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Independence measures of arithmetic functions II

by

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1. Introduction. In our earlier work, the notion of independence measure of arithmetic functions was introduced and two main results ([3, Theorems 3.2 and 3.4]) about such measure were proved. These results are proved under the hypothesis that there is a set of distinct primes for which the set of vectors of function values at points depending on these primes is linearly independent over $\mathbb{C}$, and the proofs make use of the first assertion of [3, Lemma 3.3] where the p-basic derivation is the main tool. Our first objective here is to improve upon these results by replacing the set of primes by any set of distinct natural numbers enjoying similar properties. This is accomplished by making use of the second assertion of [3, Lemma 3.3] where the log-derivation is employed instead.

To systematize our presentation, we first recall all relevant terminology. Denote by $(\mathcal{A}, +, *)$ the unique factorization domain of arithmetic functions equipped with addition and convolution (or Dirichlet product) defined by

$$(f+g)(n) := f(n) + g(n), \ (f * g)(n) = \sum_{ij=n} f(i)g(j) \quad (f, g \in \mathcal{A}, \, n \in \mathbb{N}),$$

and write $f^{*i} = f * \cdots * f$ (i terms). The convolution identity, I, is defined by $I(1) = 1$ and $I(n) = 0$ for all $n > 1$. An arithmetic function f is called a *unit* (in $\mathcal{A}$) if its convolution inverse f^{-1} exists, and this is the case if and only if $f(1) \neq 0$. It is well-known, [8, Chapter 4], that $(\mathcal{A}, +, *)$ is isomorphic to $(\mathcal{D}, +, \cdot)$, where

$$\mathcal{D} := \left\{ D(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \right\}$$

is the ring of formal Dirichlet series equipped with addition and multipli-

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cation, through the isomorphism $f \leftrightarrow D$; addition in both domains is the customary addition while the multiplication of formal Dirichlet series corresponds to the convolution of the appropriate arithmetic functions appearing as coefficients of formal Dirichlet series. For $f \in \mathcal{A}$, its *valuation*, [8, Chapter 4], is defined as

$$|f| := \frac{1}{O(f)},$$

where $O(f)$ is the least integer n for which $f(n) \neq 0$. Correspondingly, for a formal Dirichlet series $D(s) := \sum_{n \geq 1} f(n)/n^s$, its valuation is defined as

$$|D| = |f|,$$

where the same symbols are used for convenience. With this valuation, the isomorphism $(\mathcal{A}, +, *) \leftrightarrow (\mathcal{D}, +, \cdot)$ is indeed an isometry. Therefore, we often refer to these domains interchangeably.

A set of arithmetic functions $f_1, \ldots, f_r$ is said to be algebraically dependent over $\mathbb{C}$ or $\mathbb{C}$-*algebraically dependent* if there exists

$$P(X_1, \ldots, X_r) := \sum_{i_1, \ldots, i_r} a_{i_1, \ldots, i_r} X_1^{i_1} \cdots X_r^{i_r} \in \mathbb{C}[X_1, \ldots, X_r] \setminus \{0\}$$

such that

$$\sum_{i_1, \ldots, i_r} a_{i_1, \ldots, i_r} f_1^{*i_1} * \cdots * f_r^{*i_r} \equiv 0,$$

and $\mathbb{C}$-*algebraically independent* otherwise. If P is homogeneous of degree one in each variable, we say that $f_1, \ldots, f_r$ are $\mathbb{C}$-*linearly dependent*, and $\mathbb{C}$-*linearly independent* otherwise.

A *derivation*, [8], over $\mathcal{A}$ is a map $d : \mathcal{A} \to \mathcal{A}$ satisfying

$$d(f * g) = df * g + f * dg, \qquad d(c_1 f + c_2 g) = c_1 df + c_2 dg,$$

where $f, g \in \mathcal{A}$ and $c_1, c_2 \in \mathbb{C}$. Derivations of higher orders are defined in the usual manner. Two typical examples of derivation are

- the *p-basic derivation*, p prime, defined by

  $$(d_p f)(n) = f(np)\nu_p(np) \qquad (n \in \mathbb{N}),$$

  where $\nu_p(m)$ denotes the exponent of the highest power of p dividing m,
- the *log-derivation* defined by

  $$(d_L f)(n) = f(n) \log n \qquad (n \in \mathbb{N}).$$

Although there are arithmetic sequences $f(n)$ for which the corresponding Dirichlet series $D(s) := \sum_n f(n)/n^s$ are divergent, through the isometry between $\mathcal{A}$ and $\mathcal{D}$, it is legitimate to define the formal derivation $\tilde{d}$ of (formal)

Dirichlet series via the derivation d of the associated arithmetic function as

$$\tilde{d}\,D(s)=\sum_{n=1}^{\infty}\frac{df(n)}{n^s}.$$

Thus, the formal differentiation of the formal Dirichlet series, $D(s)$, with respect to the variable s, i.e.,

$$D'(s)=\sum_{n=1}^{\infty}\frac{-f(n)\log n}{n^s}=\sum_{n=1}^{\infty}\frac{-(d_Lf)(n)}{n^s},$$

corresponds to the (negative) log-derivation $-d_L$ of the associated arithmetic function f, and the p-basic derivation d_p over $\mathcal{A}$ corresponds to the formal p-basic derivation $\tilde{D_p}$ over $\mathcal{D}$ defined by

$$\tilde{d_p}D(s)=\sum_{n=1}^{\infty}\frac{(d_pf)(n)}{n^s}.$$

For convenience, we use the same derivation symbol d for both the domains $\mathcal{A}$ and $\mathcal{D}$. Our investigations concerning Dirichlet series will be **formal** throughout.

2. Algebraic independence. The following lemma, which plays a vital role in our investigation of algebraic independence, is Lemma 3.1 in [3].

LEMMA 2.1. *Let $f_1,\dots,f_r\in\mathcal{A}$ and $P(X_1,\dots,X_r)\in\mathbb{C}[X_1,\dots,X_r]\setminus\{0\}$. For $t=1,\dots,r$, define the following formal Dirichlet series*

$$D_t(s)=\sum_{n\ge1}\frac{f_t(n)}{n^s},$$

$$P(D_1,\dots,D_r)=\sum_{n\ge1}\frac{F(n)}{n^s},\qquad \frac{\partial P}{\partial X_t}(D_1,\dots,D_r)=\sum_{n\ge1}\frac{F_t(n)}{n^s}.$$

Then for each $n\in\mathbb{N}$ and for each prime p, we have

$$F(pn)\nu_p(pn)=\sum_{j=1}^{r}\sum_{k\mid n}f_j(pk)F_j(\frac{n}{k})\nu_p(pk), \tag{2.1}$$

$$F(n)\log n=\sum_{j=1}^{r}\sum_{k\mid n}f_j(k)F_j(\frac{n}{k})\log k, \tag{2.2}$$

where the Dirichlet series and their operations are considered formally.

