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A decomposition of the space of higher order modular cusp forms

by

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1. Introduction. Second order modular forms appear in the work of Goldfeld [G] on the distribution of modular symbols as well as the work of Kleban and Zagier [KZ] calculating crossing probabilities in percolation theory. A systematic theory of second order modular forms was initiated by Chinta, Diamantis, and O'Sullivan [CDO].

Let $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ be an *H-group*, that is, a Fuchsian group of the first kind which contains translations. Γ has the following presentation [Leh1, p. 236]:

$$\Gamma = \langle \gamma_1, \ldots, \gamma_{2g}, \epsilon_1, \ldots, \epsilon_r, \pi_1, \ldots, \pi_s \rangle, \tag{1.1}$$

$$\gamma_1\gamma_{g+1}\gamma_1^{-1}\gamma_{g+1}^{-1}\cdots\gamma_g\gamma_{2g}\gamma_g^{-1}\gamma_{2g}^{-1}\epsilon_1\cdots\epsilon_r\pi_1\cdots\pi_s = I, \qquad \epsilon_j^{l_j} = I. \tag{1.2}$$

Here $\gamma_j, \epsilon_j, \pi_j$ are hyperbolic, elliptic, and parabolic elements, respectively; $l_j \geq 2$ is the order of ϵ_j, g is the genus of the Riemann surface $\Gamma\backslash\mathcal{H}$ (with $\mathcal{H}$ the upper half-plane), and s is the number of inequivalent cusps. Let $\mathcal{F}$ denote a fundamental domain of Γ and $p_1 = \infty, p_2, \ldots, p_s$ a complete set of inequivalent cusps in $\mathcal{F}$. For $1 < j \leq s$, set $A_j = \left(\begin{smallmatrix} 1 & -1-p_j \\ 1 & -p_j \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$ and $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, thus $A_j p_j = \infty$. The stabilizer Γ_{p_j} is cyclic and the parabolic generator π_j can be chosen so that $A_j \pi_j A_j^{-1} = S^{\lambda_j} \doteq \left(\begin{smallmatrix} 1 & \lambda_j \\ 0 & 1 \end{smallmatrix}\right)$ with $\lambda_j \in \mathbb{R}^+$; λ_j is called the *width* of the cusp p_j. In particular $\Gamma_\infty = \langle S^{\lambda_1} \rangle$.

Definition 1.1 (slash operator). Let $\mathcal{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ be the upper half-plane and $F : \mathcal{H} \to \mathbb{C}$. We define $|_k$, the *slash operator*, by

$$(F|_k V)(\tau) = (c\tau + d)^{-k} F(\tau),$$

where $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

A function $F(\tau)$ meromorphic on $\mathcal{H}^* \doteq \mathcal{H} \cup \{\gamma p_j : \gamma \in \Gamma, 1 \leq j \leq s\}$ satisfying

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$$(F|_k\pi_j)(\tau) = F(\tau)$$

has a Fourier expansion at p_j [Leh1]:

$$F(\tau) = \frac{1}{(\tau - p_j)^k} \sum_{n=\mu_j}^{\infty} a_n^j e^{2\pi i n A_j \tau/\lambda_j}, \quad |\mu_j| < \infty. \tag{1.3}$$

$F(\tau)$ is holomorphic at p_j if $\mu_j \geq 0$ and $F(\tau)$ vanishes at p_j if $\mu_j > 1$.

Next we define the various spaces of modular forms which appear in this paper. First, the classical modular forms:

Definition 1.2. A function $F(\tau)$ meromorphic on $\mathcal{H}^*$ is a *modular form of weight* k if

$$(F|_k V)(\tau) = F(\tau), \quad \forall V \in \Gamma.$$

We denote the space of modular forms of weight k by $\{\Gamma, k\}$. A function $F(\tau) \in \{\Gamma, k\}$ which is holomorphic on $\mathcal{H}^*$ is called an *entire modular form of weight* k. The space of entire modular forms of weight k is denoted by $M_k(\Gamma)$. An entire modular form of weight k which vanishes at each cusp is called a *cusp form*. We denote by $S_k(\Gamma)$ the space of cusp forms of weight k.

We are interested in the spaces of (parabolic) higher order modular forms. They are defined iteratively as follows, with $S_k^0(\Gamma) = \{0\}$:

Definition 1.3. For $t \geq 1$, a function $F(\tau)$ holomorphic on $\mathcal{H}$ is a (parabolic) *modular cusp form of weight* k *and order* t if

(1) $F|_k(V - I)(\tau) \in S_k^{t-1}(\Gamma)$ for all $V \in \Gamma$;
(2) $(F|_k\pi_j)(\tau) = F(\tau)$ for $1 \leq j \leq s$;
(3) $F(\tau)$ vanishes at each cusp.

If we replace condition (3) with

(3′) $a_n^j = 0$ for $n < 0$,

then we call $F(\tau)$ a (parabolic) *entire modular form of weight* k *and order* t.

Remark 1.4. The term parabolic is used because of condition (2). We assume this condition throughout the paper. It allows, in particular, Fourier expansions of higher order forms.

We denote the space of (parabolic) modular cusp forms of weight k and order t by $S_k^t(\Gamma)$, and the space of (parabolic) entire modular forms of weight k and order t by $R_k^t(\Gamma)$. We also define the following space of meromorphic second order forms:

Definition 1.5. A *meromorphic second order modular form* $F(\tau)$ is a meromorphic function on $\mathcal{H}^*$ such that

(i) $F|_k(V - I)(\tau) \in \{\Gamma, k\}$,
(ii) $(F|_k\pi_j)(\tau) = F(\tau)$, $1 \leq j \leq s$.

We will denote the space of meromorphic second order modular forms by $\{\Gamma, k\}^{(2)}$.

Remark 1.6. Given $f(\tau)$ meromorphic on $\mathcal{H}^*$ with $(f|_2V)(\tau) = f(\tau)$ such that $f(\tau)$ vanishes at each cusp, the integral

$$\Phi(\tau) = \int_{\tau_0}^{\tau} f(z)\,dz \tag{1.4}$$

is called an *abelian integral*. An abelian integral satisfies

$$\Phi(V\tau) = \Phi(\tau) + c_V(f), \qquad \forall V \in \Gamma.$$

Here the *period function*

$$c_V(f) = \int_{\tau}^{V\tau} f(z)\,dz$$

is the modular symbol which is independent of τ. Eichler [Ei1] generalized abelian integrals to higher weights allowing polynomial periods. From our point of view, an abelian integral is a weight zero second order modular form. The periods are constants, that is, weight zero modular forms. Thus second order modular forms are a natural generalization of abelian integrals to weight k with modular periods.

Remark 1.7. Since $S_k^1(\Gamma) = S_k(\Gamma)$ and $R_k^1(\Gamma) = M_k(\Gamma)$, we suppress the 1 for first order (classical) modular forms.

For each $t \geq 1$, the group Γ acts on $S_k^t(\Gamma)$ by means of the slash operator; therefore, by general theory [EM], we can assign to this action the cohomology groups $H^n(\Gamma, S_k^t(\Gamma))$. In this paper we only need $H^1(\Gamma, S_k^t(\Gamma))$ which is defined as

$$H^1(\Gamma, S_k^t(\Gamma)) = Z^1(\Gamma, S_k^t(\Gamma))/B^1(\Gamma, S_k^t(\Gamma)), \qquad k > 2,$$

where

$$Z^1(\Gamma, S_k^t(\Gamma)) = \{\Omega : \Gamma \to S_k^t(\Gamma) : \Omega(VW)(\tau) = (\Omega(V)|_kW)(\tau) + \Omega(W)(\tau)\},$$
$$B^1(\Gamma, S_k^t(\Gamma)) = \{\psi \in Z^1(\Gamma, S_k^t(\Gamma)) : \psi(V)(\tau) = F|_k(V-I)(\tau) \text{ for some } F(\tau) \in S_k^t(\Gamma)\}.$$

For $k = 2$, we modify the space of 1-cocycles. Given $\Omega \in Z^1(\Gamma, S_k^t(\Gamma))$, we can construct $F(\tau) \in R_2^{t+1}(\Gamma)$ (Theorem 4.1) such that $F|_2(V-I)(\tau) = \Omega_V(\tau)$ for all $V \in \Gamma$. We denote this F by $\mathrm{Eic}(\Omega)$. Then

$$Z_*^1(\Gamma, S_2^t(\Gamma)) = \{\Omega \in Z^1(\Gamma, S_2^t(\Gamma)) : \mathrm{Eic}(\Omega)(\tau) \text{ has zero residue sum}\}. \tag{1.5}$$

This gives the modified cohomology space

$$H_*^1(\Gamma, S_2^t(\Gamma)) = Z_*^1(\Gamma, S_2^t(\Gamma))/B^1(\Gamma, S_2^t(\Gamma)).$$

REMARK 1.8. We use the convention

$$H^1(\mathit{\Gamma}, S_k^0(\mathit{\Gamma})) = S_k(\mathit{\Gamma}) \quad \text{and} \quad H_*^1(\mathit{\Gamma}, S_2^0(\mathit{\Gamma})) = S_2(\mathit{\Gamma}).$$

We may now state the main result of the paper:

THEOREM 1.9. *Let $\mathit{\Gamma}$ be an H-group. Then for $k > 2$ and $t \geq 1$,*

$$S_k^{t+1}(\mathit{\Gamma}) \cong H^1(\mathit{\Gamma}, S_k^t(\mathit{\Gamma})) \oplus H^1(\mathit{\Gamma}, S_k^{t-1}(\mathit{\Gamma})) \oplus \cdots \oplus H^1(\mathit{\Gamma}, S_k^0(\mathit{\Gamma})).$$

For $k = 2$ and $t \geq 1$,

$$S_2^{t+1}(\mathit{\Gamma}) \cong H_*^1(\mathit{\Gamma}, S_2^t(\mathit{\Gamma})) \oplus H_*^1(\mathit{\Gamma}, S_2^{t-1}(\mathit{\Gamma})) \oplus \cdots \oplus H_*^1(\mathit{\Gamma}, S_2^0(\mathit{\Gamma})).$$

COROLLARY 1.10. *For $\mathit{\Gamma}$ an H-group, $k > 2$, and $t \geq 1$,*

$$\dim S_k^{t+1}(\mathit{\Gamma}) = \sum_{\nu=0}^{t} \left(\sum_{j=0}^{[\nu/2]} (-1)^j \binom{\nu - j}{j} (2g)^{\nu-2j} \right) \dim S_k(\mathit{\Gamma}).$$

2. Second order modular forms. In this section we prove $S_k^2(\mathit{\Gamma}) \cong H^1(\mathit{\Gamma}, S_k(\mathit{\Gamma})) \oplus S_k(\mathit{\Gamma})$, $k \geq 2$. This case is straightforward and instructive. Let

$$\begin{aligned} \text{(2.1)} \qquad & \mathrm{Hom}_{\mathrm{par}}(\mathit{\Gamma}, S_k(\mathit{\Gamma})) \\ & = \{\varphi : \mathit{\Gamma} \to S_k(\mathit{\Gamma}) : \varphi(VW)(\tau) = \varphi(V)(\tau) + \varphi(W)(\tau) \text{ and} \\ & \qquad \qquad \varphi(\pi)(\tau) = 0,\ \forall \pi \text{ parabolic}\}. \end{aligned}$$

