Primes in arithmetic progressions to spaced moduli

by

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1. Introduction. Let $\Lambda$ be the von Mangoldt function. For $(a, q) = 1$, let
\[
\sum_{n \leq x, n \equiv a \pmod{q}} A(n) = \frac{x}{\phi(q)} + E(x; q, a).
\]
It is well known that for given $A > 0$, $C > 0$,
\[
E(x, q) := \max_{(a, q) = 1} |E(x; q, a)| \ll \frac{x}{q(\log x)^A}
\]
for $x \geq 2$, $q \leq (\log x)^C$. See e.g. Davenport [5].

Suppose we are given a set $S$ with some arithmetic structure. Let
\[S(Q) = \{q \in S : Q < q \leq 2Q\}.
\]
Can we prove that (1.1) holds for most $q$ in $S(Q)$, for large values of $Q$? That is, we seek bounds
\[
\sum_{q \in S(Q)} E(x, q) \ll \frac{x|S(Q)|}{Q(\log x)^A}
\]
for every $A > 0$. Here $|T|$ denotes the cardinality of a finite set $T$. If $S$ is the set $\mathbb{N}$ of natural numbers, then (1.2) holds for $Q \leq x^{1/2}(\log x)^{-A-5}$, by the Bombieri–Vinogradov theorem; see e.g. [5].

In the present paper we study the particular case
\[
S = S_f = \{f(k) : k \in \mathbb{N}\}
\]
where
\[
f(X) = a_dX^d + \cdots + a_1X + a_0, \quad a_j \in \mathbb{Z}, \; d \geq 2, \; a_d > 0.
\]
The first result for this case is due to Elliott [6]. He showed that (1.2) holds for $S = S_f$,

$$Q < x^{1/4-\varepsilon}.$$  


More is known in the special case $f(x) = x^2$. Baier and Zhao [2] used a version of the large sieve, due to Baier [1], for fractions $a/q^2$, $q \leq Q$, $(a, q) = 1$, to obtain (1.2) for $S = \{k^2 : k \geq 1\}$ whenever

$$Q < x^{4/9-\varepsilon}.$$  

In the present paper we sharpen these results.

**Theorem 1.** Let $f$ be as in (1.4). Let $\varepsilon > 0$. We have

$$\sum_{q \in S_f(Q)} E(x, q) \ll \frac{x|S_f(Q)|}{Q(\log x)^A}$$

for every $A > 0$, provided that

$$Q < x^{9/20-\varepsilon}.$$  

The implied constant depends at most on $f$, $\varepsilon$ and $A$.

**Theorem 2.** Let $f(x) = x^2$. The conclusion of Theorem 1 holds whenever

$$Q < x^{43/90-\varepsilon}.$$  

For comparison, we note that $8/19 = 0.421\ldots$, $4/9 = 0.444\ldots$, $9/20 = 0.45$, $43/90 = 0.477\ldots$.

For some applications, the following theorem is more useful than Theorem 2.

**Theorem 3.** We have

$$\sum_{Q^{1/2} < p \leq (2Q)^{1/2}} E(x, p^2) \ll xQ^{-1/2}(\log x)^{-A}$$

for every $A > 0$, provided that

$$Q < x^{1/2-\varepsilon}.$$  

To prove Theorem 1 we start from the work of Mikawa and Peneva, and import an averaging over $q$ in $S_f(Q)$ into the treatment of ‘Type 1’ sums. Theorem 2 follows the same lines, but incorporates a generalization of the large sieve inequality of Baier and Zhao [2] to obtain a new mean value bound for the relevant Dirichlet polynomials. For Theorem 3 we adapt the proof of Theorem 2 a little. The treatment of the bilinear forms in the remainder terms goes back to Iwaniec [9], and we need only adapt this to the present purpose.
In applications, it is sometimes useful to have a ‘maximal variant’ of Theorems 1, 2 or 3 in which $E(x, q)$ is replaced by $\max_{1 \leq y \leq x} E(y, q)$. We provide this maximal variant of the theorems in Section 6.

Throughout the paper, $\varepsilon$ denotes a positive number, which we suppose to be sufficiently small; furthermore, $\delta = \varepsilon^2$ and $f$ is a polynomial, as in (1.4). We assume that $Q \geq 1$, and that $N$ is a natural number.

2. The Dirichlet polynomials $\sum_{n \leq N} \chi(n)n^{-s}$. Let $\gamma$ be a constant, $0 < \gamma < 1$. We seek good bounds on

$$B(s, \chi) = \sum_{n \leq N} \chi(n)n^{-s}$$

that are valid on the critical line for all nonprincipal $\chi \pmod{q}$ and all $N \geq q^\gamma$, for $q \in S_f(Q) \setminus F(Q)$. The cardinality of the exceptional set $F(Q)$ will be small compared with $|S_f(Q)|$.

**Lemma 1.** Let $b > 0$ and let $G$ be a finite subset of $\mathbb{N} \cap [b, \infty)$. Let $F = \{q \in S_f(Q) : r \mid q \text{ for some } r \in G\}$. Then

$$|F| \ll |S_f(Q)||G|b^{-1/d + \varepsilon}.$$

The implied constant depends at most on $f$ and $\varepsilon$.

**Remark 1.** Unless otherwise stated, the dependencies of implied constants in the proof will be the same as in the statement of the lemma; similarly in subsequent proofs.

**Proof.** We may suppose that $Q$ is sufficiently large, so that $Q^{1/d} \ll |S_f(Q)| \ll Q^{1/d}$.

Fix $r \in G$. We need only show that

$$|\{q \in S_f(Q) : r \mid q\}| \ll |S_f(Q)|r^{-1/d + \varepsilon}.$$

We recall that for an irreducible polynomial $g$ in $\mathbb{Z}[x],$

$$|\{n \pmod{t} : g(n) \equiv 0 \pmod{t}\}| \ll_g t^\varepsilon$$

(see e.g. Nagell [15]). Now let

$$f = g_1 \ldots g_h$$

where $g_1, \ldots, g_h$ are irreducible, $h \leq d$. If $f(n) \equiv 0 \pmod{r}$, then

$$r = (g_1(n) \ldots g_h(n), r) \leq (g_j(n), r)^h$$
for some $j$. Hence for any interval $[a, b]$, $$|\{n \in [a, b] : f(n) \equiv 0 \pmod{r}\}|$$ $$\leq \sum_{j=1}^{h} \sum_{t \mid r} \sum_{t \geq r^{1/h}} |\{n \in [a, b] : g_j(n) \equiv 0 \pmod{t}\}|$$ $$\ll r^{\varepsilon/2} \left(\frac{b-a}{t} + 1\right) |\{n \pmod{t} : g_j(n) \equiv 0 \pmod{t}\}|$$ (for some $j$, $1 \leq j \leq h$ and $t \mid r$, $t \geq r^{1/h}$) $$\ll r^{\varepsilon} \left(\frac{b-a}{r^{1/h}} + 1\right).$$

We now obtain the lemma on noting that $$\{q \in S_f(Q) : r \mid q\} = \{f(n) : n \in [a, b], f(n) \equiv 0 \pmod{r}\}$$ with $b-a \ll Q^{1/d}$. Since $r \ll Q$ if there is some $q \in S_f(Q)$ divisible by $r$, $$|\{q \in S_f(Q) : r \mid q\}| \ll r^\varepsilon ((Q/r)^{1/d} + 1) \ll |S_f(Q)|r^{-1/d+\varepsilon}. \quad \blacksquare$$

For any nonprincipal character $\chi$ to modulus $q$, there is a divisor $$r = \text{cond } \chi$$ of $q$, the conductor of $\chi$, and a primitive character $\chi' \pmod{r}$ such that $$\chi(n) = \begin{cases} \chi'(n) & \text{if } (n, q) = 1, \\ 0 & \text{if } (n, q) > 1. \end{cases}$$ We say that $\chi$ is induced by $\chi'$ (see [5, Chapter 5]).

**Lemma 2.** Let $b > 0$, $4/5 \leq \alpha \leq 1$, $T \geq 2$. Let $$F = F(\alpha, T, b)$$ be the set of $q$ in $S_f(Q)$ for which $$L(s, \chi) = 0$$ for some nonprincipal $\chi \pmod{q}$ with $\text{cond } \chi \geq b$, and some $s$ with $\Re(s) \geq \alpha$, $|\Im(s)| \leq T$. Then $$|F| \ll |S_f(Q)|(Q^2T)^{2(1-\alpha)/\alpha}(\log QT)^{14}b^{-1/d+\varepsilon}.$$ The implied constant depends at most on $f$ and $\varepsilon$.