Our improvement of [3, Theorem 3.2] is

THEOREM 2.2. *In the notation of Lemma* 2.1, *suppose that $P(X_1,\dots,X_r)$ is of total degree* $\deg P=g$. *If there is a set of r positive integers* $\{(1<)\ n_1<$

$\cdots < n_r\}$ *such that the set of vectors*

$$\{(f_1(n_i), \ldots, f_r(n_i)) : i = 1, \ldots, r\}$$

is linearly independent over $\mathbb{C}$, *then*

$$|P(D_1, \ldots, D_r)| \geq n_r^{-g}.$$

Proof. If $\deg P = 0$, then clearly $|P(D_1, \ldots, D_r)| = 1$. If $\deg P = 1$, then

$$P(X_1, \ldots, X_r) = a_0 I + a_1 X_1 + \cdots + a_r X_r,$$

where the coefficients a_j $(j = 1, \ldots, r)$ do not vanish simultaneously. Equating coefficients, we get

$$F(n_j) = a_1 f_1(n_j) + \cdots + a_r f_r(n_j).$$

Since the set $\{(f_1(n_j), \ldots, f_r(n_j)) : j = 1, \ldots, r\}$ is linearly independent over $\mathbb{C}$, at least one of the values $F(n_1), \ldots, F(n_r)$ must be nonzero, which renders

$$|P(D_1, \ldots, D_r)| \geq n_r^{-1}.$$

Now proceed by induction on $\deg P$. Let P be of total degree $g + 1 \geq 2$, and assume that the assertion has been proved for polynomials of degree $\leq g$. Consider the polynomials $\partial P/\partial X_t$ $(t = 1, \ldots, r)$, of degree $\leq g$. Unless $\partial P/\partial X_t$ vanishes identically, by induction we have

$$\left|\frac{\partial P}{\partial X_t}(D_1, \ldots, D_r)\right| \geq n_r^{-g},$$

which implies that the n_r^g vectors

$$(F_1(1), \ldots, F_r(1)), (F_1(2), \ldots, F_r(2)), \ldots, (F_1(n_r^g), \ldots, F_r(n_r^g)) \tag{2.3}$$

cannot all be zero. Let $(F_1(m), \ldots, F_r(m))$ be the first nonzero vector in (2.3) so that

$$(F_1(d), \ldots, F_r(d)) = (0, \ldots, 0) \quad \text{for } d = 1, \ldots, m-1.$$

By the minimality of m and Lemma 2.1, for all $i = 1, \ldots, r$ we get

$$F(n_i m)\log(n_i m) = f_1(n_i)F_1(m) + \cdots + f_r(n_i)F_r(m).$$

Since the set $\{(f_1(n_j), \ldots, f_r(n_j)) : j = 1, \ldots, r\}$ is linearly independent over $\mathbb{C}$, at least one of $F(n_1 m), \ldots, F(n_r m)$ must be nonzero. This yields

$$|P(D_1, \ldots, D_r)| \geq (mn_r)^{-1} \geq (n_r^{g+1})^{-1}. \ \blacksquare$$

Recall that a formal Dirichlet series $D(s)$ is said to be *differentially algebraic* of order $r \in \mathbb{N}_0$ if D together with all its derivatives (up to order r) $D', \ldots, D^{(r)}$ $(D^{(0)} := D)$ satisfy a non-trivial algebraic equation with complex coefficients. When $r = 0$, differentially algebraic series of order 0 are simply algebraic series. The notion of differentially algebraic arithmetic functions is defined correspondingly. An immediate consequence of Theorem 2.2 is the following measure of differentially algebraic independence.

Corollary 2.3. *Let $D(s) = \sum_{n\geq 1} f(n)n^{-s} \in \mathcal{D}$ and $P(X_0,\dots,X_r) \in \mathbb{C}[X_1,\dots,X_r] \setminus \{0\}$ be of total degree g. For $r \in \mathbb{N}_0$, if there is a set of $r+1$ natural numbers $\{(1<)\ n_1 < n_2 < \cdots < n_{r+1}\}$ such that $f(n_i) \neq 0$ $(i = 1,\dots,r+1)$, then*

$$|P(D, D', \dots, D^{(r)})| \geq (n_{r+1}^g)^{-1},$$

where the Dirichlet series, their derivatives and operations are considered formally.

Proof. Formally differentiating j times the Dirichlet series with respect to s, we get

$$D^{(j)}(s) = \sum_{n\geq 1} \frac{f(n)(-\log n)^j}{n^s}.$$

For each $i \in \{1,\dots,r+1\}$, since $f(n_i)(-\log n_i)^j \neq 0$, the determinant

$$\begin{vmatrix} f(n_1) & f(n_1)(-\log n_1) & \cdots & f(n_1)(-\log n_1)^r \\ \vdots & & & \vdots \\ f(n_{r+1}) & f(n_{r+1})(-\log n_{r+1}) & \cdots & f(n_{r+1})(-\log n_{r+1})^r \end{vmatrix}$$
$$= f(n_1)\cdots f(n_{r+1}) \begin{vmatrix} 1 & (-\log n_1) & \cdots & (-\log n_1)^r \\ \vdots & & & \vdots \\ 1 & (-\log n_{r+1}) & \cdots & (-\log n_{r+1})^r \end{vmatrix},$$

being Vandermonde, is nonzero, and so the set of vectors

$$\{(f(n_1), f(n_1)(-\log n_1), \dots, f(n_1)(-\log n_1)^r), \dots,$$
$$(f(n_{r+1}), f(n_{r+1}) \log n_{r+1}, \dots, f(n_{r+1})(-\log n_{r+1})^r)\}$$

is $\mathbb{C}$-linearly independent. The assertion now follows from Theorem 2.2. ■

Corollary 2.3 reveals an interesting feature of differentially algebraic arithmetic functions:

Corollary 2.4. *Let $r \in \mathbb{N}_0$. If $f \in \mathcal{A}$ is differentially algebraic of order r, then excluding the point 1 it can be nonzero at r distinct points at most.*

Observe that the result of Corollary 2.4 when $r = 0$ is identical with that of [3, Proposition 2.1 part 1]. An even more amazing consequence of Corollary 2.4 is the next result which substantially generalizes an old theorem of Hilbert [1] stating that the Riemann zeta function does not satisfy any algebraic differential equation over $\mathbb{C}$; Ostrowski [6] showed more generally that the Riemann zeta function does not satisfy any algebraic differential-difference equation over $\mathbb{C}$.

Corollary 2.5. *An arithmetic function which is nonzero at infinitely many points is not differentially algebraic, i.e., it is* hyper-transcendental, *or equivalently, every formal Dirichlet series which is not a Dirichlet polynomial is hyper-transcendental.*

The next corollary yields a measure of algebraic independence for appropriate lacunary arithmetic functions.