Since $\mathit{\Gamma}$ acts simply on $S_k(\mathit{\Gamma})$, we have

$$B^1(\mathit{\Gamma}, S_k(\mathit{\Gamma})) = \{0\},$$
$$Z^1(\mathit{\Gamma}, S_k(\mathit{\Gamma})) = \{\mathit{\Omega} : \mathit{\Gamma} \to S_k(\mathit{\Gamma}) : \mathit{\Omega}(VW)(\tau) = \mathit{\Omega}(V)(\tau) + \mathit{\Omega}(W)(\tau)\}.$$

Thus

$$H^1(\mathit{\Gamma}, S_k(\mathit{\Gamma})) \cong \mathrm{Hom}_{\mathrm{par}}(\mathit{\Gamma}, S_k(\mathit{\Gamma})). \tag{2.2}$$

Let $\{F_1(\tau), \ldots, F_{\chi_1(k)}(\tau)\}$ be a basis for $S_k(\mathit{\Gamma})$. Then a basis $\{\omega_{il} : 1 \leq i \leq 2g,\ 1 \leq l \leq \chi_1(k)\}$ for $\mathrm{Hom}_{\mathrm{par}}(\mathit{\Gamma}, S_k(\mathit{\Gamma}))$ is determined as follows:

$$\begin{aligned} \omega_{il}(\gamma_{j_h})(\tau) &= \delta_{ij_h} F_l(\tau), && 1 \leq j_h \leq 2g; \\ \omega_{il}(\pi_{j_p})(\tau) &= 0, && 1 \leq j_p \leq s; \\ \omega_{il}(\epsilon_{j_e})(\tau) &= 0, && 1 \leq j_e \leq r. \end{aligned} \tag{2.3}$$

Thus for $V \in \mathit{\Gamma}$, we have $\omega_{il}(V)(\tau) = n_i(V) F_l(\tau)$ where $n_i(V)$ is the sum of the powers of γ_i that appear when V is expressed as a word in the generators $\{\gamma_{j_h}, \pi_{j_p}, \epsilon_{j_e} : 1 \leq j_h \leq 2g,\ 1 \leq j_p \leq s,\ 1 \leq j_e \leq r\}$; $n_i(V)$ does not depend on the factorization.

Following Eichler [Ei2], we define, for ν any even integer with $k+\nu > 2$, a weighted Eisenstein series

$$\Psi_{il}(\tau) = -\sum_{W\in(\Gamma_\infty\backslash\Gamma)} \frac{\omega_{il}(W)(\tau)}{(c\tau+d)^{k+\nu}}, \quad W = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \tag{2.4}$$
$$= -F_l(\tau)\sum_{W\in(\Gamma_\infty\backslash\Gamma)} \frac{n_i(W)}{(c\tau+d)^{k+\nu}}.$$

Here $(\Gamma_\infty\backslash\Gamma)$ denotes an arbitrary system of coset representatives of Γ with respect to Γ_∞. The sum is independent of the choice of the coset representatives since $n_i(\pi) = 0$ for all π parabolic. For $W = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$ set $\mu(W) = a^2+b^2+c^2+d^2$. Let $\mathcal{M}$ denote the choice of coset representatives such that, for some $C > 0$, $\mu(W) \le C(c^2+d^2)$ for all $W \in \Gamma$. That such an $\mathcal{M}$ exists is shown in [Leh2]. Let $\beta = \{\gamma_1,\dots,\gamma_{2g},\epsilon_1,\dots,\epsilon_r\}$ be the set of nonparabolic generators of Γ. We use the following result due to Eichler [Ei2]:

THEOREM 2.1. *$W \in \Gamma$ has a representation $W = C_1\cdots C_l$, where $C_j \in \beta$ or $C_j = \pi_{i_j}^{n_j}$ for some $1 \le i_j \le s$ and $n_j \in \mathbb{Z}$. Furthermore*

$$l \le m_1 \log\mu(W) + m_2, \tag{2.5}$$

with $m_1, m_2 > 0$ and independent of W.

PROPOSITION 2.2. *$\sum_{W\in\mathcal{M}} n_i(W)/(c\tau+d)^\sigma$ converges absolutely and uniformly on compact subsets of $\mathcal{H}$ for $\sigma > 2$.*

Proof. Let $W = C_1\cdots C_l$ be the Eichler representation of W. Then

$$n_i(W) \le l \le m_1\log\mu(W) + m_2 \le m_1\log(c^2+d^2) + m_2',$$

by our choice of $\mathcal{M}$. Thus

$$n_i(W) \le m_1\log|c\tau+d|^2 + m_2''.$$

The proposition follows from this estimate. ∎

In the above proposition we used the fact that $c^2+d^2 \le C|c\tau+d|^2$ uniformly on compact subsets of $\mathcal{H}$; this is a consequence of the following lemma proved in [K]:

LEMMA 2.3. *For real numbers c, d and $\tau = x+iy$, we have*

$$\frac{y^2}{1+4|\tau|^2}(c^2+d^2) \le |c\tau+d|^2 \le 2(|\tau|^2+y^{-2})(c^2+d^2). \tag{2.6}$$

Now $\Psi_{il}(\tau)$ satisfies, for ν an even integer and for all $V \in \Gamma$, $V = \left(\begin{smallmatrix} * & * \\ c_V & d_V \end{smallmatrix}\right)$,

$$\begin{aligned}
(\Psi_{il}|_{2k+\nu}V)(\tau) &= (c_V\tau + d_V)^{-2k-\nu}\Psi_{il}(V\tau) \\
&= -(c_V\tau + d_V)^{-k}F_l(V\tau)\sum_{W\in\mathcal{M}}\frac{n_i(W)}{(c_V\tau + d_V)^{k+\nu}(cV\tau + d)^{k+\nu}} \\
&= -F_l(\tau)\sum_{W\in\mathcal{M}}\frac{n_i(W)}{(c_{WV}\tau + d_{WV})^{k+\nu}} \\
&= -F_l(\tau)\sum_{W\in\mathcal{M}}\frac{n_i(WV)}{(c_{WV}\tau + d_{WV})^{k+\nu}} + F_l(\tau)n_i(V)\sum_{W\in\mathcal{M}}\frac{1}{(c_{WV}\tau + d_{WV})^{k+\nu}} \\
&= \Psi_{il}(\tau) + \omega_{il}(V)(\tau)E_{k+\nu}(\tau).
\end{aligned}$$

Here $E_{k+\nu}(\tau) = \sum_{W\in\mathcal{M}} 1/(c\tau + d)^{k+\nu}$ is the weight $k+\nu$ Eisenstein series associated to the cusp ∞. Set

$$\Phi_{il}(\tau) = \frac{\Psi_{il}(\tau)}{E_{k+\nu}(\tau)}. \tag{2.7}$$

Then

$$(\Phi_{il}|_kV)(\tau) = (c_V\tau + d_V)^{-k}\frac{\Psi_{il}(V\tau)}{E_{k+\nu}(V\tau)} = \frac{(c_V\tau + d_V)^{-2k-\nu}\Psi_{il}(V\tau)}{(c_V\tau + d_V)^{-k-\nu}E_{k+\nu}(V\tau)}.$$

Therefore

$$(\Phi_{il}|_kV)(\tau) = \Phi_{il}(\tau) + \omega_{il}(V)(\tau). \tag{2.8}$$

We will take $\nu = 0$ for $k > 2$ and $\nu = 2$ for $k = 2$.

From (2.8), we see that $\Phi_{il}(\tau)$ is a second order modular form ($\in\{\Gamma, k\}^{(2)}$) with period forms $\{\omega_{il}(V)(\tau)\}$. We are interested in holomorphic second order modular cusp forms as defined in the introduction. Therefore we must eliminate the poles introduced by the zeros of the Eisenstein series $E_{k+\nu}(\tau)$ in (2.7). Since adding a classical modular form to $\Phi_{il}(\tau)$ does not affect the period in (2.8), we apply the 'Mittag-Leffler' theorem for weight $k \geq 2$ classical forms [K, p. 622], which gives the existence of a $G_{il}(\tau) \in \{\Gamma, k\}$ with principal parts identical to the principal parts of $\Phi_{il}(\tau)$. Then

$$F'_{il}(\tau) = \Phi_{il}(\tau) - G_{il}(\tau) \in R^2_k(\Gamma), \quad k \geq 2, \tag{2.9}$$

that is, it is an entire second order modular form.

If $k \geq 4$ we can further add a linear combination of weight k Eisenstein series to $F'_{il}(\tau)$ in order to obtain a cuspidal form:

$$F_{il}(\tau) = F'_{il}(\tau) - \sum_{j=1}^{s} a^j_0(F'_{il})E_{k,j}(\tau) \in S^2_k(\Gamma), \tag{2.10}$$

where $E_{k,j}(\tau)$ is the standard Eisenstein series attached to the cusp p_j. In this way we obtain a second order cusp form $F_{il}(\tau)$ with the prescribed period forms $\{\omega_{il}(V)(\tau)\}$.

THEOREM 2.4. *Let Γ be an H-group. The set*

$$\{F_{il}(\tau), F_j(\tau)\}_{1\le i\le 2g,\, 1\le l\le \chi_1(k),\, 1\le j\le \chi_1(k)},$$

with $F_{il}(\tau)$ given by (2.10), *is a basis for $S_k^2(\Gamma)$, $k>2$.*

Proof. Let $k>2$. If $F(\tau)\in S_k^2(\Gamma)$, then $(V\mapsto F_V(\tau))\in \mathrm{Hom}(\Gamma, S_k(\Gamma))$. Here $F_V(\tau)=F|_k(V-I)(\tau)$. Therefore

$$F_V(\tau)=\sum_{\substack{1\le i\le 2g\\ 1\le l\le \chi_1(k)}} a_{il}\omega_{il}(V)(\tau)=\sum_{\substack{1\le i\le 2g\\ 1\le l\le \chi_1(k)}} a_{il}n_i(V)F_l(\tau)$$

for some $a_{il}\in\mathbb{C}$. Let $G(\tau)=\sum_{1\le i\le 2g,\,1\le l\le\chi_1(k)} a_{il}F_{il}(\tau)$. Then

$$(G|_kV)(\tau)=G(\tau)+\sum_{\substack{1\le i\le 2g\\ 1\le l\le \chi_1(k)}} a_{il}n_i(V)F_l(\tau)=G(\tau)+F|_k(V-I)(\tau).$$

This implies $F(\tau)-G(\tau)\in S_k(\Gamma)$ and

$$F(\tau)=\sum_{\substack{1\le i\le 2g\\ 1\le l\le \chi_1(k)}} a_{il}F_{il}(\tau)+\sum_{1\le j\le \chi_1(k)} b_jF_j(\tau).$$

This shows that $\mathrm{span}\{F_{ij}(\tau), F_j(\tau)\}=S_k^2(\Gamma)$. Suppose

$$\sum_{\substack{1\le i\le 2g\\ 1\le l\le \chi_1(k)}} a_{il}F_{il}(\tau)=\sum_{1\le j\le \chi_1(k)} b_jF_j(\tau).$$

Apply $|_kV$ to both sides to obtain, for each l,

$$\sum_i a_{il}n_i(V)=0.$$

Letting $V=\gamma_i$ gives $a_{il}=0$, and linear independence of the $F_l(\tau)$ gives $b_l=0$. ■

REMARK 2.5. The map $F_{il}\mapsto w_{il}$ and $F_j\mapsto F_j$ and (2.2) give $S_k^2(\Gamma)\cong H^1(\Gamma, S_k(\Gamma))\oplus S_k(\Gamma)$ for $k\ge 2$.