**Proof.** Let $q \in F$. Suppose that $L(s, \chi) = 0$, where $\chi$ and $s$ are as in the statement of the lemma, $\chi$ being induced by the primitive character $\chi'$ to modulus $r \geq b$. Then $$L(s, \chi') = 0$$
Let us write $N(\sigma, T, \chi')$ for the number of zeros of $L(s, \chi')$ with $\Re(s) \geq \sigma$, $|\Im(s)| \leq T$. Let

$$G = \{ r : b \leq r \leq 2Q, L(s, \chi') = 0 \text{ for some primitive character } \chi' \pmod{r} \text{ and some } s, \Re(s) \geq \alpha, |\Im(s)| \leq T \}. $$

Obviously

$$|G| \leq \sum_{r \leq 2Q} \sum_{\lambda \pmod{r}}^* N(\alpha, T, \lambda)$$

where the asterisk denotes a restriction to primitive characters. The above discussion yields

$$F \subseteq \{ q \in S_f(Q) : r \mid q \text{ for some } r \in G \}. $$

Combining Lemma 1 with (2.1), we obtain

$$|F| \ll |S_f(Q)| b^{-1/d+\varepsilon} \sum_{r \leq 2Q} \sum_{\lambda \pmod{r}}^* N(\alpha, T, \lambda). $$

We now complete the proof by appealing to the bound

$$\sum_{r \leq 2Q} \sum_{\lambda \pmod{r}}^* N(\alpha, T, \lambda) \ll (Q^{2T})^{2(1-\alpha)/\alpha} (\log QT)^{14}$$

given by Montgomery [12, Theorem 12.2].

**Lemma 3.** Let $1/2 < \alpha < 1$. Let $T \geq T_0(\alpha, \varepsilon)$. Suppose that $\chi$ is a nonprincipal character modulo $q$, and

$$L(s, \chi) \neq 0 \quad (\Re(s) \geq \alpha, |\Im(s)| \leq T). $$

Then for $\sigma \geq \alpha$, $|t| \leq T/2$,

$$\log L(\sigma + it, \chi) \ll (\log qT)^{(1-\sigma)/(1-\alpha)+\varepsilon}. $$

The implied constant depends at most on $\alpha$ and $\varepsilon$.

**Proof.** We argue as in Titchmarsh [16, proof of Theorem 14.2]. Let $\eta = \eta(\alpha, \varepsilon) > 0$ be sufficiently small, and $\sigma_1 = \sigma_1(\alpha, \varepsilon) > 0$ sufficiently large. Apply the Borel–Carathéodory theorem to the function $\log L(s, \chi)$ and the circles with center $2+it$ and radii $r, 2-\alpha$, where $|t| \leq T$ and

$$0 < r \leq 2 - \alpha - \eta. $$

On the larger circle,

$$\Re(\log L(s, \chi)) = \log |L(s, \chi)| < \log 4qT $$

([5] (14) of Chapter 12]). Hence, on the smaller circle,

$$|\log L(s, \chi)| \leq \frac{4-2\alpha}{\eta} \log 4qT + \frac{4-2\alpha - \eta}{\eta} |\log L(2+it, \chi)|.$$
Thus for \( \text{Re}(s) \geq \alpha + \eta, \ |\text{Im}(s)| \leq T \) it is clear that
\[
|\log L(s, \chi)| \ll \log qT.
\]

In proving (2.2) we may suppose that
\[
\alpha + \eta \leq \sigma \leq 1 + \eta, \quad |t| \leq T/2.
\]

We apply Hadamard’s three circles theorem to the circles with center \( \sigma_1 + it \) passing through the points \( 1 + \eta + it, \sigma + it \) and \( \alpha + \eta + it \). The radii are
\[
r_1 = \sigma_1 - (1 + \eta), \quad r_2 = \sigma_1 - \sigma, \quad r_3 = \sigma_1 - (\alpha + \eta).
\]

If the maxima of \( |\log L(s, \chi)| \) on the circles are \( M_1, M_2, M_3 \), then
\[
M_2 \leq M_1^{1-a} M_3^a, \quad \text{where} \quad a = \frac{\log(r_2/r_1)}{\log(r_3/r_1)}.
\]

Hence
\[
\log L(\sigma + it, \chi) \ll M_3^a \ll (\log qT)^a.
\]

It remains to bound \( a \). We have
\[
\log \left( \frac{r_2}{r_1} \right) = \log \left( 1 + \frac{1+\eta-\sigma}{\sigma_1-1-\eta} \right) = \frac{1+\eta-\sigma}{\sigma_1-1-\eta} \left( 1 + O(\sigma_1^{-1}) \right),
\]
\[
\log \left( \frac{r_3}{r_1} \right) = \log \left( 1 + \frac{1-\alpha}{\sigma_1-1-\eta} \right) = \frac{1-\alpha}{\sigma_1-1-\eta} \left( 1 + O(\sigma_1^{-1}) \right),
\]
where the implied constants are absolute. Hence
\[
a = \frac{1+\eta-\sigma}{1-\alpha} \left( 1 + O(\sigma_1^{-1}) \right) \ll \frac{1-\sigma}{1-\alpha} + \varepsilon
\]
as required, if \( \eta \) and \( \sigma_1 \) are chosen suitably. \( \blacksquare \)

The following version of Perron’s formula is a slight variant of [3, Lemma 13].

**Lemma 4.** Let \( b \geq 0, c > 0 \) and let \( \lambda \in \mathbb{R}, \lambda + c > 1 + b \). For \( K > 0 \) and complex numbers \( a_l \ (l \geq 1) \) with \( |a_l| \leq Kl^b \), write
\[
h(s) = \sum_{l=1}^{\infty} \frac{a_l}{l^s} \quad (\text{Re}(s) > 1 + b).
\]

Then for \( T > 1 \),
\[
\sum_{l \leq N} \frac{a_l}{l^\lambda} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} h(s+\lambda) \frac{(N+1/2)^s}{s} \, ds + O \left( \frac{KN^c}{T} \right).
\]

The implied constant depends at most on \( c, \lambda + c - 1 - b \).
Let $\chi$ be a nonprincipal character modulo $q$. We apply the lemma with $a_l = \chi(l)$, $K = 1$, $b = \lambda = 0$, $c = 1 + \varepsilon$. Thus

$$
\sum_{n \leq N} \chi(n) = \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} L(s, \chi)(N + 1/2)^s s ds + O\left(\frac{N^{1+\varepsilon}}{T}\right).
$$

This leads to the following result.

**Lemma 5.** Let $\gamma > 0$, $1/2 < \alpha < 1$ and suppose that the nonprincipal character $\chi \pmod{q}$ satisfies

$$L(s, \chi) \neq 0 \quad (\mathrm{Re}(s) \geq \alpha, |\mathrm{Im}(s)| \leq 2q).$$

Then

$$\sum_{n \leq N} \chi(n) \ll N^{\alpha+\varepsilon} \quad (N \geq q^\gamma).$$

The implied constant depends at most on $\alpha$, $\gamma$ and $\varepsilon$.

**Proof.** We may suppose that $N > T_0(\alpha, \varepsilon)$. In view of the Pólya–Vinogradov inequality, we may further suppose that $N < q$. By (2.3),

$$\sum_{n \leq N} \chi(n) = \frac{1}{2\pi i} \int_{1+\varepsilon-iN}^{1+\varepsilon+iN} L(s, \chi)(N + 1/2)^s s ds + O(N^\varepsilon).$$

We replace the integral by

$$
\int_{\alpha+\varepsilon/2-iN}^{\alpha+\varepsilon/2+iN} L(s, \chi)(N + 1/2)^s s ds,
$$

incurring an error that is the sum of the integrals over horizontal segments. On these segments the integrand is

$$O\left(\frac{N^\varepsilon}{\max_{\mathrm{Re}(s) \geq \alpha+\varepsilon/2, |\mathrm{Im}(s)| \leq q} |L(s, \chi)|}\right) = O(N^\varepsilon q^{\varepsilon}) = O(N^{2\varepsilon})$$

by an application of Lemma 3. Likewise the integral in (2.4) is

$$O\left(\frac{N^{\alpha+2\varepsilon/3}}{\sum_{-N}^{N} \frac{dt}{|\alpha + it|}}\right) = O(N^{\alpha+\varepsilon}).$$

The lemma follows on combining these estimates. ■

**Lemma 6.** Let $0 < \gamma < 1$. There is a subset $F(Q)$ of $S_f(Q)$, with

$$|F(Q)| \ll |S_f(Q)| Q^{-\beta},$$

such that for $q \in S_f(Q) \setminus F(Q)$, $\chi$ nonprincipal modulo $q$ and $\mathrm{Re}(s) = 1/2$ we have

$$\sum_{n \leq N} \chi(n)n^{-1/2+it} \ll |s|N^{1/2-\beta} \quad (N \geq q^\gamma).$$

Here $\beta = \beta(\gamma, d) > 0$. The implied constants depend only on $f$ and $\gamma$. 


Proof. Let \( s = 1/2 + it \) and
\[
T(\chi, u) = \sum_{n \leq u} \chi(n).
\]
Suppose for a moment that
\[
T(\chi, u) \ll u^{1-\beta} \quad (u \geq q^{\gamma/2}).
\]
Then for \( N \geq q^{\gamma} \),
\[
\sum_{n \leq N} \chi(n)n^{-1/2+it} = \int_{1-}^{N} u^{-1/2+it} dT(\chi, u)
\]
\[
= T(\chi, u)u^{-1/2+it} \int_{1-}^{N} \left( \frac{1}{2} + it \right) \int_{1}^{N} u^{-3/2+it} T(\chi, u) du
\]
\[
\ll |s|N^{1/2-\beta} + |s| \int_{1}^{N} u^{-1/2} du \ll |s|N^{1/2-\beta}
\]
provided that \( \beta \leq 1/4 \).