Corollary 2.6. *In the notation of Lemma* 2.1, *suppose that* $P(X_1, \dots, X_r)$ *is of total degree* g. *If there is a finite sequence of positive integers* $\{m_1 < \dots < m_r\}$ *such that for* $t \in \{1, \dots, r\}$ *we have*

$$f_t(m_t) \neq 0 \quad \textit{but} \quad f_t(k) = 0 \textit{ for } k \in \{1, \dots, m_r\} \setminus \{m_t\},$$

then

$$|P(D_1, \dots, D_r)| \geq n_r^{-g}.$$

Proof. The result follows from Theorem 2.2 by noting that the set

$$\{(f_1(m_t), \dots, f_r(m_t)) : t = 1, \dots, r\}$$

is $\mathbb{C}$-linearly independent. ∎

Corollary 2.6 leads at once to the next result which says that lacunary arithmetic functions are roughly $\mathbb{C}$-algebraically independent.

Corollary 2.7. *Let* $f_1, \dots, f_r \in \mathcal{A}$. *If there are* r *sequences of positive integers*

$$\{n_1^{(t)} < n_2^{(t)} < \cdots\} \quad (t = 1, \dots, r)$$

such that for $t \in \{1, \dots, r\}$ *we have*

$$f_t(n_j^{(t)}) \neq 0, \quad \textit{but}$$

$$f_t(k) = 0 \quad \textit{for } k \in \{1, \dots, n_1^{(t)} - 1\} \cup \bigcup_{j=1}^{\infty} \{n_j^{(t)} + 1, \dots, n_{j+1}^{(t)} - 1\},$$

then $f_1, \dots, f_r$ *are* $\mathbb{C}$*-algebraically independent.*

We end this section by comparing two measures of independence from [3] with those obtained via Theorem 2.2. Let $\{F_n\}_{n \geq 1}$ be the sequence of *Fibonacci numbers* defined by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \in \mathbb{N}).$$

The six formal *Fibonacci zeta series* are defined as (see [2])

$$\mathcal{F}^+(s) := \sum_{n=1}^{\infty}\frac{1}{F_n^s} = \sum_{n=1}^{\infty}\frac{f^+(n)}{n^s}, \qquad \mathcal{F}_e^+(s) := \sum_{n=1}^{\infty}\frac{1}{F_{2n}^s} = \sum_{n=1}^{\infty}\frac{f_e^+(n)}{n^s},$$

$$\mathcal{F}_o^+(s) := \sum_{n=1}^{\infty}\frac{1}{F_{2n-1}^s} = \sum_{n=1}^{\infty}\frac{f_o^+(n)}{n^s}, \qquad \mathcal{F}^-(s) := \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_n^s} = \sum_{n=1}^{\infty}\frac{f^-(n)}{n^s},$$

$$\mathcal{F}_e^-(s) := \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_{2n}^s} = \sum_{n=1}^{\infty}\frac{f_e^-(n)}{n^s}, \quad \mathcal{F}_o^-(s) := \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_{2n-1}^s} = \sum_{n=1}^{\infty}\frac{f_o^-(n)}{n^s},$$

Let $\{L_n\}_{n\geq 1}$ be the sequence of *Lucas numbers* defined by

$$L_1 = 1, \quad L_2 = 3, \quad L_{n+2} = L_{n+1} + L_n \quad (n \in \mathbb{N}).$$

The six formal *Lucas zeta series* are defined as

$$\mathcal{L}^+(s) := \sum_{n=1}^{\infty}\frac{1}{L_n^s} = \sum_{n=1}^{\infty}\frac{\ell^+(n)}{n^s}, \qquad \mathcal{L}_e^+(s) := \sum_{n=1}^{\infty}\frac{1}{L_{2n}^s} = \sum_{n=1}^{\infty}\frac{\ell_e^+(n)}{n^s},$$

$$\mathcal{L}_o^+(s) := \sum_{n=1}^{\infty}\frac{1}{L_{2n-1}^s} = \sum_{n=1}^{\infty}\frac{\ell_o^+(n)}{n^s}, \qquad \mathcal{L}^-(s) := \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{L_n^s} = \sum_{n=1}^{\infty}\frac{\ell^-(n)}{n^s},$$

$$\mathcal{L}_e^-(s) := \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{L_{2n}^s} = \sum_{n=1}^{\infty}\frac{\ell_e^-(n)}{n^s}, \quad \mathcal{L}_o^-(s) := \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{L_{2n-1}^s} = \sum_{n=1}^{\infty}\frac{\ell_o^-(n)}{n^s},$$

In [3, p. 10], it was shown that

$$\left|P(\mathcal{F}^+, \mathcal{F}_e^+, \mathcal{F}_o^-)\right| \geq 5^{-g}, \tag{2.4}$$

for any $P(X_1, X_2, X_3) \in \mathbb{C}[X_1, X_2, X_3] \setminus \{0\}$ of total degree g, and

$$|Q(\mathcal{L}^+, \mathcal{L}^-, \mathcal{L}_e^-, \mathcal{L}_o^-)| \geq 29^{-g}, \tag{2.5}$$

for any $Q(X_1, X_2, X_3, X_4) \in \mathbb{C}[X_1, X_2, X_3, X_4] \setminus \{0\}$ of total degree g. Since

$$\begin{vmatrix} f^+(1 = F_1) & f^+(2 = F_3) & f^+(3 = F_4) \\ f_e^+(1 = F_2) & f_e^+(2 = F_3) & f_e^+(3 = F_4) \\ f_o^-(1 = F_1) & f_o^-(2 = F_3) & f_o^-(3 = F_4) \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 2 \neq 0,$$

the set of three vectors

$$\{(f^+(1), f^+(2), f^+(3)), (f_e^+(1), f_e^+(2), f_e^+(3)), (f_o^-(1), f_o^-(2), f_o^-(3))\}$$

is $\mathbb{C}$-linearly independent and Theorem 2.2 yields

$$|P(\mathcal{F}^+, \mathcal{F}_e^+, \mathcal{F}_o^-)| \geq 3^{-g}, \tag{2.6}$$

which is much better than (2.4). A simple example of linear polynomials such as

$$P(n)\ (= P(f^+, f_e^+, f_o^-)(n)) := f^+(n) - 2f_e^+(n) + f_o^-(n)$$

shows that $P(1) = P(2) = 0, P(3) = -1 \neq 0$, i.e., the bound in (2.6) is best possible.