COROLLARY 2.6. *Let Γ be an H-group and $k>2$. Then*

$$\dim S_k^2(\Gamma)=(2g+1)\dim S_k(\Gamma).$$

REMARK 2.7. Weight 2 Eisenstein series have residue sum zero. Thus we can extend the notion of residue sum zero to higher order forms. As $F|_k(V-I)(\tau)=0$ for V parabolic (condition (2) in Definition 1.3) and for V elliptic (Proposition 6.2), the expansion and order of $F(\tau)$ at a point $\tau\in\mathcal{H}^*$ are identical to those of a classical weight k form. Although for weight 2, $F(\tau)$ can no longer be identified with a differential on the Riemann surface $\Gamma\backslash\mathcal{H}^*$, we define the residue, $\mathrm{res}(F,p_j,\Gamma)$, of $F(\tau)$ at p_j by $\mathrm{res}(F,p_j,\Gamma)=\lambda_j a_0^j$. This agrees with the residue of weight 2 forms (see [Ran, p. 122]). If

$F(\tau) \in R_2^t(\Gamma)$, the residue sum of $F(\tau)$ is $\sum_{j=1}^s \operatorname{res}(F, p_j, \Gamma) = \sum_{j=1}^s \lambda_j a_0^j$. For clarity, we will write $a_0^j(F)$ for the zeroth coefficient in the Fourier expansion of $F(\tau)$ at the cusp p_j.

We consider the map

$$\operatorname{Res} : R_2^2(\Gamma) \to \mathbb{C}, \qquad F \mapsto \sum_{j=1}^s \lambda_j a_0^j(F).$$

Remark 2.8. The map $\operatorname{Res} : R_2^2(\Gamma) \to \mathbb{C}$ is nontrivial. For example, given $f \in S_2(\Gamma)$ and a cusp $\mathfrak{a}$, the function, denoted in [DO] by $Z_{f,f}(\tau) + 2iV\,\overline{\langle f, f\rangle} P_{\mathfrak{a}0}(\tau)_2$ is shown, in the proof of Proposition 5.2 of [DO], to be in $R_2^2(\Gamma)$ with nonzero residue sum.

Let $c_{il} =$ residue sum$(F'_{il}(\tau)) = \sum_{j=1}^s \lambda_j a_0^j(F'_{il})$. At least one c_{il} is not 0 (see the remark above); without loss of generality, we assume $c_{11} \neq 0$. Let $F_{il}^{*\prime}(\tau) = c_{11}F'_{il}(\tau) - c_{il}F'_{11}(\tau)$, $(i,l) \neq (1,1)$. The $F_{il}^{*\prime}(\tau)$ have residue sum zero. Thus

$$F_{il}(\tau) = F_{il}^{*\prime}(\tau) - \sum_{j=2}^s \lambda_j a_0^j(F_{il}^{*\prime}) E_j^*(\tau) \in S_2^2(\Gamma), \quad (i,l) \neq (1,1), \tag{2.11}$$

where $E_j^*(\tau) \in M_2(\Gamma)$, $2 \le j \le s$, is the standard weight two Eisenstein series with residue -1 at p_j, residue 1 at p_1, and vanishing at the remaining cusps.

Remark 2.9. If there is only one cusp, $s = 1$, then the sum appearing in (2.11) is vacuous. Yet the result is true since for one cusp, residue sum zero is equivalent to cuspidal.

Theorem 2.10. *Let Γ be an H-group. Then*

$$\{F_{il}(\tau), F_j(\tau)\}_{1\le i\le 2g,\, 1\le l\le \chi_1(2),\, 1\le j\le \chi_1(2),\, (i,l)\neq(1,1)},$$

where $F_{il}(\tau)$ given by (2.11), *is a basis for $S_2^2(\Gamma)$ $(\chi_1(2) = g)$.*

Proof. Let $F(\tau) \in S_2^2(\Gamma)$ with $F_V(\tau) = \sum_{1\le i\le 2g,\, 1\le l\le \chi_1(k)} a_{il}\omega_{il}(V)(\tau)$. The Eichler construction above gives $F'(\tau) = \sum_{1\le i\le 2g,\, 1\le l\le \chi_1(k)} a_{il}F'_{il}(\tau) \in R_2^2(\Gamma)$ with period forms $\{F_V(\tau)\}$. Thus $F(\tau) - F'(\tau) \in M_2(\Gamma)$. We recall $M_2(\Gamma) = S_2(\Gamma) \oplus E$, where $E = \operatorname{span}\{E_j^*(\tau)\}_{j=2}^s$ (see [S]); in particular elements of $M_2(\Gamma)$ have residue sum zero. We have

$$\begin{aligned} F(\tau) - F'(\tau) &= F(\tau) - \sum_{\substack{1\le i\le 2g\\ 1\le l\le \chi_1(k)}} a_{il}F'_{il}(\tau) \\ &= F(\tau) - \sum_{\substack{1\le i\le 2g\\ 1\le l\le \chi_1(k)}} \frac{a_{il}}{c_{11}}(c_{11}F'_{il}(\tau) - c_{il}F'_{11}(\tau)) + \sum_{\substack{1\le i\le 2g\\ 1\le l\le \chi_1(k)}} \frac{a_{il}c_{il}}{c_{11}}F'_{11}(\tau). \end{aligned}$$

Recall $F_{il}^{*\prime}(\tau) = c_{11}F_{il}'(\tau) - c_{il}F_{11}'(\tau)$, so that

$$F(\tau) - F'(\tau) = F(\tau) - \sum_{\substack{1\le i\le 2g\\ 1\le l\le \chi_1(k)\\ (i,l)\ne(1,1)}} \frac{a_{il}}{c_{11}} F_{il}^{*\prime}(\tau) - \Bigg(\sum_{\substack{1\le i\le 2g\\ 1\le l\le \chi_1(k)}} \frac{a_{il}c_{il}}{c_{11}}\Bigg) F_{11}'(\tau).$$

Now as $F(\tau) - F'(\tau)$ has residue sum zero, it follows that

$$\sum_{\substack{1\le i\le 2g\\ 1\le l\le \chi_1(k)}} a_{il}c_{il} = 0.$$

Therefore

$$\begin{aligned} F(\tau) - F'(\tau) &= F(\tau) - \sum_{\substack{1\le i\le 2g\\ 1\le l\le \chi_1(2)\\ (i,l)\ne(1,1)}} \frac{a_{il}}{c_{11}} F_{il}^{*\prime}(\tau) \\ &= F(\tau) - \sum_{\substack{1\le i\le 2g\\ 1\le l\le \chi_1(2)\\ (i,l)\ne(1,1)}} \frac{a_{il}}{c_{11}} F_{il}(\tau) - \sum_{\substack{1\le i\le 2g\\ 1\le l\le \chi_1(2)\\ (i,l)\ne(1,1)}} \frac{a_{il}}{c_{11}} \sum_{j=2}^{s} \lambda_j a_0^j(F_{il}^*) E_j^*(\tau). \end{aligned}$$

We have $F(\tau) - F'(\tau), E_j^*(\tau) \in S_2(\Gamma) \oplus E$, by assumption $F(\tau) \in S_2^2(\Gamma)$ and by construction $F_{il}(\tau) \in S_2^2(\Gamma)$; therefore

$$F(\tau) - \sum_{\substack{1\le i\le 2g\\ 1\le l\le \chi_1(2)\\ (i,l)\ne(1,1)}} \frac{a_{il}}{c_{11}} F_{il}(\tau) \in S_2^2(\Gamma) \cap (S_2(\Gamma) \oplus E) = S_2(\Gamma).$$

This gives

$$\operatorname{span}\{F_{il}(\tau), F_j(\tau)\}_{1\le i\le 2g,\, 1\le l\le \chi_1(2),\, 1\le j\le \chi_1(2),\, (i,l)\ne(1,1)} = S_2^2(\Gamma).$$

The proof of linear independence is the same as for $k > 2$. ■

COROLLARY 2.11. *Let Γ be an H-group. Then*

$$\dim S_2^2(\Gamma) = (2g+1)\dim S_2(\Gamma) - 1.$$

3. Preliminary lemmas. In this section, we prove a series of lemmas which will be used, in the next section, to construct the Eichler map, Eic. For large Y, let $\mathcal{F}^Y = \{\tau \in \overline{\mathcal{F}} : y > Y\}$; it is a neighborhood of ∞ and $\mathcal{F}_j^Y = A_j^{-1}\mathcal{F}^Y$ is the corresponding neighborhood of the cusp p_j. Then $\overline{\mathcal{F}} = \bigcup_{j=1}^{s} A_j^{-1}\mathcal{F}^Y \cup \mathcal{F}_c^Y$, where $\mathcal{F}_c^Y$ is compact. For $\alpha > 0$, let $E_\alpha = \{\tau \in \mathcal{H} : |x| \le 1/\alpha,\ y \ge \alpha\}$. Let $E_\alpha^Y = \{\tau \in E_\alpha : y > Y\}$ and $E_{\alpha,Y} = \{\tau \in E_\alpha : y \le Y\}$. We use the following lemma proved in [Shi].

