. (1) The upper bound (1.4) on the ratio $\varrho_d(\mathbf{C})$ implies in the $3x+1$ case that the length of a cycle with k odd members does not exceed $2k$. Note that the only $3x+1$ cycle known at present, $\{1 \to 2 \to 1\}$, has one odd member, is of length two, and has ratio two. A slightly more elaborate argument (to be published elsewhere) shows that the length and oddlength of any other $3x+1$ cycle (in case it exists) should satisfy the inequalities:

$$1.584 < \log_2 3 \le \varrho_1(\mathbf{C}) \le 4 - \log_2 5 < 1.679. \tag{1.12}$$

(2) The bounds (1.5) and, especially, (1.6) can be apparently improved. In fact, the experimental discovery of 843 T_{14303}-cycles of oddlength 17, with perigees varying from 385057 to $1391321 < \mathbf{U}_{14303,17} = 2099280$, suggests that the bound (1.5) is apparently sharp up to a one-digit constant.

As to the bound (1.6), our calculations have unearthed 944 different T_{14303}-cycles of oddlengths, respectively, $k = 17$ (843 cycles), 34 (76), 51 (20), 68 (3), 85 (1), 1092 (1), implying the inequality

$$843 \le \varsigma_{17}(14303) < \frac{1}{3}\,\mathbf{U}_{14303,17} = 699760.$$

This estimate, far from being sharp, is at least realistic: for some d, k, the dynamical system $\mathcal{D}_d$ has "many" cycles of oddlength k.

(3) At present, the bounds (1.3), (1.5), (1.6) look useless, or at least insufficient, for a possible proof of the cyclic part of the $3x+d$ conjecture, i. e., of the finiteness of the number $\varsigma(d)$ of T_d-cycles.

However, this obstacle could possibly be circumvented by a refinement of the above scheme, to fit the purpose of yielding directly an *absolute* (i.e., not depending on k) upper bound on the number $\varsigma(d)$.

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stant C_2). His insight was completely justified, and the above derivation of (1.9) with $C_2 < 32$ from Corollary 2 of the paper [Laurent *et al.* 1995] is due to the author's discussions with Maurice Mignotte, one of the co-authors of the above paper.

2. Exponential diophantine formulae for $3x + d$ cycles. Let, as above, $\mathbf{C} = \mathbf{C}(m, d) = \tau_d(m)$ be a T_d-cycle of length l, length$(\mathbf{C}) = l$. We remind the reader that, according to (1.1), the minimal member, or *perigee* of a T_d-cycle, $n_0 = \mathrm{prg}(\mathbf{C})$, is odd, and that the total number $k \geq 1$ of odd members of a cycle is called its *oddlength.* Moreover, if n is an odd member of a cycle, then $n \in \mathbf{D}$ (see (1.2)), since no number divisible by 3 can belong to a cycle.

Note that if $m' \neq m$ is a member of a T_d-cycle $\mathbf{C} = \tau_d(m)$, or in other words, if $\mathbf{C}$ *meets* m', one should view $\mathbf{C}' = \mathbf{C}(m', d)$ as just another name for the same cycle $\mathbf{C} = \mathbf{C}(m, d)$. Since a T_d-cycle is fully characterized by its minimal member, the following notation can be adopted as the canonical one:

$$\mathbf{C} = \mathbf{C}(n_0, d) = \mathbf{C}[n_0, d] = \tau_d(n_0), \qquad n_0 = \mathrm{prg}(\mathbf{C}). \tag{2.1}$$

In this case, we also say that $\mathbf{C}$ *starts at* n_0.

For any positive integer $m \in \mathbb{N}$, let $\mathrm{odd}(m)$ be the number obtained by factoring out m by the highest possible power of 2, say 2^j, and let $\nu_2(m) = j$. Thus $\mathrm{odd}(m)$ is odd and $m = \mathrm{odd}(m) \cdot 2^{\nu_2(m)}$. Define

$$S : \mathbf{D} \times \mathbf{D} \to \mathbf{D}, \qquad (n, d) \mapsto S_d(n) = \mathrm{odd}(3n + d). \tag{2.2}$$

The function S_d speeds up the action of T_d, skipping even members of T_d-trajectories. In particular, $m = 1$ becomes the fixed point of the function $S_1 = \mathrm{odd}(3n + 1)$, $S_1(1) = 1$, corresponding to the (according to the $3x + 1$ conjecture, only) T-cycle $\mathbf{C}(1, 1) = \{1 \to 2 \to 1\}$.