Since

$$\begin{vmatrix} \ell^+(1 = L_1) & \ell^+(3 = L_2) & \ell^+(4 = L_3) & \ell^+(7 = L_4) \\ \ell^-(1 = L_1) & \ell^-(3 = L_2) & \ell^-(4 = L_3) & \ell^-(7 = L_4) \\ \ell_e^-(1 = L_1) & \ell_e^-(3 = L_2) & \ell_e^-(4 = L_3) & \ell_e^-(7 = L_4) \\ \ell_o^-(1 = L_1) & \ell_o^-(3 = L_2) & \ell_o^-(4 = L_3) & \ell_o^-(7 = L_4) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{vmatrix}$$
$$= 8 \neq 0,$$

the set of four vectors

$$\big\{ \big(\ell^+(1), \ell^+(3), \ell^+(4), \ell^+(7)\big), \big(\ell^-(1), \ell^-(3), \ell^-(4), \ell^-(7)\big), \\ \big(\ell_e^-(1), \ell_e^-(3), \ell_e^-(4), \ell_e^-(7)\big), \big(\ell_o^-(1), \ell_o^-(3), \ell_o^-(4), \ell_o^-(7)\big) \big\}$$

is $\mathbb{C}$-linearly independent and Theorem 2.2 yields

$$|Q(\mathcal{L}^+, \mathcal{L}^-, \mathcal{L}_e^-, \mathcal{L}_o^-)| \geq 7^{-g}, \tag{2.7}$$

much better than (2.5). Again a simple example of linear polynomials such as

$$Q(n)\ (= Q(\ell^+, \ell^-, \ell_e^-, \ell_o^-)(n)) := \ell^+(n) - \ell^-(n) - 2\ell_e^-(n) + 0 \cdot \ell_o^-(n)$$

shows that $Q(1) = \cdots = Q(6) = 0$ and $Q(7) = 4 \neq 0$, i.e., the bound in (2.7) is best possible.

3. Linear dependence and Wronskian. Motivated by the case of real functions, in this section, we investigate the connection between linear dependence and the notion of Wronskian in our arithmetic setting. We start with a simple proposition, whose converse, which is much more difficult, will be examined later.

Proposition 3.1. *Let $f_1, \ldots, f_r \in \mathcal{A}$ and let d be a derivation on $\mathcal{A}$. If $f_1, \ldots, f_r$ are $\mathbb{C}$-linearly dependent, then their Wronskian relative to d,*

$$W_d(f_1, \ldots, f_r) := \begin{vmatrix} f_1 & f_2 & \ldots & f_r \\ df_1 & df_2 & \ldots & df_r \\ \vdots & & & \\ d^{r-1}f_1 & d^{r-1}f_2 & \ldots & d^{r-1}f_r \end{vmatrix},$$

vanishes; here and throughout, the multiplication involved in the determinant expansion is the Dirichlet product.

Proof. Taking the derivations d^i for $i = 1, \ldots, r-1$ in the linear relation among $f_1, \ldots, f_r$, with coefficients $c_1, \ldots, c_r$ not all zero, we get a system of linear equations in the c_i's whose determinant is the Wronskian considered and the existence of nontrivial solutions forces the vanishing of this determinant. ∎

The next result gives a sufficient condition for linear dependence.

THEOREM 3.2. *Let $f_1, \ldots, f_r \in \mathcal{A}$. If the set of positive integers $\{n_1 < \cdots < n_r\}$ is such that*

$$f_t(n_t) \neq 0 \quad \textit{but} \quad f_t(k) = 0 \textit{ for } \quad k = 1, \ldots, n_t - 1 \quad (t = 1, \ldots, r),$$

then the Wronskian (*with respect to the log-derivation*)

$$W_L(f_1, \ldots, f_r) := \begin{vmatrix} f_1 & \cdots & f_r \\ d_L f_1 & \cdots & d_L f_r \\ \vdots & & \vdots \\ d_L^{r-1} f_1 & \cdots & d_L^{r-1} f_r \end{vmatrix}$$

(*where the product in the expansion of the determinant is convolution*) *does not vanish, and so $f_1, \ldots, f_r$ are $\mathbb{C}$-linearly independent.*

Proof. By the minimality of $n_1, \ldots, n_r$, we get

$$\begin{aligned} W_L(f_1, \ldots, f_r)(n_1 \cdots n_r) &= \begin{vmatrix} f_1 & \cdots & f_r \\ d_L f_1 & \cdots & d_L f_r \\ \vdots & & \vdots \\ d_L^{r-1} f_1 & \cdots & d_L^{r-1} f_r \end{vmatrix} (n_1 \cdots n_r) \\ &= \sum_{c_1 \cdots c_r = n_1 \cdots n_r} \begin{vmatrix} f_1(c_1) & \cdots & f_r(c_r) \\ f_1(c_1) \log(c_1) & \cdots & f_r(c_r) \log(c_r) \\ \vdots & & \vdots \\ f_1(c_1) \log^{r-1}(c_1) & \cdots & f_r(c_r) \log^{r-1}(c_r) \end{vmatrix} \\ &= \begin{vmatrix} f_1(n_1) & \cdots & f_r(n_r) \\ f_1(n_1) \log(n_1) & \cdots & f_r(n_r) \log(n_r) \\ \vdots & & \vdots \\ f_1(n_1) \log^{r-1}(n_1) & \cdots & f_r(n_r) \log^{r-1}(n_r) \end{vmatrix} \\ &= f_1(n_1) f_2(n_2) \cdots f_r(n_r) \begin{vmatrix} 1 & \cdots & 1 \\ \log(n_1) & \cdots & \log(n_r) \\ \vdots & & \vdots \\ \log^{r-1}(n_1) & \cdots & \log^{r-1}(n_r) \end{vmatrix} \neq 0. \end{aligned}$$

∎

Recall that the *norm* $N(f)$ of $f \in \mathcal{A}$ is defined as

$$N(f) = \min\{n \in \mathbb{N} : f(n) \neq 0\}.$$

Theorem 3.2 simply says that arithmetic functions whose norms are distinct are necessarily $\mathbb{C}$-linearly independent. This is worth comparing with Theorem 7 of [7] which asserts that the set of nonunit arithmetic functions whose norms are pairwise relatively prime is $\mathbb{C}$-algebraically independent.

For future use, we pause to establish an identity involving the Wronskian value evaluated at a general point.

THEOREM 3.3. *Let $f_1, \ldots, f_r \in \mathcal{A}$ and let $n \in \mathbb{N}$. Then*

$$W_L(f_1, \ldots, f_r)(n) := \begin{vmatrix} f_1 & \cdots & f_r \\ d_L f_1 & \cdots & d_L f_r \\ \vdots & & \vdots \\ d_L^{r-1} f_1 & \cdots & d_L^{r-1} f_r \end{vmatrix}(n)$$

$$= \sum_{n_1 \cdots n_r = n;\, n_1 < \cdots < n_r} \Big(\prod_{1 \le i < j \le r} (\log n_j - \log n_i) \Big) \begin{vmatrix} f_1(n_1) & \cdots & f_1(n_r) \\ f_2(n_1) & \cdots & f_2(n_r) \\ \vdots & & \vdots \\ f_r(n_1) & \cdots & f_r(n_r) \end{vmatrix}.$$

