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On the hybrid mean value of Cochrane sums and generalized Kloosterman sums

by

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1. Introduction. Let q be a natural number and h an integer with $(h,q)=1$. The *Cochrane sums* $C(h,q)$ are defined by

$$C(h,q)=\sum_{a=1}^{q}{}' \left(\left(\frac{\overline{a}}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-1/2 & \text{if } x \text{ is not an integer},\\ 0 & \text{if } x \text{ is an integer},\end{cases}$$

$\overline{a}$ is defined by $a\overline{a}\equiv 1 \bmod q$, and $\sum_{a=1}^{\prime q}$ denotes the summation over all $1\le a\le q$ such that $(a,q)=1$.

These sums were introduced by Todd Cochrane. In October 2000, during his visit in Xi'an, Professor Todd Cochrane suggested studying the arithmetical and mean value distribution properties of $C(h,q)$. On this subject, many interesting results have been obtained; related work can be found in [4], [7], [8] and [9]. For example, Wenpeng Zhang [8] studied the hybrid mean value properties of Cochrane sums and generalized Kloosterman sums, and proved that for any prime $p>3$, we have the asymptotic formulas

$$\sum_{h=1}^{p-1} K(h,1,1;p)C(h,p)=\frac{-1}{2\pi^2}p^2+O\left(p\exp\left(\frac{3\ln p}{\ln\ln p}\right)\right)$$

and

$$\sum_{h=1}^{p-1} K(h,1,r;p)C(h,p)=\frac{-1}{2\pi^2}p^2+O(rp^{3/2}\ln^2 p),$$

where r is a fixed positive integer, $\exp(y)=e^y$, $e(y)=e^{2\pi iy}$, and $K(m,n,r;q)$

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denotes the generalized Kloosterman sum defined as

$$K(m,n,r;q)=\sum_{a=1}^{q}{}' e\left(\frac{ma^r+n\bar{a}^r}{q}\right).$$

At the same time, Wenpeng Zhang [8] also proposed the following:

Conjecture. The asymptotic formula

$$\sum_{h=1}^{q}{}' K(h,1,r;q)C(h,q)\sim\frac{-1}{2\pi^2}q\phi(q),\qquad q\to\infty, \tag{1}$$

holds for all integers $q>2$ and any fixed positive integer r.

In this paper, we shall prove that (1) is not correct for some special positive integers q. Namely, we shall prove the following:

Theorem. *Let q be an odd square-full number ($q>1$, and prime $p\,|\,q$ if and only if $p^2\,|\,q$). Then for any fixed positive integer r,*

$$\sum_{h=1}^{q}{}' K(h,1,r;q)C(h,q)=\frac{-1}{2\pi^2}\phi^2(q)+O\left(r^{\omega(q)}q^{3/2}\exp\left(\frac{8\ln q}{\ln\ln q}\right)\right),$$

where $\omega(q)$ denotes the number of all distinct prime divisors of q.

It is clear that taking $q=9p^2$, from our theorem we can immediately deduce the asymptotic formula

$$\sum_{h=1}^{q}{}' K(h,1,r;q)C(h,q)\sim\frac{-1}{3\pi^2}q\phi(q),\qquad q\to\infty\ (p\to\infty).$$

So the asymptotic formula (1) is not correct.

For general integer $q>2$, whether there exists an asymptotic formula for $\sum_{h=1}^{\prime q}K(h,1,r;q)C(h,q)$ is still an open problem.

2. Several lemmas. In this section, we shall give several lemmas, which are necessary in the proof of our Theorem. First we have the following:

Lemma 1. *Let q be a square-full number. Then for any non-primitive character χ modulo q, we have*

$$\tau(\chi)=\sum_{a=1}^{q}\chi(a)e\left(\frac{a}{q}\right)=0.$$

Proof. Since the Gauss sum $\tau(\chi)$ is a multiplicative function, without loss of generality we can assume that $q=p^\alpha$, where p is an odd prime and α an integer with $\alpha\ge 2$. If χ is not a primitive character modulo p^α, then χ

must be a character modulo $p^{\alpha-1}$. Then from the properties of the reduced residue system modulo p^α and trigonometric sums we have

$$\begin{aligned}\tau(\chi) &= \sum_{a=1}^{p^\alpha} \chi(a) e\Big(\frac{a}{p^\alpha}\Big) = \sum_{r=0}^{p-1}\sum_{b=1}^{p^{\alpha-1}} \chi(rp^{\alpha-1}+b) e\Big(\frac{rp^{\alpha-1}+b}{p^\alpha}\Big)\\ &= \sum_{b=1}^{p^{\alpha-1}} \chi(b) e\Big(\frac{b}{p^\alpha}\Big)\sum_{r=0}^{p-1} e\Big(\frac{r}{p}\Big) = 0. \ \blacksquare\end{aligned}$$