We associate with any T_d-cycle $\mathbf{C} = \mathbf{C}[n_0, d]$ its odd *frame*, $\mathbf{F} = \mathrm{Odd}(\mathbf{C})$, the list of odd members of the cycle, in the order of their appearance in $\tau_d(n_0)$, as the T_d-iterations of n_0 proceed. By definition, the frame is an S_d-cycle starting at n_0, and its length is called the *oddlength* of $\mathbf{C}$:

$$\begin{aligned} l &= \mathrm{length}(\mathbf{C}) = \min\{i \in \mathbb{N} \mid m_i = T_d^i(n_0) = n_0\}; \\ k &= \mathrm{oddlength}(\mathbf{C}) = \min\{j \in \mathbb{N} \mid n_j = S_d^j(n_0) = n_0\}; \\ \mathbf{F} &= \mathrm{Odd}(\mathbf{C}) = \langle n_0, n_1, \ldots, n_{k-1} \rangle \in \mathbf{D}^k. \end{aligned} \tag{2.3}$$

The even members of the T_d-cycle $\mathbf{C} = \mathbf{C}[n_0, d]$ can be recovered from its frame with the help of the cycle *Collatz signature* $\mathbf{P} = \theta(\mathbf{C})$, the vector of exponents of 2 factoring out from the values of the function T_d at odd members of $\mathbf{C}$, as follows:

$$
\begin{aligned}
&\mathbf{F} = \mathrm{Odd}(\mathbf{C}) = \big\langle n_0, n_1, \ldots, n_{k-1} \big\rangle; \\
&\forall j \in [0, k-1], \quad p_{j+1} = \nu_2(T_d(n_j)) + 1 = \nu_2(3n_j + d); \\
&\mathbf{P} = \theta(\mathbf{C}) = \langle p_1, \ldots, p_k \rangle \in \mathbb{N}^k; \\
&l = \mathrm{length}(\mathbf{C}) = |\mathbf{P}| = p_1 + \ldots + p_k; \\
&\forall j \in [1, k-1], \ \begin{cases} m_{p_1+\ldots+p_j} = n_j; \\ p_j > 1 \ \Rightarrow \ \forall i \in [1, p_j - 1], \ m_{p_1+\ldots+p_j-i} = 2^i n_j. \end{cases}
\end{aligned} \tag{2.4}
$$

Moreover, the Collatz signature $\mathbf{P} = \theta(\mathbf{C})$ of a cycle $\mathbf{C} = \mathbf{C}[n_0, d]$, where $n_0, d \in \mathbf{D}$, completely characterizes it:

THEOREM 2.1 [Belaga, Mignotte 1998]. (1) *The Collatz signature* $\mathbf{P} = \theta(\mathbf{C})$ *satisfies the inequality*:

$$
l = |\mathbf{P}| = p_1 + \ldots + p_k \geq \lceil k \log_2 3 \rceil. \tag{2.5}
$$

(2) *Define the exponential diophantine function* $A = a_k : \mathbb{N}^k \to \mathbb{N}$, *as follows*: *for* $\mathbf{P} = \langle p_1, \ldots, p_k \rangle \in \mathbb{N}^k$,

$$
\begin{aligned}
A &= a_k(\mathbf{P}) \\
&= \begin{cases} 1 & \text{if } k = 1; \\ 3^{k-1} + 2^{p_1} \cdot 3^{k-2} + \ldots + 2^{p_1+\ldots+p_{k-2}} \cdot 3 + 2^{p_1+\ldots+p_{k-1}} & \text{otherwise.} \end{cases}
\end{aligned} \tag{2.6}
$$