Now let \( \alpha \) be a positive constant, \( 4/5 \leq \alpha < 1 \), to be determined below. We take \( F(Q) = F(\alpha, 4Q, (2Q)^{\gamma/2}) \) in the notation of Lemma 2. We first show that for \( q \in S_f(Q) \setminus F(Q) \) and a nonprincipal character \( \chi \) (mod \( q \)),
\[
(2.6) \quad T(\chi, u) \ll u^{1-\beta} \quad (u \geq q^{\gamma/2}).
\]
Suppose first that \( \text{cond } \chi \geq (2Q)^{\gamma/2} \). Since \( q \notin F(Q) \),
\[
L(s, \chi) \neq 0 \quad (\text{Re}(s) \geq \alpha, \ |\text{Im}(s)| \leq 4Q).
\]
By Lemma 5 with \( \gamma/2 \) in place of \( \gamma \),
\[
T(\chi, u) \ll u^{\alpha+\varepsilon} \quad (u \geq q^{\gamma/2}).
\]
This gives the bound (2.6), provided that we choose \( \beta \leq 1 - \alpha - \varepsilon \), and (2.5) follows.

Now suppose that \( \chi \) has conductor \( r < (2Q)^{\gamma/2} \) and is induced by the primitive character \( \chi' \). Let \( u \geq q^{\gamma/2} \). Then
\[
(2.7) \quad T(\chi, u) = \sum_{n \leq u} \left( \sum_{d|n} \mu(d) \right) \chi'(n) = \sum_{d|q} \mu(d) \chi'(d) \sum_{m \leq u/d} \chi'(m)
\]
\[
\ll \tau(q)r^{1/2} \log r \quad (\text{by the Pólya–Vinogradov inequality})
\]
\[
\ll q^{\gamma/4+\varepsilon} \ll u^{1-\beta} \quad (u \geq q^{\gamma/2}).
\]
This establishes that (2.5) holds for all \( \chi \) (mod \( q \)).

It remains to bound \( |F(Q)| \). According to Lemma 2
\[
|F(Q)| \ll |S_f(Q)| Q^{6(1-\alpha)/\alpha} (\log Q)^{14} Q^{-\gamma/3d}.
\]
We choose $\alpha$ so that $6(1 - \alpha)/\alpha = \gamma/(6d)$. This gives the desired bound provided that we take $\beta < \gamma/(6d)$.

3. First stage of proof of Theorems 1, 2 and 3. By the Brun–Titchmarsh theorem [13],

$$E(x, q) \ll \frac{x}{\phi(q)} \ll \frac{x}{Q} \log \log x \quad (q \in S_f(Q)).$$

With $F(Q)$ as in Lemma [6]

$$\sum_{q \in F(Q)} E(x, q) \ll \frac{x|F(Q)|}{Q} \log \log x \ll \frac{x|S_f(Q)|}{Q(log x)^A}.$$

Thus we need only show that

$$\sum_{q \in S_f(Q) \setminus F(Q)} E(x, q) \ll \frac{x|S_f(Q)|}{Q(log x)^A}.$$

We use a particular case of Vaughan’s identity (see e.g. [5, Chapter 24]). Let $Z = Q^{\varepsilon/4}$. Then

$$\Lambda(n) = a_1(n) + a_2(n) + a_3(n) + a_4(n)$$

with

$$a_1(n) = \begin{cases} 
\Lambda(n) & \text{if } n \leq Z, \\
0 & \text{if } n > Z,
\end{cases} \quad a_3(n) = \sum_{d=n, d \leq Z} \mu(d) \log h,$$

$$a_2(n) = -\sum_{m \leq Z, m \nmid n, d \leq Z} A(m) \mu(d), \quad a_4(n) = -\sum_{m > Z, k > Z \mid d, d \leq Z} A(m) \left( \sum_{d \mid k} \mu(d) \right).$$

Let

$$E_i(x; q, a) = \sum_{n \leq x \atop n \equiv a \pmod{q}} a_i(n) - \frac{1}{\phi(q)} \sum_{n \leq x \atop (n, q) = 1} a_i(n).$$

For $q \in S_f(Q)$,

$$\sum_{i=1}^{4} E_i(x; q, a) = \psi(x; q, a) - \frac{1}{\phi(q)} \sum_{n \leq x \atop (n, q) = 1} \Lambda(n)$$

$$= \psi(x; q, a) - \frac{x}{\phi(q)} + O\left( \frac{x(\log x)^{-A}}{Q} \right)$$

by the prime number theorem. Thus to prove Theorem 1 or 2 it suffices to show for $1 \leq i \leq 4$ that

$$(3.1) \quad H_i(Q) := \max_{q \in S_f(Q) \setminus F(Q) \atop (a, q) = 1} |E_i(x; q, a)| \ll \frac{x|S_f(Q)|}{Q(log x)^A}. $$
The case \( i = 1 \) is obvious from the Brun–Titchmarsh theorem. A partial
summation, together with an elementary argument, gives
\[
E_3(x; q, a) \ll Z x^\varepsilon \ll \frac{x}{Q (\log x)^A},
\]
and yields (3.1) for \( i = 3 \).

For \( i = 4 \), we appeal to the work of Mikawa and Peneva [11, Section 3.1].
Their bound \( Q < x^{8/19 - \varepsilon} \) is not used in this part of the argument, which
yields
\[
\sum_{q \in S_f(Q)} \max_{(a,q)=1} |E_4(x; q, a)| \ll \frac{x |S_f(Q)|}{Q (\log x)^A}.
\]

Turning to \( H_2(Q) \), let \( q \in S_f(Q) \), \((a,q)=1\). Then
\[
E_2(x; q, a) = - \sum_{m,n \leq Z} \frac{\Lambda(m) \mu(n)}{(mn,q)=1} \left\{ \sum_{l \leq x/mn} 1 - \frac{1}{\phi(q)} \sum_{l \leq x/mn (l,mn,q)=1} 1 \right\}.
\]

We can change the inner summation condition \((l,mn,q)=1\) to \((l,q)=1\) because \((mn,q)=1\). An easy computation yields
\[
\frac{1}{\phi(q)} \sum_{l \leq x/mn (l,q)=1} 1 - \frac{x}{qmn} = O(\tau(q)/\phi(q)),
\]

Thus it suffices for the proof of Theorem 1 or 2 to show for \( Q \) in the appro-
priate interval that
\[
\sum_{q \in S_f(Q) \setminus F(Q)} \max_{(a,q)=1} |I(x; q, a)| \ll \frac{x |S_f(Q)|}{Q (\log x)^A}.
\]

Likewise for Theorem 3 it suffices to show that
\[
\sum_{p^2 \in (Q,2Q) \setminus F(Q)} \max_{p^1 | a} |I(x, p^2, a)| \ll xQ^{-1/2} (\log x)^{-A}.
\]

4. Sums over characters of absolute values of Dirichlet polyno-
mials. Our strategy resembles that of Iwaniec [9, Section 2] in dealing with sieve remainder terms. We begin with some material about sums over sets
of characters $\chi \pmod{q}$, $q \in S_f(Q) \setminus F(Q)$, of the absolute values of certain Dirichlet polynomials.

**Proposition 1.** Let $M_1, \ldots, M_{15}$ be numbers with $M_1 \geq \cdots \geq M_{15} \geq 1$, and suppose $\{1, \ldots, 15\}$ partitions into subsets $A$ and $B$ such that

$$\prod_{i \in A} M_i \ll x^{9/20-3\varepsilon/4}, \quad \prod_{i \in B} M_i \ll x^{9/20-3\varepsilon/4}.$$  

Let $a_i(m) \ (M_i/2 < m \leq M_i)$ be a complex sequence with $|a_i(m)| \leq \log m \quad (1 \leq i \leq 15, M_i/2 < m \leq M_i)$. Suppose that whenever $M_i > x^{1/8}$ then either

$a_i(m) = 1 \quad (M_i/2 < m \leq M_i)$

or

$a_i(m) = \log m \quad (M_i/2 < m \leq M_i)$.