LEMMA 2 (see [7]). *Let a, q be two integers with $q \ge 3$ and $(a,q)=1$. Then*

$$C(a,q) = \frac{-1}{\pi^2\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \overline{\chi}(a)\Big(\sum_{n=1}^{\infty}\frac{G(\chi,n)}{n}\Big)^2,$$

where χ runs through the Dirichlet characters modulo q with $\chi(-1)=-1$, and

$$G(\chi,n) = \sum_{a=1}^{q} \chi(a)\, e\Big(\frac{an}{q}\Big)$$

denotes the Gauss sum corresponding to χ.

LEMMA 3. *Let $q > 3$ be an integer. Then*

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} L^2(1,\chi) = \frac{1}{2}J(q) + O\Big(\exp\Big(\frac{5\ln q}{\ln\ln q}\Big)\Big),$$

where the summation is restricted to all primitive odd characters χ modulo q, and $J(q)=\sum_{d|q}\mu(d)\phi(q/d)$ denotes the number of all primitive characters modulo q.

Proof. For any non-principal character χ modulo q, applying Abel's identity (see Theorem 4.2 of [1]) we have

$$L^2(1,\chi) = \sum_{n=1}^{q^3}\frac{\chi(n)d(n)}{n} + \int_{q^3}^{\infty}\frac{A(y,\chi)}{y^2}dy, \tag{2}$$

where $A(y,\chi)=\sum_{q^3<n\le y}\chi(n)d(n)$.

From [7] we know that for any real number $y > q^3$,

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} |A(y,\chi)|^2 \ll y\phi^2(q). \tag{3}$$

Applying the properties of character sums modulo q we find that for any integer n with $(n,q)=1$, we have the identity

$$\sum_{\chi \bmod q}^{*} \chi(n) = \sum_{d|(q,n-1)} \mu\left(\frac{q}{d}\right)\phi(d). \tag{4}$$

From (2)–(4) we can deduce that

$$\begin{aligned}
&\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*} L^2(1,\chi)\\
&\quad = \frac{1}{2}\sum_{n=1}^{q^3}\frac{d(n)}{n}\sum_{\chi \bmod q}^{*}(\chi(n)-\chi(-n)) + O\Bigg(\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\int_{q^3}^{\infty}\frac{A(y,\chi)}{y^2}\,dy\Bigg)\\
&\quad = \frac{1}{2}\sideset{}{'}\sum_{n=1}^{q^3}\frac{d(n)}{n}\Bigg(\sum_{d|(q,n-1)}\mu\left(\frac{q}{d}\right)\phi(d) - \sum_{d|(q,n+1)}\mu\left(\frac{q}{d}\right)\phi(d)\Bigg) + O(1)\\
&\quad = \frac{1}{2}J(q) + O\Bigg(\sum_{r|q}\phi(r)\sum_{\substack{n=2\\ n\equiv 1 \bmod r}}^{q^3}\frac{d(n)}{n}\Bigg) + O\Bigg(\sum_{r|q}\phi(r)\sum_{\substack{n=1\\ n\equiv -1 \bmod r}}^{q^3}\frac{d(n)}{n}\Bigg)\\
&\quad = \frac{1}{2}J(q) + O\Bigg(\sum_{\substack{r|q\\ r>1}}\phi(r)\sum_{l=1}^{q^3/r}\frac{d(lr\pm 1)}{lr\pm 1}\Bigg) + \ln^2 q\\
&\quad = \frac{1}{2}J(q) + O\Bigg(\sum_{\substack{r|q\\ r>1}}\frac{\phi(r)}{r}\sum_{l=1}^{q^3/r}\frac{1}{l}\exp\left(\frac{3(1+\varepsilon)\ln q}{\ln\ln q}\right)\Bigg) + \ln^2 q\\
&\quad = \frac{1}{2}J(q) + O\left(\exp\left(\frac{5\ln q}{\ln\ln q}\right)\right),
\end{aligned}$$

where $d(n)$ is the Dirichlet divisor function, and $d(n) \ll \exp\left(\frac{(1+\epsilon)\ln n}{\ln\ln n}\right)$ with $\epsilon > 0$ any fixed real number. ∎