Let $\sigma = \sigma_k$ *be the circular* (*counterclockwise*) *permutation on* k-*tuples*: *for* $\mathbf{P} = \langle p_1, \ldots, p_k \rangle \in \mathbb{N}^k$,

$$
\sigma(\mathbf{P}) = \sigma_k(\mathbf{P}) = \langle p_2, \ldots, p_k, p_1 \rangle. \tag{2.7}
$$

If now $\mathbf{P} = \theta(\mathbf{C})$ *is the Collatz signature of a cycle* $\mathbf{C} = \mathbf{C}[n_0, d]$, $n_0, d \in \mathbf{D}$, *of length* l, *oddlength* $k \geq 1$, *and with the frame* $\mathbf{F} = \langle n_0, n_1, \ldots, n_{k-1} \rangle$, *then*

$$
\begin{cases}
(1) \quad B = b_k(\mathbf{P}) = B_{k,l} = 2^l - 3^k > 0 & \text{(cf. (2.5));} \\
(2) \quad n_0 = d\,\dfrac{A}{B}, \quad A = a_k(\mathbf{P}) & \text{(cf. (2.6));} \\
(3) \quad \forall j \in [1, k-1], \quad n_j = d\,\dfrac{a_k(\sigma^j(\mathbf{P}))}{B} & \text{(cf. (2.7)).}
\end{cases} \tag{2.8}
$$

3. Upper bound on the number of $3x + d$ cycles of a given oddlength. According to the formulae (2.8)(2), (3), the odd members of a T_d-cycle of oddlength k satisfy the inequality

$$
n_j \leq \mathbf{W}_{d,k} = d \sup_{\substack{\mathbf{P} \in \mathbb{N}^k \\ |\mathbf{P}| \geq k \log_2 3}} \frac{a_k(\mathbf{P})}{2^{|\mathbf{P}|} - 3^k} = d \sup_{l \geq k \log_2 3} \frac{\max_{\mathbf{P} \in \mathbb{N}^k,\, |\mathbf{P}| = l} a_k(\mathbf{P})}{2^l - 3^k} \tag{3.1}
$$

for all $j \in [1, k-1]$. Simple calculations show that (cf. (1.10) above)

$$
\mathbf{W}_{d,k} \leq d \left(\frac{3}{2} \right)^k \frac{2^{\varepsilon_k}}{2^{\varepsilon_k} - 1}, \qquad \varepsilon_k = \lceil k \log_2 3 \rceil - k \log_2 3. \tag{3.2}
$$

We will be able to improve these bounds thanks, first, to a more careful analysis of the formulae (2.8), and then, to a remarkable inequality (3.5) proved below (Theorem 3.1). Namely, instead of evaluating from above *all members* of a T_d-cycle of oddlength k, we evaluate here its *minimal member* $n_0 = \mathrm{prg}(\mathbf{C})$. Since the different T_d-cycles have different perigees $n_0 \in \mathbf{D}$, an upper bound $n_0 \le \mathbf{V}'_{d,k}$ would imply the bound $\varsigma_k(d) \le \frac{1}{3} \cdot \mathbf{V}'_{d,k}$ to the number of T_d-cycles of the oddlength k.

More formally, if $\mathbf{P} = \theta(\mathbf{C})$ (2.4) is the Collatz signature of the cycle $\mathbf{C} = \mathbf{C}[n_0, d]$, $n_0, d \in \mathbf{D}$, of length $l = |\mathbf{P}|$ and oddlength $k \ge 1$, then, according to (2.8),

$$n_0 \le \min\{n_0, n_1, \ldots, n_{k-1}\} = d\,\frac{\min_{j\in[0,k-1]}\{a_k(\sigma^j(\mathbf{P}))\}}{2^l - 3^k}. \tag{3.3}$$