Proof. We have

$$W_L(f_1, \ldots, f_r)(n) = \begin{vmatrix} f_1 & \cdots & f_r \\ d_L f_1 & \cdots & d_L f_r \\ \vdots & & \vdots \\ d_L^{r-1} f_1 & \cdots & d_L^{r-1} f_r \end{vmatrix}(n)$$

$$= \sum_{c_1 \cdots c_r = n} \begin{vmatrix} f_1(c_1) & \cdots & f_r(c_r) \\ f_1(c_1)\log(c_1) & \cdots & f_r(c_r)\log(c_r) \\ \vdots & & \vdots \\ f_1(c_1)\log^{r-1}(c_1) & \cdots & f_r(c_r)\log^{r-1}(c_r) \end{vmatrix}$$

$$= \sum_{c_1 \cdots c_r = n} f_1(c_1) \cdots f_r(c_r) \begin{vmatrix} 1 & \cdots & 1 \\ \log(c_1) & \cdots & \log(c_r) \\ \vdots & & \vdots \\ \log^{r-1}(c_1) & \cdots & \log^{r-1}(c_r) \end{vmatrix}$$

$$= \sum_{c_1 \cdots c_r = n} f_1(c_1) \cdots f_r(c_r) \prod_{1 \le i < j \le r} (\log c_j - \log c_i)$$

$$= \sum_{n_1\cdots n_r=n;\, n_1<\cdots<n_r} \Big(\prod_{1\le i<j\le r} (\log n_j - \log n_i) \Big) \times \sum_{i_1,\ldots,i_r} \epsilon(n_{i_1},\ldots,n_{i_r}) f_1(n_{i_1})\cdots f_r(n_{i_r})$$

where the inner sum on the right hand side is taken over all permutations of $(n_1,\ldots,n_r)$ with $\epsilon(n_{i_1},\ldots,n_{i_r}) = 1$ for an even permutation and -1 for an odd one. Thus,

$$W_L(f_1,\ldots,f_r)(n_1\cdots n_r) = \sum_{n_1\cdots n_r=n;\, n_1<\cdots<n_r} \Big(\prod_{1\le i<j\le r} (\log n_j - \log n_i) \Big) \begin{vmatrix} f_1(n_1) & \cdots & f_1(n_r) \\ f_2(n_1) & \cdots & f_2(n_r) \\ \vdots & & \vdots \\ f_r(n_1) & \cdots & f_r(n_r) \end{vmatrix}. \ \blacksquare$$

In the real case it is well-known (see e.g. [4]) that the converse of Proposition 3.1 is not generally true. This is also the case in the arithmetic function setting. For example, consider the two arithmetic functions

$$I(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise}, \end{cases} \qquad g(n) = \begin{cases} 1 & \text{if } n = q \neq p, \\ 0 & \text{otherwise}, \end{cases}$$

where $q \neq p$ are primes. If $c_1 I + c_2 g = 0$ $(c_1, c_2 \in \mathbb{C})$, then

$$0 = c_1 I(1) + c_2 g(1) = c_1, \qquad 0 = c_1 I(q) + c_2 g(q) = c_2,$$

showing that I and g are $\mathbb{C}$-linearly independent. However, their Wronskian relative to the p-basic derivation d_p does vanish:

$$\begin{aligned} W(I,g)(n) &= \begin{vmatrix} I & g \\ d_p I & d_p g \end{vmatrix} (n) \\ &= \sum_{ij=n} \{ I(i) g(jp)\nu_p(jp) - g(i) I(jp)\nu_p(jp)\} = 0 \qquad (n \in \mathbb{N}). \end{aligned}$$

The converse of Theorem 3.1 does indeed hold if we stick to the log-derivation.

Theorem 3.4. *Let $f_1,\ldots,f_r \in \mathcal{A}\setminus\{0\}$. If their Wronskian $W = W_L(f_1,\ldots,f_r)$ relative to the log-derivation vanishes identically, then $f_1,\ldots,f_r$ are $\mathbb{C}$-linearly dependent.*

Proof. For brevity write d for d_L. First we consider the case $r = 2$. We consider two cases.

Case 1: $f_1(1) \neq 0$. Then f_1^{-1}, the convolution inverse of f_1, exists and so

$$0 = W_L(f_1, f_2) = W_L(f_1 * f_1 * f_1^{-1}, f_2 * f_1 * f_1^{-1}) = f_1^2 * W_L(I, f_2 * f_1^{-1}),$$

yielding

$$0 = W_L(I, f_2 * f^{-1}) = d(f_2 * f_1^{-1}).$$

Thus, $f_2 * f_1^{-1} = cI$ for some $c \in \mathbb{C}$, i.e., $f_2 = cf_1$, showing that f_1 and f_2 are $\mathbb{C}$-linearly dependent.

Case 2: $f_1(1) = f_2(1) = 0$. Since $f_1 \not\equiv 0$, let $N > 1$ be the least positive integer for which $f_1(N) \neq 0$. For $n \in \mathbb{N}$, we have

$$0 = W_L(f_1, f_2)(n) = \sum_{ab=n} (f_1(a)f_2(b) - f_1(b)f_2(a)) \log b.$$

Putting $n = 2N$, we get

$$0 = W_L(f_1, f_2)(2N) = f_1(N)f_2(2)(\log 2 - \log N),$$

i.e., $f_2(2) = 0$. By induction, for $k = 1, \ldots, N-1$, we have

$$0 = W_L(f_1, f_2)(kN) = f_1(N)f_2(k)(\log k - \log N),$$

i.e., $f_2(k) = 0$. Putting $n = N^2$ and using the previously found values, we get

$$0 = W_L(f_1, f_2)(N^2) = (f_1(N)f_2(N) - f_1(N)f_2(N)) \log N,$$

yielding $f_2(N)$ arbitrary. Putting $n = N(N+1)$ and using the previously found values, we get

$$\begin{aligned} 0 &= W_L(f_1, f_2)(N(N+1)) \\ &= (f_1(N)f_2(N+1) - f_1(N+1)f_2(N)) \log(N+1), \end{aligned}$$

i.e., $f_2(N+1) = f_1(N+1)f_2(N)/f_1(N)$. In general, for $m \geq 1$, using previously found values, we have

$$\begin{aligned} 0 = W_L(f_1, f_2)(N(N+m)) &= \sum_{ab=N(N+m)} (f_1(a)f_2(b) - f_1(b)f_2(a)) \log b \\ = \sum_{\substack{ab=N(N+m) \\ a<N}} & (f_1(a)f_2(b) - f_1(b)f_2(a)) \log b \\ & + \{f_1(N)f_2(N+m) - f_2(N)f_1(N+m)\} \log(N+m) \\ & + \sum_{\substack{ab=N(N+m) \\ N<a\leq N+m}} (f_1(a)f_2(b) - f_1(b)f_2(a)) \log b \\ = \{f_1(N)f_2(N+m) & - f_2(N)f_1(N+m)\} \log(N+m), \end{aligned}$$

i.e., $f_2(N+m) = f_1(N+m)f_2(N)/f_1(N)$. Hence, $f_2 = cf_1$, where $c := f_2(N)/f_1(N)$.

Supposing that the assertion of the theorem holds for up to $r-1$ (≥ 2) functions, we proceed to verify it for r functions. We again have two cases.