Let $M_i(s, \chi) = \sum_{M_i/2 < m \leq M_i} a_i(m) \chi(m)m^{-s}$ and

$$L = \frac{x}{M_1 \cdots M_{15}}, \quad B(s, \chi) = \sum_{n \leq L} \chi(n)n^{-s}.$$  

Then for $\Re(s) = 1/2$ and

$$Q \ll x^{9/20-\varepsilon},$$  

we have

$$\sum_{q \in S_f(Q) \setminus F(Q)} \sum_{\chi \pmod{q}} \chi(0) \prod_{i=1}^{15} |M_i(s, \chi)| \ll |s|^{3} |S_f(Q)| x^{1/2-3\delta}.$$  

**Proposition 2.** For $f(X) = X^2$, the assertion of Proposition 1 remains true if we replace $9/20$ by $43/90$ in (4.1), and replace (4.2) by

$$Q \ll x^{43/90-\varepsilon}.$$  

**Proposition 3.** Suppose that $f(X) = X^2$ and

$$Q \ll x^{1/2-\varepsilon}.$$  

The assertion of Proposition 1 remains true if we replace $9/20$ by $1/2$ in (4.1), and replace $q$ in (4.3) by $p^2$, with $p$ prime.

The following basic lemmas are needed.

**Lemma 7.** We have, for $q \geq 2$, $L \geq 1$,

$$\sum_{\chi \pmod{q}} \left| \sum_{l \leq L} \chi(l) l^{-1/2-it} \right|^4 \ll q(|t| + 1) \log^6 qL(|t| + 1).$$  

**Proof.** This is [9, Lemma 3].
LEMMA 8. For any complex numbers $a_n$ ($N < n \ll N$),
\[\sum_{\chi \pmod{q}} \left| \sum_{N < n \ll N} a_n \chi(n) \right|^2 \ll (N + q) \sum_{N < n \ll 2N} |a_n|^2.\]

Proof. See [12, Theorem 6.2].

LEMMA 9. For any complex numbers $a_n$ ($N < n \ll N$) and $V > 0$, and $G = \sum_{N < n \ll N} |a_n|^2$,
\[\left| \left\{ \chi \pmod{q} : \sum_{N < n \ll N} a_n \chi(n) > V \right\} \right| \ll GNV^{-2} + q^{1+\varepsilon}G^3NV^{-6}.
\]

Proof. See e.g. Jutila [10].

Proof of Proposition 1. We prove (4.3) simply by showing for a fixed $q$ in $S_f(Q) \setminus F(Q)$ that, writing
\[M = \prod_{i \in A} M_i, \quad N = \prod_{i \in B} M_i\]
and
\[M(s, \chi) = \prod_{i \in A} M_i(s, \chi), \quad N(s, \chi) = \prod_{i \in B} M_i(s, \chi),\]
we have
\[\sum_{\chi \pmod{q}, \chi \neq \chi_0} |B(s, \chi)M(s, \chi)N(s, \chi)| \ll |s|^3x^{1/2-3\delta}.
\]

We have trivially
\[B(s, \chi) \ll L^{1/2}, \quad M(s, \chi) \ll M^{1/2+\delta}, \quad N(s, \chi) \ll N^{1/2+\delta}.
\]
Thus the characters $\chi \neq \chi_0$ for which one of these three Dirichlet polynomials has absolute value less than $(\phi(q)x^{5\delta})^{-1}$ can be neglected. We partition the remaining characters into $O((\log x)^3)$ subsets $A_q(U, V, W)$ of characters satisfying
\[U < |B(s, \chi)| \leq 2U, \quad V < |M(s, \chi)| \leq 2V, \quad W < |N(s, \chi)| \leq 2W,
\]
where $U \ll L^{1/2}, V \ll M^{1/2+\delta}, W \ll N^{1/2+\delta}$. To prove (4.8), it suffices to show for each triple $U, V, W$ that
\[UVW|A_q(U, V, W)| \ll |s|^3x^{1/2-4\delta}.
\]

From the above lemmas applied to $B(s, \chi), M(s, \chi), N(s, \chi), B(s, \chi)^2$ we obtain
\[|A_q(U, V, W)| \ll x^\delta|s|^{1+\delta}P,\]
where
\[P = \min \left( \frac{M + Q}{V^2}, \frac{N + Q}{W^2}, \frac{Q}{U^4}, \frac{M}{V^2} + \frac{QM}{V^6}, \frac{N}{W^2} + \frac{QN}{W^6}, \frac{L^2}{U^4} + \frac{QL^2}{U^{12}} \right).
\]
Thus it suffices to show
\[ UVWP \ll x^{1/2-5\delta}. \]

We consider four cases.

**Case 1:** \( P \leq 2V^{-2}M, P \leq 2W^{-2}N \). In this case we apply Lemma \ref{lemma6} with \( \gamma = 1/10 \); we have \( MN \leq x^{9/10} \) and \( L \geq x^{1/10} \). Since \( q \in S_f(Q) \setminus F(Q) \), we obtain
\[ U \ll |s|L^{1/2}x^{-5\delta}, \]
and
\[ UVWP \leq 2UVW \min(V^{-2}M, W^{-2}N) \ll U(MN)^{1/2} \ll |s| x^{1/2-5\delta}. \]

**Case 2:** \( P > 2V^{-2}M, P > 2W^{-2}N \). In this case,
\[ P \leq 2 \min\{QV^{-2}, QW^{-2}, QMV^{-6}, QNW^{-6}, QU^{-4}, L^2U^{-4}\} \]
\[ + 2 \min\{QV^{-2}, QW^{-2}, QMV^{-6}, QNW^{-6}, QU^{-4}, QL^2U^{-12}\} \]
\[ \leq 2(QV^{-2})^{5/16}(QW^{-2})^{5/16}(QMV^{-6})^{1/16}(QNW^{-6})^{1/16} \]
\[ \times \left( \min(QU^{-4}, L^2U^{-4}) \right)^{1/4} \]
\[ + 2 \min\{(QV^{-2})^{5/16}(QW^{-2})^{5/16}(QMV^{-6})^{1/16}(QNW^{-6})^{1/16}(QU^{-4})^{1/4}, \]
\[ (QV^{-2})^716(QW^{-2})^716(QMV^{-6})^{1/48}(QNW^{-6})^{1/48}(QL^2U^{-12})^{1/12}\} \]
\[ = 2(UVW)^{-1}Q(MN)^{1/16}\left\{ \min(1, Q^{-1/4}L^{1/2}) + \min(1, L^{1/6}(MN)^{-1/24}) \right\} \]
\[ \ll (UVW)^{-1}(x^{1/16}Q^{1/32} + x^{1/20}Q) \ll (UVW)^{-1}x^{1/2-\varepsilon} \]
since \( Q \ll x^{9/20-\varepsilon} \).

**Case 3:** \( P > 2V^{-2}M, P \leq 2W^{-2}N \). In this case,
\[ P \leq 2 \min\{QV^{-2}, NW^{-2}, QMV^{-6}, QU^{-4}, L^2U^{-4}\} \]
\[ + 2 \min\{QV^{-2}, NW^{-2}, QMV^{-6}, QU^{-4}, QL^2U^{-12}\} \]
\[ \leq 2(QV^{-2})^{1/8}(NW^{-2})^{1/2}(QMV^{-6})^{1/8}\left( \min(QU^{-4}, L^2U^{-4}) \right)^{1/4} \]
\[ + 2 \min\{(QV^{-2})^{1/8}(NW^{-2})^{1/2}(QMV^{-6})^{1/8}(QU^{-4})^{1/4}, \]
\[ (QV^{-2})^{3/8}(NW^{-2})^{1/2}(QMV^{-6})^{1/24}(QL^2U^{-12})^{1/12}\} \]
\[ = 2(UVW)^{-1}(QN)^{1/2}M^{1/8}\left\{ \min(1, Q^{-1/4}L^{1/2}) + \min(1, L^{1/6}M^{-1/12}) \right\} \]
\[ \ll (UVW)^{-1}(x^{1/8}Q^{7/16}N^{3/8} + x^{1/12}Q^{1/2}N^{5/12}) \ll (UVW)^{-1}x^{1/2-\varepsilon} \]
since \( Q \ll x^{9/20-\varepsilon} \) and \( N < Qx^{\varepsilon/2} \). (There is a little to spare in Case 3.)

**Case 4:** \( P > 2W^{-2}N, P \leq 2V^{-2}M \). We proceed as in Case 3, interchanging the roles of \( M \) and \( N \).

This completes the proof of Proposition \ref{proposition}.
We break the argument for Proposition 2 into a number of lemmas. We maintain the definitions (4.6), (4.7) and let \( M = x^{\alpha_1}, N = x^{\alpha_2}, Q = x^\theta \). We may suppose that \( \theta > 9/20 - \varepsilon \) and \( \alpha_2 \leq \alpha_1 \).

It suffices to show for \( 0 \leq \lambda \leq \theta \) that

\[
(4.9) \quad \sum_{q \in S_f(Q) \setminus F(Q)} \sum_{\chi \pmod{q}, \chi \neq \chi_0 \atop x^\lambda < \text{cond} \chi \leq 2x^\lambda} |B(s, \chi) M_1(s, \chi) \ldots M_{15}(s, \chi)| \ll |s|^{3x^{1/2-4\delta}Q^{1/2}}.
\]

A strategy which works for some triples \( \lambda, \alpha_1, \alpha_2 \) is to show that, for \( q \in S_f(Q) \setminus F(Q) \),

\[
(4.10) \quad \sum_{\chi \pmod{q}, \chi \neq \chi_0 \atop x^\lambda < \text{cond} \chi \leq 2x^\lambda} |B(s, \chi) M(s, \chi) N(s, \chi)| \ll |s|^{3x^{1/2-4\delta}}.
\]

**Lemma 10.** Let \( q \in S_f(Q) \setminus F(Q) \). Suppose that

\[
(4.11) \quad \alpha_1 + \alpha_2 < 8 - 16\lambda - 200\delta,
\]

\[
(4.12) \quad \alpha_1 < 1 - 6\lambda/5 - 20\delta.
\]

Then (4.10) holds. In particular, it holds if \( \lambda \leq (5\theta + \varepsilon)/6 \).