Lemma 3.1. *There exists a positive number r, depending only on Γ, such that $|c| \geq r$ for all $V \in \Gamma - \Gamma_\infty$, where $V = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$. Moreover, for such an r, one has* $\operatorname{Im}(\tau)\operatorname{Im}(V\tau) \leq 1/r^2$ *for all $\tau \in \mathcal{H}$ and all $V \in \Gamma - \Gamma_\infty$.*

Lemma 3.2. *There exists Y_0 such that for all $Y > Y_0$,*

$$\mathcal{F}_j^Y \cap V(E_{\alpha,Y}) = \emptyset \quad \textit{for all } V \in \Gamma,\ j = 1, \ldots, s.$$

Proof. Choose $Y > 1/(\alpha r^2)$, with r as in Lemma 3.1. Let $\tau \in \mathcal{F}_j^Y$. Consider $\operatorname{Im}(A_j V\tau) = \operatorname{Im}(A_j V A_j^{-1} A_j \tau)$. By Lemma 3.1, if $A_j V A_j^{-1} \in \Gamma - \Gamma_\infty$, then

$$\operatorname{Im}(A_j V A_j^{-1} A_j \tau) \leq \frac{1}{\operatorname{Im}(A_j\tau) r^2} \leq \frac{1}{Y r^2} < \alpha.$$

If $A_j V A_j^{-1} \in \Gamma_\infty$, then $\operatorname{Im}(A_j V A_j^{-1} A_j\tau) = \operatorname{Im}(A_j\tau) > Y$. The lemma follows with $Y_0 = 1/(\alpha r^2)$. ∎

Lemma 3.3. *There exist a finite number of elements $W_1, \ldots, W_m \in \Gamma$ such that*

$$E_\alpha \subset \bigcup_{j=1}^m W_j \overline{\mathcal{F}}.$$

Proof. Choose $Y > Y_0$ as above. We have $\overline{\mathcal{F}} = \bigcup_{j=1}^s \mathcal{F}_j^Y \cup \mathcal{F}_c^Y$ and $E_\alpha = E_\alpha^Y \cup E_{\alpha,Y}$, where $\mathcal{F}_c^Y$ and E_α^Y are compact. We know $E_{\alpha,Y} \subset \bigcup_{V \in \Gamma} V\mathcal{F}$ and, by Lemma 3.2, $(VE_{\alpha,Y}) \cap \mathcal{F}_j^Y = \emptyset$ for all $V \in \Gamma$ and $j = 1, \ldots, s$; therefore

$$E_{\alpha,Y} \subset \bigcup_{V \in \Gamma} V\mathcal{F}_c^Y.$$

Since $\mathcal{F}_c^Y$ and E_α^Y are compact, $E_{\alpha,Y} \cup V\mathcal{F}_c^Y \neq \emptyset$ for only finitely many V's [Shi]. Label these $W_1, \ldots, W_n$. Also, $E_\alpha^Y \subset \bigcup_{t=-l}^{l} S^t \mathcal{F}^Y$ for some l. Label the S^i's $W_{n+1}, \ldots, W_m$. Thus $E_\alpha \subset W_1\overline{\mathcal{F}} \cup \cdots \cup W_m\overline{\mathcal{F}}$. ∎

Lemma 3.4. *Let $F(\tau) \in S_k^t(\Gamma)$. Then there exists $C = C(F,\alpha)$ such that*

$$|F(\tau)| \leq C, \quad \forall \tau \in E_\alpha.$$

Proof. By Lemma 3.3, there exist elements $W_1, \ldots, W_m \in \Gamma$ such that $E_\alpha \subset \bigcup_{j=1}^m W_j\overline{\mathcal{F}}$. Let $g(\tau) = |\operatorname{Im}(\tau)^{k/2} F(\tau)|$. Hence for $\tau \in E_\alpha$ there exist $w \in \overline{\mathcal{F}}$ and $j \in \{1, \ldots, m\}$ so that

$$\begin{aligned} g(\tau) = g(W_j w) &= \operatorname{Im}(w)^{k/2} |(c_j w + d_j)^{-k} F(W_j w)| \\ &= \operatorname{Im}(w)^{k/2} |F(w) + F_{W_j}(w)|. \end{aligned} \tag{3.1}$$

Thus $g(\tau) \leq M(F, \{W_1, \ldots, W_m\})$, where

$$M(F, \{W_1, \ldots, W_m\}) = \sup_{w \in \overline{\mathcal{F}},\, j=1,\ldots,m} \operatorname{Im}(w)^{k/2}(|F(w)| + |F_{W_j}(w)|).$$

Therefore $|F(\tau)| \leq M \operatorname{Im}(\tau)^{-k/2} \leq C(F,\alpha)$ for $\tau \in E_\alpha$. ∎

REMARK 3.5. Lemma 3.4 shows that, as in the case of classical modular cusp forms, $F(\tau) \in S_k^t(\Gamma)$ satisfies $\sup_{\tau \in \overline{\mathcal{F}}} \mathrm{Im}(\tau)^{k/2} |F(\tau)| \le C$. This follows from the exponential decay at the cusps, i.e. condition (3), and compactness away from the cusps.

Recall that

$$Z^1(\Gamma, S_k^t(\Gamma)) = \{\Omega : \Gamma \to S_k^t(\Gamma) : \Omega(VW)(\tau) = (\Omega(V)|_k W)(\tau) + \Omega(W)(\tau)\}.$$

Let $\Omega \in Z^1(\Gamma, S_k^t(\Gamma))$ and set $\Omega_V(\tau) = \Omega(V)(\tau)$. Thus $\Omega_V(\tau) \in S_k^t(\Gamma)$ for all $V \in \Gamma$ and Ω satisfies the 1-cocycle condition:

$$\Omega_{VW}(\tau) = (\Omega_V|_k W)(\tau) + \Omega_W(\tau). \tag{3.2}$$

The following statement and proof are modifications of those of Theorem 1 in [Leh2].

THEOREM 3.6. *Let $\Omega \in Z^1(\Gamma, S_k^t(\Gamma))$. For $W \in \Gamma$ and $\tau \in E_\alpha$, we have*

$$|\Omega_W(\tau)| \le C(m_1 \log \mu(W) + m_2)\mu(W)^{m_1 \log 2}. \tag{3.3}$$

Here m_1 and m_2 are as in Theorem 2.1.

Proof. Let $W = C_1 \cdots C_l$ be the Eichler factorization of W. Using (3.2) and the multiplicativity of the slash operator, $((F|_k V)|_k W)(\tau) = (F|_k VW)(\tau)$, we have

$$\begin{aligned} &\Omega_W(\tau) = (\Omega_{C_1 \cdots C_{l-1}}|_k C_l)(\tau) + \Omega_{C_l}(\tau) \\ &= (\Omega_{C_1}|_k C_2 \cdots C_l)(\tau) + (\Omega_{C_2}|_k C_3 \cdots C_l)(\tau) + \cdots + (\Omega_{C_{l-1}}|_k C_l)(\tau) + \Omega_{C_l}(\tau). \end{aligned} \tag{3.4}$$

For $\tau \in E_\alpha$ we estimate each of the above l terms. By assumption $\Omega_{C_j}(\tau) = 0$ if C_j is parabolic, thus we need only estimate the terms involving $\Omega_{C_j}(\tau)$ with C_j nonparabolic, i.e. $C_j \in \beta \doteq \{\gamma_1, \ldots, \gamma_{2g}, \epsilon_1, \ldots, \epsilon_r\}$, the set of nonparabolic generators of Γ. Since $\Omega \in Z^1(\Gamma, S_k^t(\Gamma))$, we have $\Omega_{C_j}(\tau) \in S_k^t(\Gamma)$. Thus we may write

$$\Omega_{C_j}(\tau) = \sum_{i=1}^{\chi_t(k)} b_{ij} F_i(\tau), \tag{3.5}$$

where $\{F_i(\tau)\}_{1 \le i \le \chi_t(k)}$ is a basis for $S_k^t(\Gamma)$. Now we wish to estimate $(\Omega_{C_j}|_k C_{j+1} \cdots C_l)(\tau)$; we write $C_j = \left(\begin{smallmatrix} * & * \\ c_j & d_j \end{smallmatrix}\right)$ and $C_j \cdots C_l = \left(\begin{smallmatrix} * & * \\ \gamma_j & \delta_j \end{smallmatrix}\right)$, so that

$$\begin{aligned} (\Omega_{C_j}|_k C_{j+1} \cdots C_l)(\tau) &= (\gamma_{j+1}\tau + \delta_{j+1})^{-k} \Omega_{C_j}(C_{j+1} \cdots C_l \tau) \\ &= (\gamma_{j+1}\tau + \delta_{j+1})^{-k} \sum_{i=1}^{\chi_t(k)} b_{ij} F_i(C_{j+1} \cdots C_l \tau). \end{aligned} \tag{3.6}$$

Since $F_i(\tau) \in S_k^t(\Gamma)$, we have

$$\begin{aligned}F(C_{j+1}\cdots C_l\tau) \\ &= (c_{j+1}C_{j+2}\cdots C_l\tau + d_{j+1})^k(F_i(C_{j+2}\cdots C_l\tau) + F_{iC_{j+1}}(C_{j+2}\cdots C_l\tau)),\end{aligned}$$

where $F_{iC_{j+1}}(C_{j+2}\cdots C_l\tau) = (F_i|_k(C_{j+1} - I))(C_{j+2}\cdots C_l\tau) \in S_k^{t-1}(\Gamma)$.

Remark 3.7. We use the notation

$$F_{iC_{j_1}\cdots C_{j_\nu}}(\tau) = (F_{iC_{j_1}\cdots C_{j_{\nu-1}}}|_k(C_{j_\nu} - I))(\tau).$$

Repeating the argument, we have

$$\begin{aligned}F(C_{j+1}\cdots C_l\tau) \\ &= (c_{j+1}C_{j+2}\cdots C_l\tau + d_{j+1})^k(c_{j+2}C_{j+3}\cdots C_l\tau + d_{j+2})^k \\ &\quad \times \{F_i(C_{j+3}\cdots C_l\tau) + F_{iC_{j+2}}(C_{j+3}\cdots C_l\tau) + F_{iC_{j+1}}(C_{j+3}\cdots C_l\tau) \\ &\qquad + F_{iC_{j+1}C_{j+2}}(C_{j+3}\cdots C_l\tau)\}.\end{aligned}$$

Continuing, we arrive at

$$\begin{aligned}F(C_{j+1}\cdots C_l\tau) \\ &= (c_{j+1}C_{j+2}\cdots C_l\tau + d_{j+1})^k(c_{j+2}C_{j+3}\cdots C_l\tau + d_{j+2})^k \\ &\qquad \cdots (c_{l-1}C_l\tau + d_{l-1})^k(c_l\tau + d_l)^k \\ &\quad \times \Big\{F_i(\tau) + \sum_{1\le\nu_1\le l-j} F_{iC_{j+\nu_1}}(\tau) + \sum_{1\le\nu_1<\nu_2\le l-j} F_{iC_{j+\nu_1}C_{j+\nu_2}}(\tau) \\ &\qquad + \cdots + F_{iC_{j+1}C_{j+2}\cdots C_l}(\tau)\Big\}.\end{aligned}$$

Thus

$$\begin{aligned}F(C_{j+1}\cdots C_l\tau) &= (\gamma_{j+1}\tau + \delta_{j+1})^k \\ &\times \Big\{F_i(\tau) + \sum_{1\le\nu_1\le l-j} F_{iC_{j+\nu_1}}(\tau) + \sum_{1\le\nu_1<\nu_2\le l-j} F_{iC_{j+\nu_1}C_{j+\nu_2}}(\tau) \\ &\qquad + \cdots + F_{iC_{j+1}C_{j+2}\cdots C_l}(\tau)\Big\}.\end{aligned} \tag{3.7}$$

Since $F_i(\tau) \in S_k^t(\Gamma)$, $F_{iC_{j_1}\cdots C_{j_\nu}}(\tau) = 0$ when $\nu > t$. Thus we set

$$N = \max_{\substack{1\le i\le\chi_t(k) \\ C_{j_1},\ldots,C_{j_\nu}\in\beta \\ 0\le\nu\le t}} C(F_{iC_{j_1}\cdots C_{j_\nu}}, \alpha)$$

where $C(F, \alpha)$ is the bound given in Lemma 3.4. Here we have used the convention $F_{iC_0} = F_i$. N is a bound, independent of j or l, for each of the 2^{l-j} higher order forms appearing in (3.7). Therefore for $\tau \in E_\alpha$, we have

$$|F(C_{j+1}\cdots C_l\tau)| \le 2^{l-j}N|\gamma_{j+1}\tau + \delta_{j+1}|^k. \tag{3.8}$$

This implies, by (3.6) and (3.8), that

$$|(\Omega_{C_j}|_k C_{j+1}\cdots C_l)(\tau)| \le N \sum_{i=1}^{\chi_t(k)} |b_{ij}|2^{l-j} \le N_1 2^l. \tag{3.9}$$

Here $N_1 = N \max_{C_j\in\beta}\sum_{i=1}^{\chi_t(k)} |b_{ij}|$. Since there are l terms in (3.4), we have

$$|\Omega_W(\tau)| \le N_1 l 2^l. \tag{3.10}$$

Recall l is the number of factors in the Eichler factorization of W. We use the Eichler estimate on l given in Theorem 2.1 to obtain

$$\begin{aligned}|\Omega_W(\tau)| &\le N_1(m_1\log\mu(W)+m_2)2^{m_1\log\mu(W)+m_2}\\ &\le C(m_1\log\mu(W)+m_2)\mu(W)^{m_1\log 2}. \ \blacksquare\end{aligned} \tag{3.11}$$

4. Higher order forms with preassigned periods; the Eichler map. Let $\Omega \in Z^1(\Gamma, S_k^t(\Gamma))$. We use the Eichler construction to obtain a $t+1$ order form with period Ω. The exposition follows Lehner [Leh2].