For any k-tuple $\mathbf{P}$ of positive integers define its *average* $\overline{\mathbf{P}}$ to be the arithmetical mean of all its counterclockwise permutations. This k-tuple of positive (generally speaking, rational) numbers depends only on the dimension k and *length* $l = |\mathbf{P}|$ of $\mathbf{P}$:

$$\overline{\mathbf{P}} = \frac{1}{k} \sum_{j\in[0,k-1]} \sigma^j(\mathbf{P}) = \left\{\frac{l}{k}, \ldots, \frac{l}{k}\right\}. \tag{3.4}$$

Extending the definition of the function a_k (see (2.6)) to k-tuples of positive reals, we will prove below (Theorem 3.2) the inequality

$$\widetilde{a}_k(\mathbf{P}) = \min_{j\in[0,k-1]}\{a_k(\sigma^j(\mathbf{P}))\} \le a_k(\overline{\mathbf{P}}) = \frac{2^l - 3^k}{2^{l/k} - 3} \tag{3.5}$$

for $\mathbf{P} \in \mathbb{N}^k$. The inequalities (3.3) and (3.5) imply the general upper bound (1.5), depending only on d and k, for the minimal member $n_0 = \mathrm{prg}(\mathbf{C})$ of any T_d-cycle of oddlength k:

$$\forall n, d \in \mathbf{D}, \quad \mathbf{C} = \mathbf{C}[n,d] \;\Rightarrow\; n \le \mathbf{U}_{d,k} = \frac{d}{2^{l/k} - 3}, \tag{3.6}$$

and, finally, the upper bound (1.6).

Definition 3.1. (1) Let Λ be the set of pairs of positive integers (k,l) satisfying the inequality implied by (2.5),

$$\Lambda = \{(k,l) \in \mathbb{N}^2 \mid \lambda(k,l) = l - \lceil k \log_2 3 \rceil \ge 0\}. \tag{3.7}$$

Extend the definition of the function $A = a_k$ (see (2.6)) to k-tuples of positive reals from the $(k-1)$-dimensional tetrahedron $\mathbf{T}_{k,l}$, $(k,l) \in \Lambda$,

$$\mathbf{T}_{k,l} = \{\mathbf{X} \in \mathbb{R}^k \mid |\mathbf{X}| = x_1 + \ldots + x_k = l \wedge \forall j \in [1,k],\ x_j \ge 1\}, \tag{3.8}$$

with k vertices $\mathbf{V}_1, \ldots, \mathbf{V}_k$,

$$\begin{aligned} \mathbf{V}_1 &= \{l-k+1, 1, \ldots, 1\}, \\ \mathbf{V}_2 &= \{1, l-k+1, \ldots, 1\}, \ \ldots, \\ \mathbf{V}_k &= \{1, 1, \ldots, l-k+1\}. \end{aligned} \tag{3.9}$$

(2) The permutation σ (see (2.7)) induces on $\mathbf{T}_{k,l}$ the rotation σ, with the center $\mathbf{O}$ of the tetrahedron being the only fixed point: for $\mathbf{X} = \langle x_1, x_2, \ldots, x_{k-1}, x_k\rangle \in \mathbf{T}_{k,l}$,

$$\begin{aligned} &\sigma(\mathbf{X}) = \langle x_2, x_3, \ldots, x_k, x_1\rangle; \\ &\overline{\mathbf{X}} = \frac{1}{k}\sum_{j\in[1,k]} \sigma^j(\mathbf{X}) = \left\langle \frac{l}{k}, \ldots, \frac{l}{k}\right\rangle = \mathbf{O}; \\ &\sigma(\mathbf{O}) = \mathbf{O}; \\ &a_k(\mathbf{O}) = \sum_{j=1}^{k} 3^{k-j}2^{l/k} = \frac{2^l - 3^k}{2^{l/k}-3}. \end{aligned} \tag{3.10}$$

THEOREM 3.2. *For any k-tuple $\mathbf{X}$ from $\mathbf{T}_{k,l}$, we have*

$$\begin{cases} (1) \quad \overline{a}_k(\mathbf{X}) = \dfrac{1}{k}\displaystyle\sum_{j=0}^{k-1} a_k(\sigma^j(\mathbf{X})) \geq a_k(\mathbf{O}) = \dfrac{2^l-3^k}{2^{l/k}-3}, \\ (2) \quad \widetilde{a}_k(\mathbf{X}) = \displaystyle\min_{j\in[0,k-1]}\{a_k(\sigma^j(\mathbf{X}))\} \leq a_k(\mathbf{O}) = \dfrac{2^l-3^k}{2^{l/k}-3}, \end{cases} \tag{3.11}$$

with equalities holding only in the case $\mathbf{X} = \mathbf{O}$.