**Proof.** When \( \chi \) is counted in the sum in (4.10),

\[
M(s, \chi) = \sum_{(n, q) = 1} a(n) \chi'(n)n^{-s}
\]

with \( a(n) \ll x^\delta \) and some primitive character \( \chi' \pmod{r} \), \( r \leq 2x^\lambda \); similarly for \( N(s, \chi) \). We may improve our bounds for mean and large values of these Dirichlet polynomials, replacing \( q \) by \( x^\lambda \) in each case. Thus

\[
|A_q(U, V, W)| \ll \min(MV^{-2} + x^{\lambda+\delta}V^{-2}, NW^{-2} + x^{\lambda+\delta}W^{-2}, MV^{-2} + x^{\lambda+\delta}MV^{-6}, NW^{-2} + x^{\lambda+\delta}NW^{-6}).
\]

To get variants of the other quantities in the definition of \( P \), we observe that

\[
B(s, \chi) = \sum_{n \leq L} \left( \sum_{d \mid q \atop d \mid n} \mu(d) \right) \chi'(n)n^{-s} = \sum_{d \mid q} \frac{\mu(d)\chi'(d)}{d^s} \sum_{k \leq L/d} \chi'(k)k^{-s}.
\]

If \( |B(s, \chi)| \geq U \), then

\[
\left| \sum_{k \leq L/d} \chi'(k)k^{-s} \right| \geq Ux^{-\delta/12}
\]

for some \( d \) with \( d \mid q \), and consequently

\[
|A_q(U, V, W)| \ll \min(x^{\lambda+\delta}U^{-4}|s|^{1+\delta}, x^\delta L^2U^{-4} + x^{\lambda+\delta}L^2U^{-12}).
\]
Let
\[ P' = \min \left( \frac{M + x^\lambda}{V^2}, \frac{N + x^\lambda}{W^2}, \frac{x^\lambda M}{U^4}, \frac{M}{V^2} + \frac{x^\lambda N}{W^2} + \frac{x^\lambda N}{U^4} + \frac{x^\lambda L^2}{U^12} \right). \]

The bound (4.10) will follow if we show that
\[ UVWP' \ll |s|^{1+ \delta} x^{1/2 - 7\delta}. \]

As in the preceding proof, we break the argument into Cases 1–4, defined exactly as before with \( P \) replaced by \( P' \). Case 1 proceeds as before. In Case 2,
\[ P' \leq 2 \min \{x^\lambda V^{-2}, x^\lambda W^{-2}, x^\lambda MV^{-6}, x^\lambda NW^{-6}, x^\lambda U^{-4}\} \leq 2(UVW)^{-1}x^\lambda (MN)^{1/16} \ll (UVW)^{-1}x^{1/2 - 7\delta} \]
from (4.11). In Case 3, the argument used in proving (4.8) yields
\[ P' \ll (UVW)^{-1}(x^{1/8+7\lambda/16}N^{3/8} + x^{1/12+\lambda/2}N^{5/12}) \ll (UVW)^{-1}x^{1/2 - 7\delta}. \]

To see this, note that
\[ \frac{1}{8} + \frac{7\lambda}{16} + \frac{3\alpha_2}{8} < \frac{1}{2} - 7\delta \]
since \( \alpha_2 < 1 - 7\lambda/6 - 20\delta \), and
\[ \frac{1}{12} + \frac{\lambda}{2} + \frac{5\alpha_2}{12} < \frac{1}{2} - 7\delta \]
since \( \alpha_2 < 1 - 6\lambda/5 - 20\delta \). In Case 4, proceed as in Case 3, with \( M \) and \( N \) interchanged.

This proves the first assertion of the lemma. For the second assertion, we observe that if \( \lambda \leq (5\theta + \varepsilon)/6 \), then
\[ \alpha_1 \leq \theta + \varepsilon/4 < 1 - \theta - \varepsilon < 1 - 6\lambda/5 - 20\delta, \]
\[ \alpha_1 + \alpha_2 \leq 2\theta + \varepsilon/2 < 8 - 80\theta/6 - 20\varepsilon < 8 - 16\lambda - 200\delta. \]
We obtain (4.10) in view of the first assertion of the lemma.

In view of Lemma 10, we suppose for the remainder of the proof of Proposition 2 that
\[ \lambda > (5\theta + \varepsilon)/6. \]

We now bring the work of Baier and Zhao into play.

**Lemma 11.** Let \( a_1, \ldots, a_N \) be complex numbers and
\[ T(\alpha) = \sum_{n=1}^{N} a_n e(n\alpha), \quad G = \sum_{n=1}^{N} |a_n|^2. \]
Let $g \in \mathbb{N}$, $g \leq Q$. Then
\[
\sum_{q \leq Q} \sum_{a=1}^{gq^2} \left| T\left( \frac{a}{gq^2} \right) \right|^2 \ll (QN)^\varepsilon (g^2Q^3 + gQ^{1/2}N)G.
\]

**Proof.** We deduce this from the work of Baier and Zhao \cite{??}, where the case $g = 1$ is treated. By \cite[Theorem 2.1]{??},

\[
(4.14) \quad \sum_{q \leq Q} \sum_{a=1}^{gq^2} \left| T\left( \frac{a}{gq^2} \right) \right|^2 \ll K(\Delta)(N + \Delta^{-1})G.
\]

Here

\[
K(\Delta) = \max_{\alpha \in \mathbb{R}} \sum_{q \leq Q} \sum_{a=1}^{gq^2} \|a/(gq^2) - \alpha\| \leq \Delta
\]

(4.15)

We observe that the conditions of summation in (4.15) imply

\[
(4.16) \quad \left\| \frac{a}{q^2} - g\alpha \right\| \leq g\Delta.
\]

If there are $\mathcal{N}(\alpha)$ solutions of (4.16) with $(a, q) = 1$, $1 \leq a \leq q^2$, $q \leq Q$, then there are $g\mathcal{N}(\alpha)$ solutions with $(a, q) = 1$, $1 \leq a \leq gq^2$, $q \leq Q$. Now according to \cite[Section 11]{??}, with $g\Delta$ in place of $\Delta$,

\[
\mathcal{N}(\alpha) \ll (Q\Delta^{-1})^\varepsilon (Q^3(g\Delta) + Q^{7/4}(g\Delta)^{1/2} + Q(g\Delta)^{1/4} + Q^{1/2}).
\]

Take $\Delta = N^{-1}$ to obtain

\[
(N + \Delta^{-1})\mathcal{N}(N^{-1}) \ll (QN)^\varepsilon (g^2Q^3 + g^{3/2}Q^{7/4}N^{1/2} + g^{5/4}QN^{3/4} + gQ^{1/2}N).
\]

The lemma follows on combining this with (4.14), since

\[
g^{3/2}N^{1/2}Q^{7/4} = (g^2Q^3)^{1/2}(gQ^{1/2}N)^{1/2},
\]

\[
g^{5/4}QN^{3/4} \leq (g^2Q^3)^{1/4}(gQ^{1/2}N)^{3/4}.
\]

**Lemma 12.** Let $c_1, \ldots, c_J$ be complex numbers. Let

\[
T(J, \lambda) = \sum_{Q^{1/2} < q \leq 2Q^{1/2}} \sum_{\chi \pmod{q^2}, \chi \neq \chi_0 \atop x^\lambda < \text{cond } \chi \leq 2x^\lambda} \left| \sum_{m=1}^J c_m\chi(m) \right|^2.
\]

Then

\[
T(J, \lambda) \ll (QJ)^{2\delta} (Q^{3/2} + Q^{7/4}x^{-3\lambda/2}J) \sum_{m=1}^J |c_m|^2.
\]
Proof. The conductor of a character $\chi$ counted in $T(J, \lambda)$ may be written as $gk^2$ where $g$ is square-free, $gk^2 \in (x^\lambda, 2x^\lambda]$. These $\chi$ counted by $T(J, \chi)$ arising from a given primitive character $\chi'$ to modulus $gk^2$ may be written as

$$\chi'_v(m) = \begin{cases} \chi'(m) & \text{if } (m, v) = 1, \\ 0 & \text{if } (m, v) > 1, \end{cases}$$

where $v$ takes integer values such that

$$vgk^2 = q^2 \in (Q, 2Q].$$

Clearly all such $v$ have

$$g \mid v, \quad v \in (Qx^{-\lambda}/2, 2Qx^{-\lambda}).$$

Let

$$a_{v,m} = \begin{cases} c_m & \text{if } (m, v) = 1, \\ 0 & \text{if } (m, v) > 1. \end{cases}$$