Case 1: there is an $i \in \{1, \ldots, r\}$ for which $f_i(1) \neq 0$. We may assume that $f_1(1) \neq 0$. Then f_1^{-1} exists and so

$$\begin{aligned} 0 = W_L(f_1, \ldots, f_r) &= f_1^r * W_L(I, f_2 * f_1^{-1}, \ldots, f_r * f_1^{-1}) \\ &= f_1^r * W_L(d(f_2 * f_1^{-1}), \ldots, d(f_r * f_1^{-1})). \end{aligned}$$

By the induction hypothesis, $d(f_2 * f_1^{-1}), \ldots, d(f_r * f_1^{-1})$ are $\mathbb{C}$-linearly dependent, which implies that so are $f_1, \ldots, f_r$.

Case 2: $f_1(1) = \cdots = f_r(1) = 0$. For brevity, write

$$A(i) = (f_1(i), \ldots, f_r(i)).$$

Thus, $A(1) = (0, \ldots, 0)$. Since $f_1, \ldots, f_r \in \mathcal{A} \setminus \{0\}$, let N_1 be the least positive integer such that

$$A(N_1) \neq (0, \ldots, 0).$$

There are two subcases.

Subcase 1: All the vectors $A(n)$ with $n > N_1$ are $\mathbb{C}$-multiples of $A(N_1)$, so there exist $c(n) \in \mathbb{C}$ such that $A(n) = c(n)A(N_1)$, i.e.,

$$f_1(n) = c(n)f_1(N_1), \ \ldots, \ f_r(n) = c(n)f_r(N_1).$$

Observe that the (single) linear equation in r (≥ 4) unknowns $x_1, \ldots, x_r$,

$$0 = x_1 f_1(N_1) + \cdots + x_r f_r(N_1),$$

has a nontrivial solution $(x_1, \ldots, x_r) \neq (0, \ldots, 0)$. This shows that

$$x_1 f_1(n) + \cdots + x_r f_r(n) = 0 \quad \text{for all } n \in \mathbb{N},$$

i.e., $f_1, \ldots, f_r$ are $\mathbb{C}$-linearly dependent.

Subcase 2: There exists a least positive integer N_2 ($> N_1$) such that $A(N_1), A(N_2)$ are $\mathbb{C}$-linearly independent. Again we treat two possibilities.

If all the vectors $A(n)$ with $n > N_2$ are $\mathbb{C}$-linear combinations of $A(N_1)$ and $A(N_2)$, so there exist $c_1(n), c_2(n) \in \mathbb{C}$ such that $A(n) = c_1(n)A(N_1) + c_2(n)A(N_2)$, i.e.,

$$f_1(n) = c_1(n)f_1(N_1) + c_2(n)f_1(N_2), \ \ldots, \ f_r(n) = c_1(n)f_r(N_1) + c_2(n)f_r(N_2),$$

then the system of two equations in r (≥ 4) unknowns $x_1, \ldots, x_r$,

$$\begin{aligned} 0 &= x_1 f_1(N_1) + \cdots + x_r f_r(N_1), \\ 0 &= x_1 f_1(N_2) + \cdots + x_r f_r(N_2), \end{aligned}$$

has a nontrivial solution $(x_1, \ldots, x_r) \neq (0, \ldots, 0)$. Then

$$x_1 f_1(n) + \cdots + x_r f_r(n) = 0 \quad \text{for all } n \in \mathbb{N},$$

showing that $f_1, \ldots, f_r$ are $\mathbb{C}$-linearly dependent.

Otherwise, there exists a least positive integer N_3 ($> N_2 > N_1$) such that $A(N_1), A(N_2), A(N_3)$ are $\mathbb{C}$-linearly independent and we continue as above. In general, assume that there is a set of $1 \le j$ $(\le r-1)$ (lexicographically least) positive integers $N_1 < \cdots < N_j$ such that the vectors $A(N_1), \ldots, A(N_j)$ are $\mathbb{C}$-linearly independent. If all the vectors $A(n)$ with $n > N_j$ $(> N_{j-1} > \cdots > N_1)$ are $\mathbb{C}$-linear combinations of $A(N_1), \ldots, A(N_j)$, so there exist $c_1(n), \ldots, c_j(n) \in \mathbb{C}$ such that $A(n) = c_1(n)A(N_1) + \cdots + c_j(n)A(N_j)$, i.e.,

$$f_1(n) = c_1(n)f_1(N_1) + \cdots + c_j(n)f_1(N_j), \ \ldots,$$
$$f_r(n) = c_1(n)f_r(N_1) + \cdots + c_j(n)f_r(N_j),$$

then the system of j $(\le r-1)$ equations in r unknowns $x_1, \ldots, x_r$,

$$\begin{aligned} 0 &= x_1 f_1(N_1) + \cdots + x_r f_r(N_1), \\ &\vdots \\ 0 &= x_1 f_1(N_j) + \cdots + x_r f_r(N_j), \end{aligned}$$

has a nontrivial solution $(x_1, \ldots, x_r) \neq (0, \ldots, 0)$. Then

$$x_1 f_1(n) + \cdots + x_r f_r(n) = 0 \quad \text{for all } n \in \mathbb{N},$$

showing that $f_1, \ldots, f_r$ are $\mathbb{C}$-linearly dependent.

There remains the case where there are (lexicographically) least positive integers $N_1 < \cdots < N_r$ such that $A(N_1), \ldots, A(N_r)$ are $\mathbb{C}$-linearly independent and so

$$\begin{vmatrix} f_1(N_1) & \cdots & f_1(N_r) \\ f_2(N_1) & \cdots & f_2(N_r) \\ \vdots & & \vdots \\ f_r(N_1) & \cdots & f_r(N_r) \end{vmatrix} \neq 0. \tag{3.1}$$

Using Theorem 3.3 together with the (lexicographically) minimal property of $N_1 < \cdots < N_r$, the hypothesis that the Wronskian vanishes shows that so does the determinant on the left hand side of (3.1). This contradiction finishes the proof. ∎

Proposition 3.1 together with Theorem 3.4 provides us with a satisfactory necessary and sufficient condition for $\mathbb{C}$-linear dependence of arithmetic functions through the use of Wronskian. This should be compared with the use of Jacobian for testing $\mathbb{C}$-algebraic independence in [9], which only works in one direction. Though Proposition 3.1 and Theorem 3.4 are not so easy to use, they do yield several independence tests; we next give an example.

Theorem 3.5. *Let* $\alpha, \beta \in \mathbb{N}$ *and*

$$S = \{s_1, \ldots, s_\alpha\} \subseteq \mathbb{C}, \qquad K = \{0 \le k_1 \le \cdots \le k_\beta\} \subseteq \mathbb{N}_0,$$
$$T = \{f_{s,k} : s \in S,\ k \in K\} \subseteq \mathcal{A},$$

with $f_{s,k}(1) \neq 0$ $(s \in S,\, k \in K)$. *Assume that for all sufficiently large primes* p,

(1) $f_{s,k}(p) \neq 0$ $(s \in S,\, k \in K)$;

(2) $\lim_{p\to\infty} \dfrac{f_{s,k_i}(p)}{f_{s,k_u}(p)} = 0$ *for* $1 \le i < u \le \beta$;

(3) $\lim_{p\to\infty} \dfrac{f_{s_j,k_a}(p)}{f_{s_v,k_b}(p)} = 0$ *for* $1 \le j < v \le \alpha$ *and* $a, b \in \{1, \dots, \beta\}$.