Lemma 4. *Let $q > 3$ be an integer. Then*

$$\sum_{a=1}^{q-1}(a+1,q)^{1/2}\Bigg|\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\chi(a)L^2(1,\overline{\chi})\Bigg| = O\left(q\exp\left(\frac{6\ln q}{\ln\ln q}\right)\right).$$

Proof. From the method of proof of Lemma 3 we have

$$\begin{aligned}
\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\chi(a)L^2(1,\overline{\chi}) &= \frac{1}{2}\sum_{n=1}^{q^3}\frac{d(n)}{n}\sum_{\chi \bmod q}^{*}(\chi(a\overline{n}) - \chi(-a\overline{n})) + O(1)\\
&= O\left(\frac{d(a)}{a}J(q)\right) + O\left(\exp\left(\frac{5\ln q}{\ln\ln q}\right)\right).
\end{aligned} \tag{5}$$

Hence we obtain the estimate

$$\sum_{a=1}^{q-1}(a+1,q)^{1/2}\Big|\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\chi(a)L^2(1,\overline{\chi})\Big|$$
$$= O\biggl(\sum_{a=1}^{q-1}\frac{d(a)(a+1,q)^{1/2}}{a}J(q)\biggr)+O\biggl(\sum_{a=1}^{q-1}(a+1,q)^{1/2}\exp\biggl(\frac{5\ln q}{\ln\ln q}\biggr)\biggr)$$
$$= O\biggl(J(q)\sum_{h\mid q}h^{1/2}\sum_{l=2}^{q/h}\frac{d(lh-1)}{lh-1}\biggr)+O\biggl(\sum_{h\mid q}h^{1/2}\sum_{l=1}^{q/h}\exp\biggl(\frac{5\ln q}{\ln\ln q}\biggr)\biggr)$$
$$= O\biggl(q\exp\biggl(\frac{6\ln q}{\ln\ln q}\biggr)\biggr). \blacksquare$$

Lemma 5. *Let $q>3$ be an integer and r a fixed positive integer. Then for any integer n,*

$$\sum_{b=1}^{q}{}' e\biggl(\frac{nb^r}{q}\biggr)=O(r^{\omega(q)}(n,q)^{1/2}q^{1/2}d(q)),$$

where (n,q) denotes the GCD of n and q, and $d(q)$ is the Dirichlet divisor function.

Proof. Let $C(n,r,q)=\sum_{b=1}^{q}e(nb^r/q)$. As $|C(n,r,q)|$ is clearly a multiplicative function, we only have to prove the assertion for $q=p^\alpha$, where p is a prime and α a positive integer. From A. Weil's classical work [6] or T. Cochrane [2], [3] we know that for any integer n with $(n,p^\alpha)=1$, we have the estimate

$$|C(n,r,p^\alpha)|=\biggl|\sum_{b=1}^{p^\alpha}e\biggl(\frac{nb^r}{p^\alpha}\biggr)\biggr|\le rp^{\alpha/2}.$$

If $(n,p^\alpha)=p^\beta$, then $(n/p^\beta,p^{\alpha-\beta})=1$. Hence from the above estimate we deduce that

$$|C(n,r,p^\alpha)|=\biggl|\sum_{b=1}^{p^\alpha}e\biggl(\frac{(n/p^\beta)b^r}{p^{\alpha-\beta}}\biggr)\biggr|=p^\beta\biggl|\sum_{b=1}^{p^{\alpha-\beta}}e\biggl(\frac{(n/p^\beta)b^r}{p^{\alpha-\beta}}\biggr)\biggr|$$
$$\le p^\beta rp^{(\alpha-\beta)/2}=r(n,p^\alpha)^{1/2}p^{\alpha/2}.$$

Now the Möbius inversion formula yields

$$\biggl|\sum_{b=1}^{q}{}' e\biggl(\frac{nb^r}{q}\biggr)\biggr|=\biggl|\sum_{d\mid q}\mu(d)C\biggl(n,r,\frac{q}{d}\biggr)\biggr|=O(r^{\omega(q)}(n,q)^{1/2}q^{1/2}d(q)). \blacksquare$$