THEOREM 4.1. *Let Γ be an H-group and $\Omega \in Z^1(\Gamma, S_k^t(\Gamma))$.*

(1) *For $k > 2$, there exists $F(\tau) \in S_k^{t+1}(\Gamma)$ such that $F|_k(V-I)(\tau) = \Omega_V(\tau)$ for all $V \in \Gamma$.*
(2) *For $k = 2$, there exists $F(\tau) \in R_2^{t+1}(\tau)$ such that $F|_2(V-I)(\tau) = \Omega_V(\tau)$ for all $V \in \Gamma$.*

Proof. Let δ be a positive even integer $> 2m_1\log 2$ and

$$\Psi(\tau) = -\sum_{W\in\mathcal{M}} \frac{\Omega_W(\tau)}{(c\tau+d)^{k+\delta}}.$$

As in the proof of Proposition 2.2, the estimate (3.3) implies

$$|\Omega_W(\tau)| \le C(m_1\log|c\tau+d|^2+m_2'')|c\tau+d|^{2m_1\log 2} \tag{4.1}$$

on compact subsets of H. It follows that the sum $\sum_{W\in\mathcal{M}}\Omega_W(\tau)/(c\tau+d)^{k+\delta}$ converges absolutely uniformly on compact subsets of H for $k \ge 2$. We have, as before,

$$(\Psi|_{2k+\delta}V)(\tau) = \Psi(\tau) + \Omega_V(\tau)E_{k+\delta}(\tau).$$

Here $E_{k+\delta}(\tau) = \sum_{W\in\mathcal{M}} 1/(c\tau+d)^{k+\delta}$. Also, with $\Phi(\tau) = \psi(\tau)/E_{k+\delta}(\tau)$, we have

$$(\Phi|_k V)(\tau) = \Phi(\tau) + \Omega_V(\tau).$$

As in the second order case, there exists a classical modular form, $G(\tau)$, with exactly the same poles and principal parts as $\Phi(\tau)$. If we set $F'(\tau) = \Phi(\tau) - G(\tau)$, then $F'(\tau) \in R_k^{t+1}(\Gamma)$ and $F'(\tau)|_k(V-I)(\tau) = \Omega_V(\tau)$ for all $k \ge 2$. For $k > 2$, we let $F(\tau) = F'(\tau) - \sum_{j=1}^{s} a_0^j(F')E_j(\tau)$ so that $F(\tau)|_k(V-I)(\tau) = \Omega_V(\tau)$ and $F(\tau) \in S_k^{t+1}(\Gamma)$. For $k = 2$, in order to obtain a cuspidal form, we restrict to $F'(\tau)$ having residue sum zero; then $F(\tau) = F'(\tau) - \sum_{j=2}^{s} a_j(F')E_j^*(\tau)$ is in $S_2^{t+1}(\Gamma)$ and has the correct period. ∎

Fixing the choices made in Theorem 4.1 gives us the maps

$$\mathrm{Eic} : Z^1(\Gamma, S_k^t(\Gamma)) \to S_k^{t+1}(\Gamma), \quad k > 2, \tag{4.2}$$

$$\mathrm{Eic} : Z^1(\Gamma, S_2^t(\Gamma)) \to R_2^{t+1}(\tau) \tag{4.3}$$

with $\mathrm{Eic}(\Omega)|_k(V - I)(\tau) = \Omega_V(\tau)$ for all $V \in \Gamma$ and $k \geq 2$. Next we define

$$Z_*^1(\Gamma, S_2^t(\Gamma)) = \{\Omega \in Z^1(\Gamma, S_2^t(\Gamma)) : \mathrm{Eic}(\Omega)(\tau) \text{ has zero residue sum}\}. \tag{4.4}$$

We have $B^1(\Gamma, S_2^t(\Gamma)) \subset Z_*^1(\Gamma, S_2^t(\Gamma))$, thus we define

$$H_*^1(\Gamma, S_2^t(\Gamma)) = Z_*^1(\Gamma, S_2^t(\Gamma))/B^1(\Gamma, S_2^t(\Gamma)).$$

Finally, we have

$$\mathrm{Eic} : Z_*^1(\Gamma, S_2^t(\Gamma)) \to S_2^{t+1}(\Gamma). \tag{4.5}$$

5. The decomposition. In this section, we establish

THEOREM 5.1. *Let Γ be an H-group.*

(1) *For $k > 2$ and $t \geq 1$,*

$$S_k^{t+1}(\Gamma) \cong H^1(\Gamma, S_k^t(\Gamma)) \oplus \cdots \oplus H^1(\Gamma, S_k^0(\Gamma)).$$

(2) *For $t \geq 1$,*

$$S_2^{t+1}(\Gamma) \cong H_*^1(\Gamma, S_2^t(\Gamma)) \oplus \cdots \oplus H_*^1(\Gamma, S_2^0(\Gamma)).$$

Proof. Let $\pi : S_k^{t+1}(\Gamma) \to H^1(\Gamma, S_k^t(\Gamma))$ be the map $F \mapsto [\{F_V\}]$. Here $[\{F_V\}]$ is the equivalence class of the 1-cocycle F_V. We must show that π is onto, π is a homomorphism, and $\operatorname{Ker}\pi = S_k^t(\Gamma)$. That π is onto is given by the Eichler map defined above: given $\Omega + B^1(\Gamma, S_k^t(\Gamma)) \in H^1(\Gamma, S_k^t(\Gamma))$ (or $H_*^1(\Gamma, S_2^t(\Gamma))$, for $k = 2$), $F(\tau) = \mathrm{Eic}(\Omega)(\tau)$ has the property that $F|_k(V - I)(\tau) = \Omega_V(\tau)$ for $k \geq 2$. That $\operatorname{Ker}\pi = S_k^t(\Gamma)$ follows from the definition of $B^1(\Gamma, S_k^t(\Gamma))$. ∎

6. The dimension of $H^1(\Gamma, S_k^t(\Gamma))$. In this section we calculate the dimension of $H^1(\Gamma, S_k^t(\Gamma))$ under the assumption $\Omega_\pi = 0$ for all π parabolic. We use the following notation:

$$\begin{aligned}
R_t(k) &= \dim H^1(\Gamma, S_k^t(\Gamma)), && k > 2,\\
R_t(2) &= \dim H_*^1(\Gamma, S_2^{t-1}(\Gamma)), && k = 2,\\
r_t(k) &= \dim Z^1(\Gamma, S_k^t(\Gamma)), && k > 2,\\
r_t(2) &= \dim Z_*^1(\Gamma, S_2^t(\Gamma)), && k = 2,\\
r_t^0(k) &= \dim B^1(\Gamma, S_k^t(\Gamma)), && k \geq 2.
\end{aligned}$$

We have

$$R_t(k) = r_t(k) - r_t^0(k).$$

PROPOSITION 6.1. *Let Γ be an H-group.*

(1) *If $k > 2$ and $t \geq 1$, then*

$$B^1(\Gamma, S_k^t(\Gamma)) = Z^1(\Gamma, S_k^{t-1}(\Gamma)).$$

(2) *For $t \geq 1$,*

$$B^1(\Gamma, S_2^t(\Gamma)) = Z_*^1(\Gamma, S_2^{t-1}(\Gamma)).$$

Proof. Let $k > 2$ and $\Omega \in B^1(\Gamma, S_k^t(\Gamma))$. By definition, there exists $F(\tau) \in S_k^t(\Gamma)$ such that $\Omega_V(\tau) = F|_k(V-I)(\tau)$ for all $V \in \Gamma$. Therefore $\Omega \in Z^1(\Gamma, S_k^{t-1}(\Gamma))$. If $\Omega \in Z^1(\Gamma, S_k^{t-1}(\Gamma))$, then $G(\tau) = \mathrm{Eic}(\Omega)(\tau) \in S_k^t(\Gamma)$ satisfies $G|_k(V-I)(\tau) = \Omega_V(\tau)$ for all $V \in \Gamma$. That is, $\Omega \in B^1(\Gamma, S_k^t(\Gamma))$. This gives (1).

For $k = 2$, as before, if $\Omega \in B^1(\Gamma, S_2^t(\Gamma))$ then $\Omega \in Z^1(\Gamma, S_2^{t-1}(\Gamma))$ and there exists $F(\tau) \in S_2^t(\Gamma)$ such that $\Omega_V(\tau) = F|_2(V-I)(\tau)$. We must show $\Omega \in Z_*^1(\Gamma, S_2^{t-1}(\Gamma))$. As $\Omega \in Z^1(\Gamma, S_2^{t-1}(\Gamma))$ we have $\mathrm{Eic}(\Omega)(\tau) \in R_2^t(\Gamma)$ by (4.3), and $\mathrm{Eic}(\Omega)|_k(V-I)(\tau) = F|_k(V-I)(\tau)$. Therefore $\mathrm{Eic}(\Omega)(\tau) - F(\tau) \in M_2(\Gamma)$, which implies

$$0 = \text{residue sum}(\mathrm{Eic}(\Omega)(\tau) - F(\tau)) = \text{residue sum}(\mathrm{Eic}(\Omega)(\tau)),$$

since $F(\tau) \in S_2^t(\Gamma)$. Hence $\Omega \in Z_*^1(\Gamma, S_2^{t-1}(\Gamma))$. Conversely, $Z_*^1(\Gamma, S_2^{t-1}(\Gamma)) \subset B^1(\Gamma, S_2^t(\Gamma))$ follows from (4.5). ∎

The above proposition implies $R_t(k) = r_t(k) - r_{t-1}(k)$ for $k \geq 2$. Hence we are left to determine the dimension, $r_t(k)$, of $Z^1(\Gamma, S_k^t(\Gamma))$. We calculate $\dim Z^1(\Gamma, S_k^t(\Gamma))$ following the method described in [Ei1]. Given $\Omega \in Z^1(\Gamma, S_k^t(\Gamma))$, we have $\Omega_V(\tau) \in S_k^t(\Gamma)$ and Ω is a 1-cocycle, i.e.,

$$\Omega_{VW}(\tau) = (\Omega_V|_k W)(\tau) + \Omega_W(\tau). \tag{6.1}$$

Essentially we calculate the number of choices for Ω_δ, where δ is a generator of Γ, subject to the relations (1.2).