Then the elements of T *are* $\mathbb{C}$*-linearly independent.*

Proof. Suppose that the elements of T are $\mathbb{C}$-linearly dependent and so their Wronskian vanishes by Proposition 3.1. Write W for

$$W_L(f_{s_1,k_1}, \dots, f_{s_1,k_\beta}, \dots, f_{s_\alpha,k_1}, \dots, f_{s_\alpha,k_\beta}).$$

Let $A(i) = (f_{s_1,k_1}(i) \ \cdots \ f_{s_1,k_\beta}(i) \ \cdots \ f_{s_\alpha,k_1}(i) \ \cdots \ f_{s_\alpha,k_\beta}(i))$, and

$$\det(A(i_0), A(i_1), \cdots, A(i_{r-1}))$$
$$:= \begin{vmatrix} f_{s_1,k_1}(i_0) & \dots & f_{s_1,k_\beta}(i_0) & \dots & f_{s_\alpha,k_1}(i_0) & \dots & f_{s_\alpha,k_\beta}(i_0) \\ f_{s_1,k_1}(i_1) & \dots & f_{s_1,k_\beta}(i_1) & \dots & f_{s_\alpha,k_1}(i_1) & \dots & f_{s_\alpha,k_\beta}(i_1) \\ \vdots & & & & & & \\ f_{s_1,k_1}(i_{r-1}) & \dots & f_{s_1,k_\beta}(i_{r-1}) & \dots & f_{s_\alpha,k_1}(i_{r-1}) & \dots & f_{s_\alpha,k_\beta}(i_{r-1}) \end{vmatrix},$$

where $r = \alpha\beta$. Then, for $\nu \in \mathbb{N}$,

$$W(\nu) = \sum_{i_0 i_1 \cdots i_{r-1} = \nu} (\log i_1)(\log i_2)^2 \dots (\log i_{r-1})^{r-1} \det(A(i_0), A(i_1), \cdots, A(i_{r-1})).$$

Taking $\nu = p_1 \cdots p_{r-1}$, where $p_1 < \cdots < p_{r-1}$ are distinct primes, we get

$$W(p_1 \cdots p_{r-1}) = C(p_1, \dots, p_{r-1}) \det(A(1), A(p_1), \dots, A(p_{r-1})),$$

where $C(p_1, \dots, p_{r-1}) \neq 0$ is the Vandermonde determinant defined by

$$C(i_1, \dots, i_{r-1}) = \begin{vmatrix} \log i_1 & \log i_2 & \dots & \log i_{r-1} \\ (\log i_1)^2 & (\log i_2)^2 & \dots & (\log i_{r-1})^2 \\ \vdots & & & \\ (\log i_1)^{r-1} & (\log i_2)^{r-1} & \dots & (\log i_{r-1})^{r-1} \end{vmatrix} \tag{3.2}$$
$$= \sum_\sigma \operatorname{sgn}(\sigma)(\log i_{\sigma(1)})^1 (\log i_{\sigma(2)})^2 (\log i_{\sigma(3)})^3 \cdots (\log i_{\sigma(r-1)})^{r-1},$$

where the summation is over all permutations σ of $\{1, \dots, r-1\}$ with $\operatorname{sgn}(\sigma) = \pm 1$ depending on whether σ is even or odd. Since $C(i_1, \dots, i_{r-1})$ is a Vandermonde determinant, we have $C(i_1, \dots, i_{r-1}) \neq 0$ if and only if

$i_1, \ldots, i_{r-1}$ are distinct and not equal to 1. We wish to derive a contradiction by showing that there are primes $p_1, \ldots, p_{r-1}$ such that

$$D := \det(A(1), A(p_1), \ldots, A(p_{r-1})) \neq 0.$$

For primes $p_1, \ldots, p_{r-1}$ sufficiently large, since the function values are non-zero by condition (1), we can write

$$D = f_{s_\alpha,k_\beta}(p_{r-1}) f_{s_\alpha,k_\beta}(p_{r-2}) \cdots f_{s_\alpha,k_\beta}(p_1) f_{s_\alpha,k_\beta}(1) D^*,$$

where

$$D^* = \begin{vmatrix} \frac{f_{s_1,k_1}(1)}{f_{s_\alpha,k_\beta}(1)} & \cdots & \frac{f_{s_1,k_\beta}(1)}{f_{s_\alpha,k_\beta}(1)} & \cdots & \frac{f_{s_\alpha,k_1}(1)}{f_{s_\alpha,k_\beta}(1)} & \cdots & \frac{f_{s_\alpha,k_{\beta-1}}(1)}{f_{s_\alpha,k_\beta}(1)} & 1 \\ \frac{f_{s_1,k_1}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} & \cdots & \frac{f_{s_1,k_\beta}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} & \cdots & \frac{f_{s_\alpha,k_1}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} & \cdots & \frac{f_{s_\alpha,k_{\beta-1}}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} & 1 \\ \vdots & & & & & & & \\ \frac{f_{s_1,k_1}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} & \cdots & \frac{f_{s_1,k_\beta}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} & \cdots & \frac{f_{s_\alpha,k_1}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} & \cdots & \frac{f_{s_\alpha,k_{\beta-1}}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} & 1 \\ \frac{f_{s_1,k_1}(p_{r-1})}{f_{s_\alpha,k_\beta}(p_{r-1})} & \cdots & \frac{f_{s_1,k_\beta}(p_{r-1})}{f_{s_\alpha,k_\beta}(p_{r-1})} & \cdots & \frac{f_{s_\alpha,k_1}(p_{r-1})}{f_{s_\alpha,k_\beta}(p_{r-1})} & \cdots & \frac{f_{s_\alpha,k_{\beta-1}}(p_{r-1})}{f_{s_\alpha,k_\beta}(p_{r-1})} & 1 \end{vmatrix}.$$

It thus suffices to show that $D^* \neq 0$. Expanding D^* along the last row, keeping $p_1, \ldots, p_{r-2}$ fixed for the moment and letting $p_{r-1} \to \infty$, by the asymptotic assumptions (2) and (3), we see that

$$D^* = D_1 + o(p_{r-1}),$$

where

$$D_1 = \begin{vmatrix} \frac{f_{s_1,k_1}(1)}{f_{s_\alpha,k_\beta}(1)} & \cdots & \frac{f_{s_1,k_\beta}(1)}{f_{s_\alpha,k_\beta}(1)} & \cdots & \frac{f_{s_\alpha,k_1}(1)}{f_{s_\alpha,k_\beta}(1)} & \cdots & \frac{f_{s_\alpha,k_{\beta-1}}(1)}{f_{s_\alpha,k_\beta}(1)} \\ \frac{f_{s_1,k_1}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} & \cdots & \frac{f_{s_1,k_\beta}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} & \cdots & \frac{f_{s_\alpha,k_1}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} & \cdots & \frac{f_{s_\alpha,k_{\beta-1}}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} \\ \vdots & & & & & & \\ \frac{f_{s_1,k_1}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} & \cdots & \frac{f_{s_1,k_\beta}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} & \cdots & \frac{f_{s_\alpha,k_1}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} & \cdots & \frac{f_{s_\alpha,k_{\beta-1}}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} \end{vmatrix}.$$