For a given triple $k, g, v$ satisfying (4.17), (4.18), we have

$$\sum_{J} \sum_{m=1}^{\star} c_m \chi'_v(m)^2 = \sum_{J} \sum_{m=1}^{\star} a_{v,m} \chi'(m)^2 \leq \frac{\phi(gk^2)}{gk^2} \sum_{a=1 \atop (a, gk^2) = 1}^{gk^2} \left| T_v \left( \frac{a}{gk^2} \right) \right|^2,$$

where

$$T_v(\alpha) = \sum_{m=1}^{J} a_{v,m} e(m\alpha).$$

Here we appeal to (10) in [5, Section 27]. Combining this with Lemma 11, we find that, for a given pair $g, v$ satisfying (4.18),

$$\sum_{Q^{1/2}/(vg)^{1/2} \leq k \leq (2Q)^{1/2}/(vg)^{1/2}} \sum_{J} \sum_{m=1}^{\star} c_m \chi'_v(m)^2 \ll (QJ)^{\delta} \left( \frac{g^2Q^{3/2}}{(vg)^{3/2}} + \frac{gQ^{1/4}}{(vg)^{1/4}}J \right) \sum_{m=1}^{J} |c_m|^2 \ll (QJ)^{\delta} (Q^{3/2}v^{-1} + Q^{1/4}Jv^{1/2}) \sum_{m=1}^{J} |c_m|^2.$$ 

Summing over all pairs $v, g$ satisfying (4.18), we obtain

$$T(J, \lambda) \ll (QJ)^{2\delta} (Q^{3/2} + Q^{1/4}J(Qx^{-\lambda})^{3/2}) \sum_{m=1}^{J} |c_m|^2,$$

as claimed. \blacksquare
Lemma 13. Let
\[ H(s, \chi) = \sum_{n \leq H} a_n \chi(n)n^{-s}, \quad K(s, \chi) = \sum_{n \leq K} b_n \chi(n)n^{-s}, \]
with \(|a_n| \leq \tau(n)^B\), \(|b_n| \leq \tau(n)^B\) for an absolute constant \(B\). If
\[ HK \ll x, \quad K \leq H \ll x^{1 + 3\lambda/2 - 9\theta/4 - 16\delta}, \]
then
\[ (4.19) \quad \sum_{q \in S_f(Q)} \sum_{\chi \not\equiv \chi_0 \pmod{q}, \chi^x < \text{cond} \chi} |H(s, \chi)K(s, \chi)| \ll x^{1/2 - 6\delta}Q^{1/2}. \]

Proof. By Lemma 12 and the Cauchy–Schwarz inequality, the left-hand side of (4.19) is
\[ \ll x^{2\delta}(Q^{3/4} + Q^{7/8} x^{-3\lambda/4} H^{1/2})(Q^{3/4} + Q^{7/8} x^{-3\lambda/4} K^{1/2}) \]
\[ \ll x^{2\delta}(Q^{3/2} + Q^{7/4} x^{-3\lambda/2+1/2} + Q^{13/8} x^{-3\lambda/4} H^{1/2}). \]
Now
\[ x^{2\delta} Q^{3/2} \ll Q^{1/2 + 1/2 - 6\delta} \]
since \(\theta < 1/2 - \varepsilon\). Also
\[ x^{2\delta} Q^{7/4} x^{-3\lambda/2+1/2} \ll Q^{1/2 + 1/2 - 6\delta} \]
from (4.13). Finally,
\[ x^{2\delta} Q^{13/8} x^{-3\lambda/4} H^{1/2} \ll Q^{1/2 + 1/2 - 6\delta} \]
since \(H \ll x^{1-9\theta/4+3\lambda/2-16\delta}\). 

Lemma 14. Let \(\beta_1 \geq \cdots \geq \beta_R \geq 0\), \(\sum \beta_j = 1/2\), \(R \geq 2\). Suppose that \(\sum \beta_j \leq 3/5\). Then there is a sum
\[ \sigma = \sum_{j=1}^r \beta_j, \quad 2 \leq r \leq R, \]
such that \(\sigma \in [2/5, 3/5]\).

Proof. Suppose the contrary; then \(\beta_1 + \beta_2 < 2/5\),
\[ \beta_1 + \beta_2 + \beta_3 \leq \frac{3}{2}(\beta_1 + \beta_2) < \frac{3}{5}, \quad \text{hence} \quad \beta_1 + \beta_2 + \beta_3 < \frac{2}{5}. \]
Arguing in this way we prove for \(j = 4, \ldots, R\) that
\[ \beta_1 + \cdots + \beta_j \leq \frac{j}{j-1}(\beta_1 + \cdots + \beta_{j-1}) < \frac{3}{5}, \quad \text{hence} \quad \beta_1 + \cdots + \beta_j < \frac{2}{5}. \]
When \(j = R\), we have a contradiction. 

Lemma 15. Suppose that
\[ \lambda \geq -\frac{4}{15} + \frac{3\theta}{2} + 12\delta. \]
Then (4.9) holds.

Proof. We decompose \( B(s, \chi) \) into \( O(\log x) \) Dirichlet polynomials of the form
\[ M_{16}(s, \chi) = \sum_{M_{16}/2 < m \leq M_{16}} \chi(m)m^{-s}. \]
It suffices to prove the analog of (4.9) with \( M_{16} \) in place of \( B \) and \( 6\delta \) in place of \( 4\delta \). Fix \( M_{16} \) and rearrange \( M_1, \ldots, M_{16} \) as \( N_1 \geq \cdots \geq N_{16} \); write \( N_i(s) \) for the corresponding Dirichlet polynomials and
\[ N_i = x^{\beta_i}. \]
Thus \( \beta_1 \geq \cdots \geq \beta_{16} \geq 0, \beta_1 + \cdots + \beta_{16} \leq 1. \)
We treat the rather trivial case
\[ \beta_1 + \cdots + \beta_{16} < 1/2 \]
by applying Lemma 13 with \( K(s, \chi) = 1, \)
\[ H(s, \chi) = N_1(s, \chi) \ldots N_{16}(s, \chi), \quad H = x^{\beta_1 + \cdots + \beta_{16}} < x^{1/2} < x^{1 + 3\lambda/2 - 9\theta/4 - \varepsilon} \]
since \( 3\lambda/2 > 5\theta/4 \) and \( \theta < 1/2 - \varepsilon \).
Now suppose that \( \beta_1 + \cdots + \beta_{16} \geq 1/2 \), so that Lemma 14 is applicable.
Suppose first that \( \beta_1 + \beta_2 > 3/5 \). We write \( N_0(s) = N_3(s) \ldots N_{16}(s), \)
\[ A(U_0, U_1, U_2) = \{ \chi \pmod{q} : q \in S_f(Q), \chi \neq \chi_0, x^\lambda < \text{cond} \chi \leq 2x^\lambda, \]
\[ U_j < |N_j(s)| \leq 2U_j \ (j = 0, 1, 2) \} \].
Arguing as in the proof of Proposition 1, it suffices to show that
(4.21) \[ U_0 U_1 U_2 |A(U_0, U_1, U_2)| \ll Q^{1/2}|s|^{3x^{1/2-6\delta}}. \]
Since \( N_1 \geq x^{3/10} \), we have
\[ |A(U_0, U_1, U_2)| \ll Q^{1/2}|s|^{1+\delta}x^{\theta+\delta}U_1^{-4} \]
from Lemma 7 (and, if needed, a partial summation). Next
\[ |A(U_0, U_1, U_2)| \ll Q^{1/2}|s|^{1+\delta}x^{\theta+\delta}U_2^{-4} \]
from Lemma 7 (if \( N_2^2 > x^\theta \)) and Lemma 8 (if \( N_2^2 \leq x^\theta \)). We have
\[ |A(U_0, U_1, U_2)| \ll Q^{1/2}x^{\theta+\delta}U_0^{-2} \]
from Lemma 8 since \( N_0 \ll x^{2/5} \ll x^\theta \). Hence
\[ |A(U_0, U_1, U_2)| \ll Q^{1/2}|s|^{1+\delta}x^{\theta+\delta}(U_1^{-4})^{1/4}(U_2^{-4})^{1/4}(U_0^{-2})^{1/2}, \]
and (4.21) follows at once.
Now suppose that \( \beta_1 + \beta_2 \leq 3/5 \). By Lemma \[14\] there is a subset \( W \) of \( \{1, \ldots, 16\} \) such that
\[
x^{2/5} \ll \prod_{j \in W} M_j \ll x^{3/5}.
\]
We now apply Lemma \[13\] with \( \{H, K\} = \{\prod_{j \in W} (2M_j), \prod_{j \leq 16, j \notin W} (2M_j)\} \), \( H \geq K \). We have
\[
x^{1/2} \ll H \ll x^{3/5} \ll x^{1 + 3\lambda/2 - 9\theta/4 - 16\delta}
\]
by hypothesis. This gives the analog of (4.9) with \( M_{16} \) in place of \( B \) and \( 6\delta \) in place of \( 4\delta \), and the lemma follows at once.