PROPOSITION 6.2. *Let $\Omega \in Z^1(\Gamma, S_k^t(\Gamma))$, $k \geq 2$. If $\epsilon \in \Gamma$ is elliptic, then $\Omega_\epsilon(\tau) = 0$.*

Proof. Suppose $\epsilon^l = I$, l a positive integer. Applying (6.1), we have

$$0 = \Omega_I(\tau) = \Omega_{\epsilon^l}(\tau) = \Omega_\epsilon|_k\{I + \epsilon + \cdots + \epsilon^{l-1}\}(\tau). \tag{6.2}$$

Now let (see [Ei1])

$$F(\tau) = \frac{-1}{l-1}\Omega_\epsilon|_k\{(l-1)I + (l-2)\epsilon + (l-3)\epsilon^2 + \cdots + \epsilon^{l-1}\}(\tau). \tag{6.3}$$

This choice is such that $F(\tau) \in S_k^t(\Gamma)$ and

$$F|_k(\epsilon - I)(\tau) = \Omega_\epsilon(\tau). \tag{6.4}$$

This relation implies, by the definition of $S_k^t(\Gamma)$, that $\Omega_\epsilon(\tau) \in S_k^{t-1}(\Gamma)$, which in turn implies, by (6.3), $F(\tau) \in S_k^{t-1}(\Gamma)$. Iterating, we arrive at $F(\tau) \in S_k(\Gamma)$ such that

$$\Omega_\epsilon(\tau) = F|_k(\epsilon - I)(\tau) = 0. \blacksquare$$

REMARK 6.3. The above proposition together with the parabolic assumption, $F|_k(\pi - I)(\tau) = 0$ for all π parabolic, necessitates Γ to have positive genus ($g > 0$) in order for there to exist nontrivial higher order (parabolic) forms.

From the decomposition given in Theorem 5.1, we can write

$$S_k^{t+1}(\Gamma) = P_{k,t+1}(\Gamma) \oplus \cdots \oplus P_{k,1}(\Gamma)$$

where $P_{k,t}(\Gamma) = \pi^{-1}(H^1(\Gamma, S_k^{t-1}(\Gamma))) \cong H^1(\Gamma, S_k^{t-1}), k > 2$ and $P_{2,t}(\Gamma) \cong H^1_*(\Gamma, S_2^{t-1}(\Gamma))$.

We want to enumerate basis elements for each component $P_t(\Gamma)$. We use the index sets $I_t(k)$, $k > 2$, and $I_t(2)$ defined, iteratively, as follows:

$$\begin{aligned}
&I_0(k) = \{i : 1 \le i \le \chi_1(k)\},\\
&I_1(k) = \{(i_1, i) : 1 \le i_1 \le 2g,\ 1 \le i \le \chi_1(k)\},\\
&I_t(k) = \{(i_t, i_{t-1}, \ldots, i_1, i) : 1 \le i_t \le 2g,\ (i_t, i_{t-1}) \ne (1, g+1),\\
&\qquad\qquad\qquad\qquad\qquad\qquad (i_{t-1}, \ldots, i_1, i) \in I_{t-1}(k)\};\\
&I_t(2) = \{(i_t, i_{t-1}, \ldots, i_1, i) : 1 \le i_t \le 2g,\ (i_t, i_{t-1}) \ne (1, g+1),\\
&\qquad\qquad (i_t, \ldots, i_1, i) \ne (1,1),\ (i_{t-1}, \ldots, i_1, i) \in I_{t-1}(2)\},\\
&I_0(2) = \{i : 1 \le i \le \chi_1(2)\},\\
&I_1(2) = \{(i_1, i) : 1 \le i_1 \le 2g,\ 1 \le i \le \chi_1(2),\ (i_1, i) \ne (1,1)\}.
\end{aligned}$$

The condition $(i_t, \ldots, i_1, i) \ne (1,1)$ is vacuous for $t > 1$.

THEOREM 6.4. *Let Γ be an H-group.*

(1) *For $k > 2$ and $t \ge 1$, there exists a basis $\{F_{i_t,\ldots,i_1,i}(\tau)\}_{(i_t,\ldots,i_1,i)\in I_t(k)}$ of $P_{k,t+1}(\Gamma)$ such that*

$$F_{i_t,\ldots,i_1,i}|_k(\gamma_j - I)(\tau) = \delta_{i_t j} F_{i_{t-1},\ldots,i_1,i}(\tau), \quad 1 \le i_t, j \le 2g,\ (i_t, i_{t-1}) \ne (1, g+1). \tag{6.5}$$

Here $\{F_{i_{t-1},\ldots,i_1,i}(\tau)\}_{(i_{t-1},\ldots,i_1,i)\in I_{t-1}(k)}$ is a basis for $P_{k,t}(\Gamma)$.

(2) *There exists a basis $\{F_{i_t,\ldots,i_1,i}(\tau)\}_{(i_t,\ldots,i_1,i)\in I_t(2)}$ of $P_{2,t+1}(\Gamma)$, $t \ge 1$, such that*

$$F_{i_t,\ldots,i_1,i}|_k(\gamma_j - I)(\tau) = \delta_{i_t j} c_{11} F_{i_{t-1},\ldots,i_1,i}(\tau) - \delta_{1j} c_{i_t\ldots i_1 1} F'_{11}(\tau), \quad 1 \le i_t, j \le 2g,\ (i_t, i_{t-1}) \ne (1, g+1). \tag{6.6}$$

Here $\{F_{i_{t-1},\dots,i_1,i}(\tau)\}_{(i_{t-1},\dots,i_1,i)\in I_{t-1}(k)}$ *is a basis for* $P_{2,t}(\Gamma) \cong H^1_*(\Gamma, S_2^{t-1}(\Gamma))$ *and* $F'_{11}(\tau)\in R_2^2(\Gamma)$, *constructed in Section* 2, *satisfies* $F'_{11}|_2(V-I)(\tau)=n_1(V)F_1(\tau)$ *and* $c_{11}=\text{residue sum}(F'_{11}(\tau))\neq 0$.

By convention, $F_{i_0,i}(\tau) = F_i(\tau)$.

Proof. The proof is by induction on t. For $t = 1$, we have shown there exists a basis $\{F_{i_1,i}(\tau)\}$ for $P_{k,2}(\Gamma) \cong H^1(\Gamma, S_k(\Gamma))$ with $F_{i_1,i}|_k(\gamma_j - I)(\tau) = \delta_{i_1 j}F_i(\tau)$. The condition $i_0 \neq g + 1$ is vacuous. Next we assume a basis $\{F_{i_{t-1},\dots,i_1,1}(\tau)\}$ for $P_{k,t}(\Gamma)$, satisfying (6.5), has been chosen. We have, by Proposition 6.1, $H^1(\Gamma, S_k^t(\Gamma)) = Z^1(\Gamma, S_k^t(\Gamma))/Z^1(\Gamma, S_k^{t-1}(\Gamma))$. Thus we first find a basis for $Z^1(\Gamma, S_k^t(\Gamma))$. Let $\Omega \in Z^1(\Gamma, S_k^t(\Gamma))$, $k \geq 2$, and consider the hyperbolic generators $\gamma_1,\dots,\gamma_{2g}$. The relation (1.2) and the cocycle condition (6.1) imply

$$\Omega_{\gamma_1\gamma_{g+1}\gamma_1^{-1}\gamma_{g+1}^{-1}}|_k\gamma_2\gamma_{g+2}\cdots\gamma_g^{-1}\gamma_{2g}^{-1}\epsilon_1\cdots\epsilon_r\pi_1\cdots\pi_s(\tau) + \Omega_{\gamma_2\gamma_{g+2}\cdots\gamma_g^{-1}\gamma_{2g}^{-1}\epsilon_1\cdots\epsilon_r\pi_1\cdots\pi_s}(\tau) = 0. \tag{6.7}$$

Also, the cocycle condition implies

$$\Omega_{\alpha^{-1}}(\tau) = -\Omega_\alpha|_k\alpha^{-1}(\tau). \tag{6.8}$$

Therefore

$$\Omega_{\gamma_1\gamma_{g+1}\gamma_1^{-1}\gamma_{g+1}^{-1}}(\tau) = \{\Omega_{\gamma_1}|_k(\gamma_{g+1}-I) - \Omega_{\gamma_{g+1}}|_k(\gamma_1 - I)\}|_k\gamma_1^{-1}\gamma_{g+1}^{-1}(\tau).$$

We write the above equation as

$$C(\tau) = N_{g+1}(\Omega_{\gamma_1})(\tau), \tag{6.9}$$

where N_i denotes the map

$$N_i : S_k^t(\Gamma) \to S_k^{t-1}(\Gamma), \qquad F(\tau) \mapsto F|_k(\gamma_i - I)(\tau), \tag{6.10}$$

and

$$\begin{aligned} C(\tau) &= N_1(\Omega_{\gamma_{g+2}})(\tau) \\ &\quad + \Omega_{\gamma_2\gamma_{g+2}\cdots\gamma_g^{-1}\gamma_{2g}^{-1}\epsilon_1\cdots\epsilon_r\pi_1\cdots\pi_s}|_k(\gamma_2\gamma_{g+2}\cdots\gamma_g^{-1}\gamma_{2g}^{-1}\epsilon_1\cdots\epsilon_r\pi_1\cdots\pi_s)^{-1}(\tau) \\ &= N_1(\Omega_{\gamma_{g+2}})(\tau) + \Omega_{\pi_s^{-1}\cdots\pi_1^{-1}\epsilon_r^{-1}\cdots\epsilon_1^{-1}\gamma_{2g}\cdots\gamma_{g+2}^{-1}\gamma_2^{-1}}(\tau), \end{aligned}$$

using (6.8). Thus, since $\Omega_\delta = 0$ for δ parabolic or elliptic, we have

$$C(\tau) = N_1(\Omega_{\gamma_{g+2}})(\tau) + \Omega_{\gamma_{2g}\gamma_g\gamma_{2g}^{-1}\gamma_g^{-1}\cdots\gamma_{g+2}^{-1}\gamma_2^{-1}}(\tau).$$

Remark 6.5. The importance, for us, of $C(\tau)$ is that it depends only on $\Omega_{\gamma_2}(\tau),\dots,\Omega_{\gamma_{2g}}(\tau)$. Also note that $\Omega_{\gamma_{2g}\gamma_g\gamma_{2g}^{-1}\gamma_g^{-1}\cdots\gamma_{g+2}^{-1}\gamma_2^{-1}}(\tau)$, $N_1(\Omega_{\gamma_{g+2}})(\tau)$, and $C(\tau)$ are in $S_k^{t-1}(\Gamma)$.