4. Proof of Theorem 3.2. Note that, according to (2.6), if $k = 1$, then $l \geq 2$ and for $\mathbf{X} \in \mathbf{T}_{1,l}$,

$$\overline{\mathbf{X}} = \mathbf{X} = \mathbf{O}, \quad \overline{a}_k(\mathbf{X}) = \widetilde{a}_k(\mathbf{X}) = \frac{2^l - 3}{2^l - 3} = 1. \tag{4.1}$$

Thus, it can be henceforth assumed that $k \geq 2$.

(1) The inequality (3.11)(1) is implied by the standard inequality $\frac{1}{k}(a + b + \ldots) \geq \sqrt[k]{a\cdot b\cdot \ldots}$, as follows: for all $k \geq 2$, $(k,l) \in \Lambda$, and $\mathbf{X} \in \mathbf{T}_{k,l}$,

$$\begin{aligned} \frac{1}{k}\sum_{0\leq j\leq k-1} a_k(\sigma^j(\mathbf{X})) &= 3^{k-1} + \sum_{1\leq j\leq k-1} \frac{3^{k-j-1}}{k}\sum_{0\leq r\leq k-1} 2^{\sigma^r(x_1+\ldots+x_j)} \\ &\geq 3^{k-1} + \sum_{1\leq j\leq k-1} 3^{k-j-1}\cdot 2^{\frac{1}{k}\sum_{0\leq r\leq k-1}\sigma^r(x_1+\ldots+x_j)} \\ &= a_k(\mathbf{O}) = \frac{2^l-3^k}{2^{l/k}-3} \quad \text{(cf. (3.10))}. \end{aligned}$$

(2) If $\mathbf{X} = \mathbf{O} \in \mathbf{T}_{k,l}$, then (3.11)(2) becomes a trivial identity. Otherwise, $\mathbf{O} \neq \mathbf{X} \in \mathbf{T}_{k,l}$ $(k \geq 2,\ (k,l) \in \Lambda)$, and among the k k-tuples $\sigma^j(\mathbf{X}),\ 0 \leq j \leq k-1$, there exist at least two different ones:

$$\forall j \in [0, k-1], \quad \sigma^j(\mathbf{X}) \neq \overline{\mathbf{X}} = \mathbf{O};$$
$$\exists j \in [1, k-1], \quad \mathbf{X} \neq \sigma^j(\mathbf{X}). \tag{4.2}$$

Now the proof proceeds *ad absurdum*: the assumption $a_k(\sigma^j(\mathbf{X})) > a_k(\mathbf{O})$ for all $j \in [0, k-1]$ would imply that $a_k(\overline{\mathbf{X}}) > a_k(\mathbf{O})$ as well—a contradiction, since $\overline{\mathbf{X}} = \mathbf{O}$ (see (3.10), (4.2)).

The equation $a_k(\mathbf{X}) = a_k(\mathbf{O})$ induces a break up of the $(k-1)$-dimensional tetrahedron $\mathbf{T}_{k,l}$ (see (3.8)) into three disjoint subsets: the closed $(k-2)$-dimensional submanifold $\mathbf{T}^0 = \mathbf{T}^0_{k,l}$ defined by this equation, and two $(k-1)$-dimensional submanifolds $\mathbf{T}^+ = \mathbf{T}^+_{k,l}$ and $\mathbf{T}^- = \mathbf{T}^-_{k,l}$, open in $\mathbf{T}_{k,l}$, defined by the inequalities $a_k(\mathbf{X}) > a_k(\mathbf{O})$ and $a_k(\mathbf{X}) < a_k(\mathbf{O})$, respectively:

$$\begin{aligned} \mathbf{T}^0 &= \mathbf{T}^0_{k,l} = \{\mathbf{X} \in \mathbf{T}_{k,l} \mid a_k(\mathbf{X}) = a_k(\mathbf{O}) = (2^l - 3)/(2^{l/k} - 3)\}; \\ \mathbf{T}^+ &= \mathbf{T}^+_{k,l} = \{\mathbf{X} \in \mathbf{T}_{k,l} \mid a_k(\mathbf{X}) > a_k(\mathbf{O})\}; \\ \mathbf{T}^- &= \mathbf{T}^-_{k,l} = \{\mathbf{X} \in \mathbf{T}_{k,l} \mid a_k(\mathbf{X}) < a_k(\mathbf{O})\}. \end{aligned} \tag{4.3}$$

Below we prove the following properties of these three submanifolds:

(A) $\mathbf{T}^0$ is a smooth (in fact, analytical) submanifold.

(B) The submanifolds $\mathbf{T}^0$, $\mathbf{T}^+$, $\mathbf{T}^-$ are connected and simply connected.

(C) The closed set $\mathbf{T}^{0+} = \mathbf{T}^0 \cup \mathbf{T}^+$ is strictly convex: the convex closure $\mathcal{P}(S)$ of a finite set S of k-tuples from $\mathbf{T}^{0+}$ is contained in $\mathbf{T}^+$, excluding, if necessary, the tuples from S belonging to $\mathbf{T}^0$.

The last property immediately implies the validity of the above argument *ad absurdum*.

To prove (A)–(C), one needs to look at the first and second partial derivatives of the function $a_k(\mathbf{X})$: for all $k \geq 2$, $(k,l) \in \Lambda$ and $\mathbf{X} = \langle x_1, \ldots, x_k \rangle \in \mathbf{T}_{k,l}$,

$$\begin{aligned} a_k(\mathbf{X}) &= \left\{ \frac{\partial a_k}{\partial x_1}, \ldots, \frac{\partial a_k}{\partial x_{k-1}}, \frac{\partial a_k}{\partial x_k} \right\}(\mathbf{X}) \\ &\qquad = \ln 2 \cdot \{2^{x_1} a_{k-1}(x_2, \ldots, x_k), \ldots, 2^{x_1 + \ldots + x_{k-1}} a_1(x_k), 0\}; \\ \frac{\partial a_k(\mathbf{X})}{\partial x_i} &= \left(\frac{\partial^2 a_k}{\partial x_i \partial x_j} \right)_{i,j \in [1,k]} (\mathbf{X}); \\ \frac{1}{(\ln 2)^2} &\left(\frac{\partial^2 a_k}{\partial x_i \partial x_j} \right)(\mathbf{X}) = 2^{x_1 + \ldots + x_r} a_{k-r}(x_{r+1}, \ldots, x_k), \\ &\qquad\qquad r = \max(i,j). \end{aligned} \tag{4.4}$$

Properties (A), (B) of the submanifolds $\mathbf{T}^0$, $\mathbf{T}^+$, $\mathbf{T}^-$ follow immediately from the character of the first derivative $a_k(\mathbf{X})$. To prove (C), consider the second differential of the function a_k, the quadratic form

$$\begin{aligned}(4.5)\qquad d^2a_k(\mathbf{X}) &= \sum_{i,j\in[1,k]} \frac{\partial^2 a_k}{\partial x_i \partial x_j}(\mathbf{X})dx_i dx_j \\ &= (\ln 2)^2 \sum_{r\in[1,k]} 2^{x_1+\ldots+x_r} a_{k-r}(x_{r+1},\ldots,x_k)\Big(\sum_{i,j\in[1,r]} dx_i dx_j\Big) \\ &= (\ln 2)^2 \sum_{r\in[1,k]} 2^{x_1+\ldots+x_r} a_{k-r}(x_{r+1},\ldots,x_k)(dx_1+\ldots+dx_r)^2 > 0.\end{aligned}$$

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