**Lemma 16.** Suppose that
\[
\alpha_1 \geq \frac{9\theta}{4} - \frac{3\lambda}{2} + 16\delta.
\]
Then (4.9) holds.

**Proof.** Since \( \alpha_1 < 1/2 \), this is a straightforward consequence of Lemma \[13\] with \( K(x, \chi) = M(s, \chi), H(s, \chi) = N(s, \chi)B(s, \chi) \).

**Lemma 17.** Suppose that
\[
\alpha_1 < 4 - 8\lambda - 100\delta.
\]
Then (4.9) holds.

**Proof.** We have (4.11) since \( \alpha_2 \leq \alpha_1 \). In view of Lemma \[10\] we need only show that
\[
\alpha_1 < 1 - \frac{6\lambda}{5} - 20\delta.
\]
By Lemma \[16\] we may suppose that
\[
\alpha_1 < \frac{9\theta}{4} - \frac{3\lambda}{2} + 16\delta.
\]
Hence we can establish
\[
\alpha_1 < 1 - \frac{6\lambda}{5} - 20\delta
\]
by using \( \lambda > (5\theta + \varepsilon)/6, \theta < 1/2 \) to obtain
\[
1 + \frac{3\lambda}{10} > \frac{9\theta}{4} + 40\delta.\]

**Proof of Proposition 2** We recall that it suffices to prove (4.9). By Lemma \[15\] we may suppose that
\[
\lambda < -\frac{4}{15} + \frac{3\theta}{2} + 12\delta.
\]
In view of Lemmas 16 and 17, it remains to show that the intervals \([9\theta/4 - 3\lambda/2 + 16\delta, \theta + \varepsilon/4]\) and \([0, 4 - 8\lambda - 100\delta]\) overlap. That is, we need to show

\[4 - 8\lambda - 100\delta > \frac{9\theta}{4} - \frac{3\lambda}{2} + 16\delta,\]

or

\[\frac{13\lambda}{2} < 4 - \frac{9\theta}{4} - 116\delta.\]

Indeed, from (4.22),

\[\frac{13\lambda}{2} < -\frac{26}{15} + \frac{39\theta}{4} + 78\delta < 4 - \frac{9\theta}{4} - 116\delta\]

since \(\theta < 43/90 - \varepsilon\). ■

**Proof of Proposition 3** As in the preceding proof it suffices to show that for each tuple \(M_1, \ldots, M_{15}\),

\[
\sum_{p^2 \in (Q, 2Q]\setminus F(Q)} \sum_{\chi \mod p^2, \chi \neq \chi_0, x^\lambda < \text{cond} \leq 2x^\lambda} |B(s, \chi)M_1(s, \chi) \cdots M_{15}(s, \chi)| \ll |s|^{3}x^{1/2-4\delta}Q^{1/2}.
\]

The conductor of each character counted in (4.23) is either \(p\) or \(p^2\), so that

\[\text{cond} \chi \in \left(Q^{1/2}, (2Q)^{1/2}\right] \cup (Q, 2Q).\]

Thus the sum in (4.23) is empty unless

\[\lambda = \theta/2 \quad \text{or} \quad \lambda = \theta.\]

For \(\lambda = \theta/2\), we obtain (4.23) as a consequence of Lemma 10. (Note that no inequality stronger than \(\theta < 1/2 - \varepsilon\) was used in the proofs of Lemmas 10–17.) For \(\lambda = \theta\), we have

\[\lambda > -\frac{4}{15} + \frac{3\theta}{2} + 12\delta,\]

with something to spare. Now (4.23) is a consequence of Lemma 15 ■

**5. Proofs of Theorems 1, 2 and 3** We work with the Riesz means

\[A_k(x, q, a, d) = \frac{1}{k!} \sum_{\substack{l \leq x \mod q, x \mod d \mod l \equiv a \mod q \mod d}} \left(\log \frac{x}{l}\right)^k.\]

Ultimately we are interested in \(A_0\); the presence of the factor \(s^{-5}\) in (5.5) below is the reason for working initially with \(A_4\).

Let us write the associated remainder term as

\[r_k(x, q, a, d) = A_k(x, q, a, d) - \frac{x}{qd}.\]
We borrow from Iwaniec [9, (2.5)] the inequalities

\[ r_{k-1}(x, q, a, d) \leq \left( \frac{e^\lambda - 1}{\lambda} - 1 \right) \frac{x}{qd} + \frac{1}{\lambda} \left[ r_k(e^\lambda x, q, a, d) - r_k(x, q, a, d) \right], \]

(5.1) \[ r_{k-1}(x, q, a, d) \geq \left( \frac{1 - e^{-\lambda}}{\lambda} - 1 \right) \frac{x}{qd} + \frac{1}{\lambda} \left[ r_k(x, q, a, d) - r_k(e^{-\lambda} x, q, a, d) \right]. \]

If \( u_d \geq 0 \) \((D_1 < d \leq D)\), it follows that

\[ \sum_{D_1 < d \leq D} u_d r_{k-1}(x, q, a, d) \leq \left( \frac{e^\lambda - 1}{\lambda} - 1 \right) \frac{x}{q} \sum_{D_1 < d \leq D} u_d \frac{d}{d}
+ \frac{1}{\lambda} \left[ \sum_{D_1 < d \leq D} u_d r_k(e^\lambda x, q, a, d) - \sum_{D_1 < d \leq D} u_d r_k(x, q, a, d) \right]. \]

(5.2)

There is a similar lower bound for the left-hand side of (5.2), which follows from (5.1). We see that for \( 0 < \lambda < 1 \),

\[ \sum_{q \in S_f(Q) \setminus F(Q)} u_d r_{k-1}(x, q, a, d) \leq \left( \frac{e^\lambda - 1}{\lambda} - 1 \right) \frac{x}{q} \sum_{D_1 < d \leq D} u_d \frac{d}{d}
+ \frac{1}{\lambda} \left[ \sum_{D_1 < d \leq D} u_d r_k(e^\lambda x, q, a, d) - \sum_{D_1 < d \leq D} u_d r_k(x, q, a, d) \right]. \]

(5.3)

For \( q \in S_f(Q) \setminus F(Q) \), let \( a^{(q)} \) be an integer coprime to \( q \). Suppose that

\[ \sum_{D_1 < d \leq D} \frac{u_d}{d} \ll x^{\eta/6} \]

and

\[ \sum_{q \in S_f(Q) \setminus F(Q)} \left| \sum_{D_1 < d \leq D} u_d r_k(x, q, a^{(q)}, d) \right| \ll \frac{|S_f(Q)|}{Q} x^{1-\eta} \]

for some \( \eta > 0 \), whenever \( Q \ll x^\alpha \). Taking \( \lambda = x^{-\eta/2} \), we deduce from (5.3) that

\[ \sum_{q \in S_f(Q) \setminus F(Q)} \left| \sum_{D_1 < d \leq D} u_d r_{k-1}(x, q, a^{(q)}, d) \right| \ll \frac{|S_f(Q)|}{Q} x^{1-\eta/3} \]

for \( Q \ll x^\alpha \).

We are now ready to make a suitable inference from the work of Section 4 about remainders \( r_0(x, q, a^{(q)}, d) \).
Lemma 18. Let \( a_i(m) \) \((M_i/2 < m \leq M_i)\) be nonnegative sequences satisfying the hypotheses of Proposition 1. Let
\[
u_d = \sum_{d=m_1 \ldots m_{15}} a_1(m_1) \ldots a_{15}(m_{15})
\]
for \( D_1 < d \leq D \), with \( D = M_1 \ldots M_{15}, D_1 = 2^{-15}D \). Suppose that (4.2) holds. Then for every \( A > 0 \),
\[
\sum_{q \in S_f \setminus F(Q)} \sum_{D_1 < d \leq D} |u_d r_0(x, q, a(q), d)| \ll \frac{x |S_f(Q)|}{Q (\log x)^A}.
\]