We choose $\Omega_{\gamma_j}(\tau) \in S_k^t(\Gamma)$, $2 \leq j \leq 2g$, arbitrarily; $\Omega_{\gamma_1}(\tau)$ is then subject to the constraint (6.9). As we noted, $C(\tau) \in S_k^{t-1}(\Gamma)$, thus, by

Theorem 5.1 and the induction hypothesis, we write

$$C(\tau) = \sum_{\nu=0}^{t-2} \sum_{(i_\nu,\ldots,i_1,i)\in I_\nu(k)} C_{i_\nu,\ldots,i_1,i} F_{i_\nu,\ldots,i_1,i}(\tau). \tag{6.11}$$

Also, since $\Omega_{\gamma_1}(\tau) \in S_k^t(\Gamma)$,

$$\Omega_{\gamma_1}(\tau) = \sum_{\nu=0}^{t-1} \sum_{(i_\nu,\ldots,i_1,i)\in I_\nu(k)} \omega_{i_\nu,\ldots,i_1,i} F_{i_\nu,\ldots,i_1,i}(\tau). \tag{6.12}$$

Thus we have

$$\begin{aligned} C(\tau) &= \sum_{(i_{t-2},\ldots,i_1,i)\in I_{t-2}(k)} C_{i_{t-2},\ldots,i_1,i} F_{i_{t-2},\ldots,i_1,i}(\tau) \\ &\quad + \cdots + \sum_{(i_1,i)\in I_1(k)} C_{i_1,i} F_{i_1,i}(\tau) + \sum_{i\in I_0(k)} C_i F_i(\tau) \\ &= N_{g+1}(\Omega_{\gamma_1})(\tau) \qquad \text{(by (6.9))} \\ &= \sum_{(i_{t-1},\ldots,i_1,i)\in I_{t-1}(k)} \omega_{i_{t-1},\ldots,i_1,i} N_{g+1}(F_{i_{t-1},\ldots,i_1,i})(\tau) \\ &\quad + \cdots + \sum_{(i_1,i)\in I_1(k)} \omega_{i_1,i} N_{g+1}(F_{i_1,i})(\tau) \\ &= \sum_{(i_{t-2},\ldots,i_1,i)\in I_{t-2}(k)} \omega_{g+1,i_{t-2},\ldots,i_1,i} F_{i_{t-2},\ldots,i_1,i}(\tau) \\ &\quad + \cdots + \sum_{i\in I_0(k)} \omega_{g+1,i} F_i(\tau). \end{aligned}$$

We have used (6.10) and (6.5), which is valid by the induction hypothesis. Therefore, equating coefficients, we have

$$\omega_{g+1,i_{\nu-1},\ldots,i_1,i} = C_{i_{\nu-1},\ldots,i_1,i}, \quad (i_{\nu-1},\ldots,i_1,i) \in I_{\nu-1}(k),\ 1 \le \nu \le t-1. \tag{6.13}$$

All other coefficients appearing in (6.12) may be arbitrarily chosen. Thus a basis for $Z^1(\Gamma, S_k^t(\Gamma))$, $k \ge 2$, is given by $\bigcup_{\nu=1}^{t} \{\Omega^{i_\nu,i_{\nu-1},\ldots,i_1,i}\}_{(i_\nu,i_{\nu-1},\ldots,i_1,i)\in I_\nu}$ where

$$\Omega_{\gamma_j}^{i_\nu,i_{\nu-1},\ldots,i_1,i}(\tau) = \delta_{i_\nu j} F_{i_{\nu-1},\ldots,i_1,i}(\tau), \quad 2 \le i_\nu \le 2g,\ 2 \le j \le 2g, \quad (i_{\nu-1},\ldots,i_1,i) \in I_{\nu-1}(k), \tag{6.14}$$

$$\Omega_{\gamma_1}^{i_\nu,i_{\nu-1},\ldots,i_1,i}(\tau) = \sum_{\mu=1}^{t-2} \sum_{(i_\mu,\ldots,i_1,i)\in I_\mu(k)} C_{i_\mu,\ldots,i_1,i} F_{g+1,i_\mu,\ldots,i_1,i}(\tau), \tag{6.15}$$

and

$$\Omega_{\gamma_j}^{1,i_{\nu-1},\ldots,i_1,i}(\tau) = 0, \quad 2 \le j \le 2g, \tag{6.16}$$

$$\Omega_{\gamma_1}^{1,i_{\nu-1},\ldots,i_1,i}(\tau) = F_{i_{\nu-1},\ldots,i_1,i}(\tau), \tag{6.17}$$
$$(i_{\nu-1},\ldots,i_1,i) \in I_{\nu-1}(k),\ i_{\nu-1} \neq g+1.$$

Thus (6.14) represents all possible choices for $\Omega_{\gamma_j}(\tau)$, $j \ge 2$. The $C_{i_\mu,\ldots,i_1,i}$ are then determined by these choices. The $(i_{\nu-1},\ldots,i_1,i)$ in (6.17) parameterize the remaining degrees of freedom for $\Omega_{\gamma_1}(\tau)$.

A basis for $H^1(\Gamma, S_k^t(\Gamma))$, $k > 2$, is given by the cocycle representatives $\{\Omega^{i_t,\ldots,i_1,i}\}$ satisfying the conditions (6.14)–(6.17) with $\nu = t$. The corresponding basis of P_{t+1} is $\{F_{i_t,\ldots,i_1,i}(\tau)\}_{(i_t,\ldots,i_1,i)\in I_t(k)}$ where $F_{i_t,\ldots,i_1,i}(\tau) = \mathrm{Eic}(\Omega^{i_t,\ldots,i_1,i})(\tau)$. To see this, we need only check that the above set is linearly independent. Suppose

$$\begin{aligned} 0 &= \sum_{(i_t,\ldots,i_1,i)\in I_t(k)} a_{i_t,\ldots,i_1,i} F_{i_t,\ldots,i_1,i}(\tau) \\ &= \sum_{(i_t,\ldots,i_1,i)\in I_t(k)} a_{i_t,\ldots,i_1,i}\, \mathrm{Eic}(\Omega^{i_t,\ldots,i_1,i})(\tau) \\ &= \mathrm{Eic}\Big(\sum_{(i_t,\ldots,i_1,i)\in I_t(k)} a_{i_t,\ldots,i_1,i} \Omega^{i_t,\ldots,i_1,i}\Big)(\tau) + F(\tau), \end{aligned} \tag{6.18}$$

where $F(\tau) \in S_k(\Gamma)$. Then all periods of $\sum_{(i_t,\ldots,i_1,i)\in I_t(k)} a_{i_t,\ldots,i_1,i}\Omega^{i_t,i_{t-1},\ldots,i_1,i}$ are zero. Hence

$$0 = \sum_{(i_t,\ldots,i_1,i)\in I_t(k)} a_{i_t,\ldots,i_1,i} \Omega^{i_t,\ldots,i_1,i}, \tag{6.19}$$

which implies $a_{i_t,\ldots,i_1,i} = 0$ since the set $\{\Omega^{i_t,\ldots,i_1,i}\}$ is linearly independent.

For $k = 2$ we fix the element $\Omega^{11} = n_1(\cdot)F_1(\tau) \in Z^1(\Gamma, S_2^t(\Gamma))$; it has the property that $F'{}_{11}(\tau) = \mathrm{Eic}(\Omega^{11})(\tau)$ and the residue sum of $F'_{11}(\tau)$ is $c_{11} \neq 0$. A basis for $Z_*^1(\Gamma, S_2^t(\Gamma))$ is then given by $\bigcup_{\nu=1}^t \{c_{11}\Omega^{i_\nu,i_{\nu-1},\ldots,i_1,i} - c_{i_\nu,i_{\nu-1},\ldots,i_1,i}\Omega^{11}\}$ where $\{\Omega^{i_\nu,i_{\nu-1},\ldots,i_1,i}\}$ is a basis for $Z^1(\Gamma, S_2^t(\Gamma))$ given by (6.14)–(6.17) with $\nu = t$, $k = 2$ and $\{F_{i_{\nu-1},\ldots,i_1,i}(\tau)\}_{(i_{\nu-1},\ldots,i_1,i)\in I_{\nu-1}(2)}$ a basis for $P_{2,t}(\Gamma) \subset S_2^t(\Gamma)$. A basis for $H_*^1(\Gamma, S_2^t(\Gamma))$ is given by the cocycle representatives $\{c_{11}\Omega^{i_t,\ldots,i_1,i} - c_{i_t,\ldots,i_1,i}\Omega^{1,1}\}$. The corresponding basis of $P_{2,t+1}(\Gamma)$ is $\{F_{i_t,\ldots,i_1,i}(\tau)\}_{(i_t,\ldots,i_1,i)\in I_t(2)}$ where

$$F_{i_t,\ldots,i_1,i}(\tau) = \mathrm{Eic}(c_{11}\Omega^{i_t,\ldots,i_1,i} - c_{i_t,\ldots,i_1,i}\Omega^{11})(\tau). \ \blacksquare$$

By counting the number of elements satisfying (6.14)–(6.17), for $\nu = t$, we obtain

COROLLARY 6.6. *Let Γ be an H-group.*

(1) *For $k > 2$ and $t \geq 1$,*

$$\dim H^1(\Gamma, S_k^t(\Gamma)) = \sum_{j=0}^{[t/2]} (-1)^j \binom{t-j}{j} (2g)^{t-2j} \dim S_k(\Gamma). \tag{6.20}$$

(2) *For $k = 2$ and $t \geq 1$,*

$$\begin{aligned}\dim H^1_*(\Gamma, S_2^t(\Gamma)) &= \sum_{j=0}^{[t/2]} (-1)^j \binom{t-j}{j} (2g)^{t-2j} \dim S_2(\Gamma) \\ &\quad - \sum_{j=0}^{[(t-1)/2]} (-1)^j \binom{t-1-j}{j} (2g)^{t-1-2j}.\end{aligned} \tag{6.21}$$

Proof. Let $|I_t(k)|$ denote the cardinality of $I_t(k)$. The number of $\Omega^{i_\nu,\dots,i_1,i}$ satisfying (6.14)–(6.15), with $\nu = t$ is

$$(2g-1)|I_{t-1}(k)| = (2g-1)\dim H^1(\Gamma, S_k^{t-1}(\Gamma)),$$

and the number of $\Omega^{i_\nu,\dots,i_1,i}$ satisfying (6.16)–(6.17) with $\nu = t$ is

$$|I_{t-1}(k)| - |I_{t-2}(k)| = \dim H^1(\Gamma, S_k^{t-1}(\Gamma)) - \dim H^1(\Gamma, S_k^{t-2}(\Gamma)).$$

Hence the dimension satisfies the recursive formula

$$\dim H^1(\Gamma, S_k^t(\Gamma)) = 2g \dim H^1(\Gamma, S_k^{t-1}(\Gamma)) - \dim H^1(\Gamma, S_k^{t-2}(\Gamma)). \tag{6.22}$$