Observe that D_1 is independent of p_{r-1} and $\dim D_1 = \dim D - 1$. It is thus enough to show that $D_1 \neq 0$. Now we repeat the above steps by writing

$$D_1 = \frac{f_{s_\alpha,k_{\beta-1}}(p_{r-2})}{f_{s_\alpha,k_\beta}(p_{r-2})} \cdots \frac{f_{s_\alpha,k_{\beta-1}}(p_1)}{f_{s_\alpha,k_\beta}(p_1)} \frac{f_{s_\alpha,k_{\beta-1}}(1)}{f_{s_\alpha,k_\beta}(1)} D_1^*,$$

where

$$D_1^* = \begin{vmatrix} \frac{f_{s_1,k_1}(1)}{f_{s_\alpha,k_{\beta-1}}(1)} & \cdots & \frac{f_{s_1,k_\beta}(1)}{f_{s_\alpha,k_{\beta-1}}(1)} & \cdots & \frac{f_{s_\alpha,k_1}(1)}{f_{s_\alpha,k_{\beta-1}}(1)} & \cdots & \frac{f_{s_\alpha,k_{\beta-2}}(1)}{f_{s_\alpha,k_{\beta-1}}(1)} & 1 \\ \frac{f_{s_1,k_1}(p_1)}{f_{s_\alpha,k_{\beta-1}}(p_1)} & \cdots & \frac{f_{s_1,k_\beta}(p_1)}{f_{s_\alpha,k_{\beta-1}}(p_1)} & \cdots & \frac{f_{s_\alpha,k_1}(p_1)}{f_{s_\alpha,k_{\beta-1}}(p_1)} & \cdots & \frac{f_{s_\alpha,k_{\beta-2}}(p_1)}{f_{s_\alpha,k_{\beta-1}}(p_1)} & 1 \\ \vdots & & & & & & & \\ \frac{f_{s_1,k_1}(p_{r-2})}{f_{s_\alpha,k_{\beta-1}}(p_{r-2})} & \cdots & \frac{f_{s_1,k_\beta}(p_{r-2})}{f_{s_\alpha,k_{\beta-1}}(p_{r-2})} & \cdots & \frac{f_{s_\alpha,k_1}(p_{r-2})}{f_{s_\alpha,k_{\beta-1}}(p_{r-2})} & \cdots & \frac{f_{s_\alpha,k_{\beta-2}}(p_{r-2})}{f_{s_\alpha,k_{\beta-1}}(p_{r-2})} & 1 \end{vmatrix}.$$

It thus suffices to show that $D_1^* \neq 0$. Expanding D_1^* along the last row, keeping $p_1, \dots, p_{r-3}$ fixed for the time being and letting $p_{r-2} \to \infty$, by the asymptotic assumptions (2) and (3), we get

$$D_1^* = D_2 + o(p_{r-2}),$$

where

$$D_2 = \begin{vmatrix} \frac{f_{s_1,k_1}(1)}{f_{s_\alpha,k_{\beta-1}}(1)} & \cdots & \frac{f_{s_1,k_\beta}(1)}{f_{s_\alpha,k_{\beta-1}}(1)} & \cdots & \frac{f_{s_\alpha,k_1}(1)}{f_{s_\alpha,k_{\beta-1}}(1)} & \cdots & \frac{f_{s_\alpha,k_{\beta-2}}(1)}{f_{s_\alpha,k_{\beta-1}}(1)} \\ \frac{f_{s_1,k_1}(p_1)}{f_{s_\alpha,k_{\beta-1}}(p_1)} & \cdots & \frac{f_{s_1,k_\beta}(p_1)}{f_{s_\alpha,k_{\beta-1}}(p_1)} & \cdots & \frac{f_{s_\alpha,k_1}(p_1)}{f_{s_\alpha,k_{\beta-1}}(p_1)} & \cdots & \frac{f_{s_\alpha,k_{\beta-2}}(p_1)}{f_{s_\alpha,k_{\beta-1}}(p_1)} \\ \vdots & & & & & & \\ \frac{f_{s_1,k_1}(p_{r-3})}{f_{s_\alpha,k_{\beta-1}}(p_{r-3})} & \cdots & \frac{f_{s_1,k_\beta}(p_{r-3})}{f_{s_\alpha,k_{\beta-1}}(p_{r-3})} & \cdots & \frac{f_{s_\alpha,k_1}(p_{r-3})}{f_{s_\alpha,k_{\beta-1}}(p_{r-3})} & \cdots & \frac{f_{s_\alpha,k_{\beta-2}}(p_{r-3})}{f_{s_\alpha,k_{\beta-1}}(p_{r-3})} \end{vmatrix}.$$

Observe again that D_2 is independent of p_{r-2} and $\dim D_2 = \dim D_1 - 1$. It is again enough to show that $D_2 \neq 0$. Repeating the same reduction steps, we finally reach a nonzero determinant of dimension 1 as desired. ∎

Theorem 3.5 yields another proof of the following, slightly modified, Lemma 3 of Lucht–Schmalmack [5].

Corollary 3.6. *Let $\alpha \in \mathbb{N}$, $S = \{s_1, \dots, s_\alpha\} \subseteq \mathbb{C}$ with $\Re(s_1) < \cdots < \Re(s_\alpha)$, and let $K = \{0, 1, \dots, \beta\} \subseteq \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a fixed $a \in \mathbb{N} \setminus \{1\}$, let $T = \{a^\nu : \nu \in \mathbb{N}\}$ be a geometric progression such that $n^s \neq n^{s'}$ for all $n \in T$ and distinct $s, s' \in S$. Then the set*

$$\{I^s \log^k |_T : s \in S,\, k \in K\}$$

of arithmetic functions $(I^s \log^k)(n) := n^s (\log n)^k$, whose domain is restricted to the set T, is $\mathbb{C}$-linearly independent.

Proof. This follows immediately from Theorem 3.5 applied to the arithmetic functions

$$f_{sk}(\nu) = (I^s \log^k)(a^\nu) = a^{\nu s} (\log a^\nu)^k \qquad (\nu \in \mathbb{N}). \ ∎$$

In contrast to the $\mathbb{C}$-linear independence over the domain T, it is known (see e.g. [9] or [7]) that the functions $I^s \log^k$ are indeed $\mathbb{C}$-algebraically independent over the whole $\mathbb{N}$.

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