Proof. In view of the above discussion, it suffices to prove that
\[
\sum_{q \in S_f \setminus F(Q)} \sum_{D_1 < d \leq D} u_d r_4(x, q, a(q), d) \ll \frac{x^{1-\delta} |S_f(Q)|}{Q}
\]
for \( Q \ll x^{9/20-\varepsilon} \). We represent \( r_4(x, q, a(q), d) \) in the form
\[
r_4(x, q, a(q), d) = \frac{1}{24 \phi(q)} \sum_{\chi \pmod{q}} \sum_{b \leq x/d} \chi(b) \left( \log \frac{x}{bd} \right)^4 - \frac{x}{qd}
\]
\[
= \frac{1}{24 \phi(q)} \sum_{\chi(q) \pmod{q}} \sum_{b \leq x/d \chi \neq \chi_0} \chi(b) \left( \log \frac{x}{bd} \right)^4 + O\left( \frac{x^\delta}{q} \right)
\]
for \((d, q) = 1\). Since \( D < x^{1-\varepsilon} \), it suffices to show that
\[
(5.4) \sum_{q \in S_f \setminus F(Q)} \sum_{\chi \pmod{q}, \chi \neq \chi_0} \sum_{D_1 < d \leq D} u_d \chi(d) \chi(b) \left( \log \frac{x}{bd} \right)^4 \ll |S_f(Q)| x^{1-\delta}.
\]
We now use the integral representation
\[
(5.5) \int_{(1/2)} \frac{y^s}{s^5} ds = \begin{cases} (\log y)^4 & \text{if } y > 1, \\ 0 & \text{if } y \leq 1 \end{cases}
\]
(e.g. Montgomery and Vaughan [14, p. 143]). This gives
\[
\sum_{D_1 < d \leq D} u_d \chi(d) \sum_{b \leq x/d} \chi(b) \left( \log \frac{x}{bd} \right)^4 = \int_{(1/2)} x^s \sum_{D_1 < d \leq D} u_d \chi(d) d^{-s} \sum_{b \leq x/D_1} \chi(b)d^{-s} \frac{ds}{s^5}.
\]
and
\[ \sum_{q \in S_f(Q) \setminus F(Q) \setminus \chi \not \equiv \chi_0} \sum_{\chi \equiv \chi(q)} \left| \sum_{D_1 < d \leq D} u_d \chi(d) \sum_{b \leq x/d} \chi(b) \left( \log \frac{x}{bd} \right)^4 \right| \leq x^{1/2} \int_{1/2} (q \in S_f(Q) \setminus F(Q) \setminus \chi \not \equiv \chi_0} \sum_{\chi \equiv \chi(q)} \left| \sum_{D_1 < d \leq D} u_d \chi(d) d^{-s} \left| B(s, \chi) \right| \right| \frac{ds}{s^5}. \]

Now (5.4) follows from Proposition 1. ■

Proof of Theorem 4 Let \( a(q) \) be an integer coprime to \( q \) for which \( I(s; q, a) \) is maximal. The left-hand side of (3.2) is
\[ \sum_{q \in S_f(Q) \setminus F(Q) \setminus \chi \not \equiv \chi_0} \left| \sum_{m,n \leq Z} \Lambda(m) \mu(n) r(x, q, a(m), mn) \right|. \]

We recall Heath-Brown’s decomposition \( \mathcal{S} \) of \( \Lambda(m) \) and the slight variant, used e.g. in [4], for the arithmetic function \( \mu(n) \). Taking \( k = 4 \) in both cases, we see that
\[ \Lambda(m) = \sum_{(I_1, \ldots, I_8)} \sum_{m_i \in I_i} (\log m_1) \mu(m_5) \mu(m_6) \mu(m_7) \mu(m_8) \quad (1 \leq m \leq Z), \]
\[ \mu(n) = \sum_{(J_1, \ldots, J_7)} \sum_{n_i \in I_i} \mu(n_4) \ldots \mu(n_7) \quad (1 \leq n \leq Z). \]

Here \( I_i = (a_i, 2a_i] \), \( J_j = (b_j, 2b_j] \), \( \prod_i a_i < Z \), \( \prod_j b_j < Z \), \( 2a_i \leq Z^{1/4} \) if \( i > 4 \), \( 2b_j \leq Z^{1/4} \) if \( j > 3 \). Some of the intervals \( I_i, J_j \) may contain only the integer 1. There are \( O((\log x)^8) \) tuples \( (I_1, \ldots, I_8) \) and \( O((\log x)^7) \) tuples \( (J_1, \ldots, J_7) \) in these expressions. Now write \( \mu(m) = a(m) + b(m) \) where \( a(m) = \max(\mu(m), 0) \). Then
\[ \sum_{m \leq Z, n \leq Z} \Lambda(m) \mu(n) r_0(x, q, a(q), mn) \]
\[ = \sum_{(I_1, \ldots, I_8)} \sum_{(J_1, \ldots, J_7)} \sum_{m_i \in I_i, n_j \in J_j} (\log m_1) (a(m_5) + b(m_5)) \]
\[ \times \ldots (a(n_7) + b(n_7)) r_0(x, q, a(q), m_1 \ldots m_8 n_1 \ldots n_7). \]

This splits in an obvious way into \( O((\log x)^{15}) \) sums with an attached \( \pm \) sign, in each of which the coefficients are nonnegative. Now (3.2) follows on applying Lemma 18 to each of the sums. This completes the proof of Theorem 1. ■

In just the same way, Theorem 2 follows from Proposition 2 and Theorem 3 follows from Proposition 3.
6. A maximal variant of Theorems 1, 2 and 3

Theorem 4. The results of Theorems 1 and 2 remain valid when \( E(x, q) \) is replaced by

\[
\max_{1 \leq y \leq x} E(y, q).
\]

The result of Theorem 3 remains valid when \( E(x, p^2) \) is replaced by

\[
\max_{1 \leq y \leq x} E(y, p^2).
\]

Proof. As above, we write \( \theta = 9/20 - \varepsilon \) (Theorem 1), \( \theta = 43/90 - \varepsilon \) (Theorem 2).

We write \( v = x/(\log x)^A \).

For \( q < x^{1/2}, 1 \leq t \leq x \), we have

\[
\max_{(a,q)=1} \left| \{ p : p \equiv a \pmod{q}, t < p \leq t + v \} \right| \ll \frac{v}{\phi(q) \log x}.
\]

This can easily be deduced from [7, Theorem 2.2], for example.

Let \( v = x_0, x_1, \ldots, x_N \) be a sequence of equally spaced positive numbers,

\[
x_j - x_{j-1} = v \quad (j = 1, \ldots, N), \quad x \leq x_N < x + v.
\]

By Theorem 1 or 2 for \( Q < x^\theta \),

\[
\sum_{q \in S_f(Q)} E(x_j, q) \ll \frac{x_j |S_f(Q)|}{Q (\log x)^{3A+1}} \quad (0 \leq j \leq N).
\]

Let

\[
G_j = \left\{ q \in S_f(Q) : E(x_j, q) > \frac{x_j}{Q (\log x)^{A+1}} \right\}.
\]

From (6.3),

\[
|G_j| \ll \frac{|S_f(Q)|}{(\log x)^{2A}}.
\]

The union \( G = \bigcup_{j=1}^N G_j \) thus satisfies

\[
|G| \ll \frac{N |S_f(Q)|}{(\log x)^{2A}} \ll \frac{x}{v} \frac{|S_f(Q)|}{(\log x)^{2A}} \ll \frac{|S_f(Q)|}{(\log x)^A}
\]

from (6.2).

Now suppose that \( q \in S_f(Q) \setminus G \) and let \( 1 \leq y \leq x \). If \( y < v \), then (6.1) yields

\[
E(y, q) \ll \frac{v}{\phi(q) \log x}.
\]

If \( v < y \leq x \), then \( y \in (x_{j-1}, x_j] \) for some \( j, 1 \leq j \leq N \). Thus, for some \( \lambda \)
in \((0, 1]\),
\[
|\{p : p \equiv a \pmod{q}, p \leq y\}| = |\{p : p \equiv a \pmod{q}, p \leq x_{j-1}\}|
\]
\[
+ \lambda|\{p : p \equiv a \pmod{q}, x_{j-1} < p \leq x_j\}|
\]
\[
= \frac{x_{j-1}}{\phi(q) \log x_{j-1}} + O\left(\frac{x}{Q(\log x)^{A+1}}\right) + O\left(\frac{v}{\phi(q) \log x}\right)
\]
by (6.1) and the condition \(q \in S_f(Q) \setminus G_j\). After an application of the mean value theorem, we obtain
\[
|\{p : p \equiv a \pmod{q}, p \leq y\}| = \frac{y}{\phi(q) \log x} + O\left(\frac{v}{\phi(q) \log x}\right).
\]
We have established that, for \(q \in S_f(Q) \setminus G\),
\[
\max_{1 \leq y \leq x} E(y, q) \ll \frac{v}{\phi(q) \log x},
\]
and so
\[
\sum_{q \in S_f(Q) \setminus G} \max_{1 \leq y \leq x} E(y, q) \ll \frac{v}{\log x} \sum_{q \in S_f(Q)} \frac{1}{\phi(q)}
\]
\[
\ll \frac{|S_f(Q)| \log \log x}{Q \log x} \ll x |S_f(Q)| Q(\log x)^A.
\]
On the other hand, for \(q \in G\),
\[
\max_{1 \leq y \leq x} E(y, q) \ll \frac{x}{\phi(q) \log x} \ll \frac{x \log log x}{Q \log x}
\]
from (5.1). Recalling (6.4), we get
\[
\sum_{q \in G} \max_{1 \leq y \leq x} E(y, q) \ll \frac{|G|x \log \log x}{Q \log x} \ll \frac{x|S_f(Q)|}{Q(\log x)^A}.
\]
The maximal variant of Theorems 1 and 2 follows on combining (6.5), (6.6). The maximal variant of Theorem 3 is proved in similar fashion. ■

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