3. Proof of the Theorem. In this section, we shall use the lemmas proved in Section 2 to complete the proof of our Theorem. For any odd square-full number q, note the identity

$$\sum_{h=1}^{q}{}' \overline{\chi}(h)K(h,1,r;q) = \sum_{b=1}^{q}{}' e\left(\frac{b^r}{q}\right)\sum_{h=1}^{q}{}' \overline{\chi}(h)e\left(\frac{h\overline{b}^r}{q}\right) = \tau(\overline{\chi})\sum_{b=1}^{q}\overline{\chi}(b^r)e\left(\frac{b^r}{q}\right).$$

From Lemmas 1 and 2 and the properties of Gauss sums $G(\chi,n)$ we have

$$\begin{aligned}(6)\quad & \sum_{h=1}^{q}{}' K(h,1,r;q)C(h,q)\\ &= \frac{-1}{\pi^2\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\sum_{h=1}^{q}{}' \overline{\chi}(h)K(h,1,r;q)\left(\sum_{n=1}^{\infty}\frac{G(\chi,n)}{n}\right)^2\\ &= \frac{-1}{\pi^2\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}{}^{*} \tau(\overline{\chi})\tau^2(\chi)\left(\sum_{b=1}^{q}\overline{\chi}(b^r)e\left(\frac{b^r}{q}\right)\right)L^2(1,\overline{\chi})\\ &= \frac{q}{\pi^2\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}{}^{*} \tau(\chi)\left(\sum_{b=1}^{q}\overline{\chi}(b^r)e\left(\frac{b^r}{q}\right)\right)L^2(1,\overline{\chi}),\end{aligned}$$

where we have used the fact that $\tau(\overline{\chi})\cdot\tau(\chi) = \overline{\chi}(-1)\overline{\tau(\chi)}\cdot\tau(\chi) = -q$ if χ is a primitive character modulo q with $\chi(-1) = -1$.

For any primitive character χ modulo q with $\chi(-1) = -1$, note the identity

$$\begin{aligned}\tau(\chi)\sum_{b=1}^{q}\overline{\chi}(b^r)e\left(\frac{b^r}{q}\right) &= \sum_{a=1}^{q}\sum_{b=1}^{q}\chi(a)\overline{\chi}^r(b)e\left(\frac{b^r+a}{q}\right)\\ &= \sum_{a=1}^{q}\chi(a)\sum_{b=1}^{q}{}' e\left(\frac{(a+1)b^r}{q}\right) = -\phi(q) + \sum_{a=1}^{q-2}\chi(a)\sum_{b=1}^{q}{}' e\left(\frac{(a+1)b^r}{q}\right).\end{aligned}$$

From Lemmas 3–5 and (2) we have

$$\begin{aligned}&\sum_{h=1}^{q}{}' K(h,1,r;q)C(h,q)\\ &= \frac{-q}{\pi^2}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}{}^{*} L^2(1,\overline{\chi}) + \frac{q}{\pi^2\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}{}^{*}\sum_{a=1}^{q-2}\sum_{b=1}^{q}{}' e\left(\frac{(a+1)b^r}{q}\right)\chi(a)L^2(1,\overline{\chi})\end{aligned}$$

$$= \frac{-q}{\pi^2} \sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} L^2(1,\overline{\chi})$$
$$+ O\left(\frac{q}{\phi(q)} \sum_{a=1}^{q-2} \left| \sideset{}{'}\sum_{b=1}^{q} e\left(\frac{(a+1)b^r}{q} \right) \right| \cdot \left| \sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(a) L^2(1,\overline{\chi}) \right| \right)$$
$$= \frac{-q}{2\pi^2} J(q) + O\left(\frac{q^{3/2} r^{\omega(q)} d(q)}{\phi(q)} \sum_{a=1}^{q-2} (a+1,q)^{1/2} \left| \sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(a) L^2(1,\overline{\chi}) \right| \right)$$
$$= \frac{-\phi^2(q)}{2\pi^2} + O\left(q^{3/2} r^{\omega(q)} \exp\left(\frac{8 \ln q}{\ln \ln q} \right) \right),$$

where we have used the identity $J(q) = \phi^2(q)/q$ if q is a square-full number. This completes the proof of our Theorem.

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