We use this formula and induction to prove (6.20). As noted in Section 2, $H^1(\Gamma, S_k(\Gamma)) \cong \mathrm{Hom}(\Gamma, S_k(\Gamma))$. Hence $\dim H^1(\Gamma, S_k(\Gamma)) = 2g \dim S_k(\Gamma)$. Also, recall $H^1(\Gamma, S_k^0(\Gamma)) = S_k(\Gamma)$. Thus for $t = 2$, (6.22) gives

$$\begin{aligned}\dim H^1(\Gamma, S_k^2(\Gamma)) &= 2g \dim H^1(\Gamma, S_k(\Gamma)) - \dim H^1(\Gamma, S_k^0(\Gamma)) \\ &= ((2g)^2 - 1)\dim S_k(\Gamma).\end{aligned} \tag{6.23}$$

Assume (6.20) holds for $\nu < t$. Applying (6.22), we have

$$\begin{aligned}&\dim H^1(\Gamma, S_k^t(\Gamma))/\dim S_k(\Gamma) \\ &\quad = 2g \sum_{j=0}^{[(t-1)/2]} (-1)^j \tbinom{t-1-j}{j} (2g)^{t-1-2j} - \sum_{j=0}^{[(t-2)/2]} (-1)^j \tbinom{t-2-j}{j} (2g)^{t-2-2j} \\ &\quad = (2g)^t + \sum_{j=1}^{[(t-1)/2]} (-1)^j \tbinom{t-1-j}{j} (2g)^{t-2j} + \sum_{j=1}^{[(t-2)/2]+1} (-1)^j \tbinom{t-1-j}{j-1} (2g)^{t-2j}.\end{aligned}$$

For t odd, $\left[\frac{t}{2}\right] = \left[\frac{t-1}{2}\right] = \left[\frac{t-2}{2}\right] + 1$, thus the above sum is

$$(2g)^t + \sum_{j=1}^{[(t-1)/2]} (-1)^j \binom{t-j}{j} (2g)^{t-2j} = \sum_{j=0}^{[t/2]} (-1)^j \binom{t-j}{j} (2g)^{t-2j}.$$

Here we have used $\binom{t-1-j}{j} + \binom{t-1-j}{j-1} = \binom{t-j}{j}$. For t even, $\left[\frac{t}{2}\right] = \left[\frac{t-1}{2}\right] + 1 = \left[\frac{t-2}{2}\right] + 1$, thus the sum becomes

$$(2g)^t + \sum_{j=1}^{[t/2-1]} (-1)^j \binom{t-j}{j} (2g)^{t-2j} + (-1)^{t/2} = \sum_{j=0}^{[t/2]} (-1)^j \binom{t-j}{j} (2g)^{t-2j}.$$

For $k = 2$, $t \geq 2$, we count the number of elements $\{c_{1,1}\Omega^{i_t,\dots,i_1,i} - c_{i_t,\dots,i_1,i}\Omega^{1,1}\}$ with $\Omega^{i_t,\dots,i_1,i}$ satisfying (6.14) with $\nu = t$. The number of $\{\Omega^{i_t,\dots,i_1,i}\}$ satisfying $\Omega^{i_t,i_{t-1},\dots,i_1,i}_{\gamma_j} = \delta_{i_t j} F_{i_{t-1},\dots,i_1,i}(\tau)$, $2 \leq i_t, j \leq 2g$, $(i_{t-1},\dots,i_1,i) \in I_{t-1}(2)$, is $(2g-1)|I_{t-1}(2)| = (2g-1)\dim H^1_*(\Gamma, S_2^{t-1}(\Gamma))$ and the number of elements satisfying $\Omega^{1,i_{t-1},\dots,i_1,i}_{\gamma_1} = F_{i_{t-1},\dots,i_1,i}(\tau)$, $i_{t-1} \neq g+1$, is $|I_{t-1}(2)| - |I_{t-2}(2)| = \dim H^1_*(\Gamma, S_2^{t-1}(\Gamma)) - \dim H^1_*(\Gamma, S_2^{t-2}(\Gamma))$. In particular, $H^1_*(\Gamma, S_2^t(\Gamma))$ satisfies the recursive relation (6.22) with initial data $\dim H^1_*(\Gamma, S_2^0(\Gamma)) = \dim S_2(\Gamma) = g$ and $\dim H^1_*(\Gamma, S_2(\Gamma)) = 2g\dim S_2(\Gamma) - 1$. Since the solution to the recursive relation (6.22) depends linearly on the initial data, $H^1_*(\Gamma, S_2^t(\Gamma))$ is the sum of the solutions with initial data $(\dim S_2(\Gamma), 2g\dim S_2(\Gamma))$ and $(0, -1)$. This yields (6.21). ∎

Corollary 6.7. *Let Γ be an H-group.*

(1) *For $k > 2$ and $t \geq 1$,*

$$\dim S_k^{t+1}(\Gamma) = \sum_{\nu=0}^{t} \sum_{j=0}^{[\nu/2]} (-1)^j \binom{\nu-j}{j} (2g)^{\nu-2j} \dim S_k(\Gamma). \tag{6.24}$$

(2) *For $k = 2$ and $t \geq 1$,*

$$\dim S_2^{t+1}(\Gamma) = \sum_{\nu=0}^{t} \sum_{j=0}^{[\nu/2]} (-1)^j \binom{\nu-j}{j} (2g)^{\nu-2j} \dim S_2(\Gamma) - \sum_{\nu=0}^{t-1} \sum_{j=0}^{[\nu/2]} (-1)^j \binom{\nu-j}{j} (2g)^{\nu-2j}.$$

Finally, we express these results in the form given in [DS]:

Corollary 6.8. *Let Γ be an H-group.*

(1) *For $k > 2$ and $t \geq 1$,*

$$\dim(S_k^{t+1}(\Gamma)/S_k^t(\Gamma)) = \frac{\dim S_k(\Gamma)}{2\sqrt{g^2-1}} \big((g+\sqrt{g^2-1})^{t+1} - (g-\sqrt{g^2-1})^{t+1}\big). \tag{6.25}$$

(2) *For $k=2$ and $t\geq 1$,*

$$\dim(S_2^{t+1}(\Gamma)/S_2^t(\Gamma)) = \frac{1}{2}\big((g+\sqrt{g^2-1})^{t+1} + (g-\sqrt{g^2-1})^{t+1}\big). \tag{6.26}$$

To show that Corollary 6.8 follows from Corollary 6.7, use the binomial formula to write

$$\frac{(g+\sqrt{g^2-1})^{t+1} - (g-\sqrt{g^2-1})^{t+1}}{2\sqrt{g^2-1}} = \sum_{j=0}^{[t/2]} \binom{t+1}{2j+1} g^{t-2j}(g^2-1)^j. \tag{6.27}$$

Then we use the fact, to be proved in the next section, that

$$\sum_{j=0}^{[t/2]} \binom{t+1}{2j+1} g^{t-2j}(g^2-1)^j = \sum_{j=0}^{[t/2]} (-1)^j \binom{t-j}{j} (2g)^{t-2j}. \tag{6.28}$$

7. A binomial identity. In this section we prove (6.28). Expanding and equating like terms reduces the problem to showing

$$\binom{t-m}{m} 2^{t-2m} = \sum_{j=m}^{[t/2]} \binom{t+1}{2j+1}\binom{j}{m}, \quad 0\leq m\leq [t/2]. \tag{7.1}$$

We introduce, for a a positive integer, the notation $(a)_n = a(a-1)\cdots(a-n)$ if $n\geq 0$ and $(a)_n = 1$ if $n<0$. Thus we want to prove the following

Proposition 7.1.

$$(t-m)_{m-1} 2^{t-2m} = \sum_{j=m}^{[t/2]} \binom{t+1}{2j+1} (j)_{m-1}. \tag{7.2}$$

Proof. We introduce the auxiliary function

$$F(x,y;t) = \frac{(\sqrt{x}+y)^{t+1} - (-\sqrt{x}+y)^{t+1}}{\sqrt{x}}.$$

Applying the binomial expansion and differentiating, we have

$$\frac{\partial^m F}{\partial x^m}(1,1;t) = 2\sum_{j=m}^{[t/2]} \binom{t+1}{2j+1} (j)_{m-1}. \tag{7.3}$$

Therefore we want to prove

$$\frac{\partial^m F}{\partial x^m}(1,1;t) = (t-m)_{m-1} 2^{t-2m+1}. \tag{7.4}$$

We prove (7.4) by induction. Our starting point is $(0,0)$ where we have $F(1,1;0)=2$. Next we assume (7.4) holds for any $(m',t')<(m,t)$, that is, either $t' < t$, or $t' = t$ and $m' < m$. The function $F(x, y; t)$ satisfies the recursive differential equation

$$\frac{\partial^2 F}{\partial x^2}(x, y; t) = -\frac{3}{2}\frac{\partial F}{\partial x}(x, y; t) + \frac{(t+1)t}{2^2 x}F(x, y; t-2).$$

Therefore, after applying Leibniz's rule and evaluating at $x = y = 1$, we have

$$\begin{aligned}(7.5)\qquad \frac{\partial^m F}{\partial x^m}(1, 1; t) &= \frac{(-1)^{m-2}(m-2)!}{2^2} \\ &\times \sum_{j=0}^{m-2} \frac{(-1)^j}{j!}\left\{(t+1)t\frac{\partial^j F}{\partial x^j}(1, 1; t-2) - 6\frac{\partial^{j+1} F}{\partial x^{j+1}}(1, 1; t)\right\}.\end{aligned}$$

Our induction hypothesis holds for all derivatives appearing in the right hand side of (7.5), thus

$$\begin{aligned}(7.6)\qquad \frac{\partial^m F}{\partial x^m}(1, 1; t) &= (-1)^{m-2}(m-2)! \\ &\times \sum_{j=0}^{m-2} \frac{(-1)^j}{j!} 2^{-2j}(t^2 - 5t + 6(j+1))(t-(j+2))_{j-1}.\end{aligned}$$

Now the problem is reduced to showing, for $m \geq 3$,

$$\begin{aligned}(7.7)\qquad (-1)^{m-2}(m-2)!2^{t-3} \sum_{j=0}^{m-2} \frac{(-1)^j}{j!} 2^{-2j}(t^2 - 5t + 6(j+1))(t-(j+2))_{j-1} \\ = 2^{-2(m-2)}(t-m)_{m-1}.\end{aligned}$$

We denote the left hand side of (7.7) by G_m and again apply induction. Direct calculation gives the result for $m = 3$. We then rewrite G_m as

$$G_m = 2^{-2(m-2)}(t^2 - 5t + 6(l+1))(t-m)_{m-3} - (m-2)G_{m-1}.$$

By the induction hypothesis, $G_{m-1} = 2^{-2(m-3)}(t-(m-1))_{m-3}$, so that

$$\begin{aligned}G_m &= 2^{-2(m-2)}(t-m)_{m-3}(t^2 + (3-4m)t + (2m-1)(2m-2)) \\ &= 2^{-2(m-2)}(t-m)_{2m-1}. \blacksquare\end{aligned}$$

REMARK 7.2. Cormac O'Sullivan has pointed out that identity (7.1) is the combinatorial identity counting the number of odd committees one can make from n people with subcommittees of size m chosen from the first half (i.e. below the median $j + 1$st element in a committee with $2j + 1$ elements) [BQ].

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