Comparison of $L^1$- and $L^\infty$-norms
of squares of polynomials

by

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1. Introduction. Let $\mathcal{P}(n)$ be the set of polynomials $P(X) = Q(X)^2$ where $Q$ is a nonzero polynomial of degree $< n$ with nonnegative real coefficients. We are interested in

$$A(n) = n^{-1} \sup_{P \in \mathcal{P}(n)} |P|_1 / |P|_\infty,$$

where $|P|_1$ is the sum, and $|P|_\infty$ the maximum of the coefficients of $P$. Let $\mathcal{F}$ be the set of functions $f = g * g$ where $*$ denotes convolution and $g$ runs through nonnegative, not identically zero, integrable functions with support in $[0, 1]$. Functions in $\mathcal{F}$ have support in $[0, 2]$. We set

$$B = \sup_{f \in \mathcal{F}} |f|_1 / |f|_\infty$$

where $|f|_1$ is the $L^1$-norm and $|f|_\infty$ the sup norm of $f$.

It is fairly obvious that

$$1 \leq A(n) \leq 2 - 1/n.$$

Indeed, the left inequality follows on taking $P = Q^2$ with $Q(X) = 1 + X + \ldots + X^{n-1}$, the right inequality is obtained by noting that $P \in \mathcal{P}(n)$ has at most $2n - 1$ nonzero coefficients, so that $|P|_1 / |P|_\infty \leq 2n - 1$. In a similar way one sees that

$$1 \leq B \leq 2.$$

Theorem 1. For natural $n, l$,

(i) $A(n) \leq A(nl)$,
(ii) $A(n) \leq B$,
(iii) $A(n) > B(1 - 6n^{-1/3})$.

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It follows that

\[ B = \lim_{n \to \infty} A(n) = \sup_n A(n). \]

The determination of \( B \) appears to be difficult.

**Theorem 2.** \( 4/\pi \leq B < 1.7373. \)

A slightly better upper bound will in fact be proved. We should mention that Ben Green [1] showed in effect that

\[ \left( \frac{|f|_1}{|f|_2} \right)^2 < \frac{7}{4} \]

for \( f \in \mathcal{F} \), where \( |f|_2 \) denotes the \( L^2 \)-norm. In fact he has the slightly better bound \( 1.74998 \ldots \). Since \( |f|^2_2 \leq |f|_1 |f|_{\infty} \), this yields \( B < 1.74998 \ldots \), which is only slightly weaker than the upper bound in Theorem 2. However, Green’s result is valid without the assumption \( g \geq 0 \).

On the other hand, Prof. Stanisław Kwapień (private communication) proved that

\[ A(n) \geq B(1 - 3(B/4)^{1/3}n^{-1/3}). \]

**2. Assertions (i), (ii) of Theorem 1.** When \( R \) is a polynomial or power series \( a_0 + a_1 X + \ldots \), set \( |R|_{\infty} \) for the maximum modulus of its coefficients. For such \( R \), and for a polynomial \( S \),

\[ |RS|_{\infty} \leq |R|_{\infty} |S|_1. \]

When \( P \in \mathcal{P}(n) \), say \( P = Q^2 \), set

\[ \tilde{Q} = (1 + X + \ldots + X^{l-1})Q(X^l) \quad \text{and} \quad \tilde{P} = \tilde{Q}^2. \]

Then \( \deg \tilde{Q} \leq l-1+l(n-1) = ln-1 \), so that \( \tilde{P} \in \mathcal{P}(ln) \). Further \( |\tilde{Q}|_1 = l|Q|_1 \), yielding

\[ |\tilde{P}|_1 = |\tilde{Q}|^2_1 = l^2|Q|_1^2 = l^2|P|_1. \]

For polynomials or series \( R = a_0 + a_1 X + \ldots \), \( S = b_0 + b_1 X + \ldots \) with nonnegative coefficients, write \( R \preceq S \) if \( a_i \leq b_i \) \( (i = 0, 1, \ldots) \). Then

\[ Q(X^l)^2 \preceq |Q|_{\infty}^2 (1 + X^l + X^{2l} + \ldots) = |P|_{\infty} (1 + X^l + X^{2l} + \ldots). \]

Therefore

\[ \tilde{P} = (1 + X + \ldots + X^{l-1})^2 Q(X^l)^2 \preceq |P|_{\infty} (1 + X^l + X^{2l} + \ldots)(1 + X + \ldots + X^{l-1})^2 = |P|_{\infty} (1 + X + X^2 + \ldots)(1 + X + \ldots + X^{l-1}). \]
Norms of squares of polynomials

Now (2.1) gives $|\tilde{P}|_\infty \leq |P|_\infty l$. Together with (2.2) this yields $n^{-1}|P|_1/|P|_\infty \leq (ln)^{-1}|P|_1/|\tilde{P}|_\infty \leq A(nl)$. Assertion (i) follows.

We now turn to (ii). Let $P \in \mathcal{P}(n)$ be given, say $P = Q^2$ with $Q = a_0 + a_1X + \ldots + a_{n-1}X^{n-1}$. Let $g$ be the function with support in $[0,1)$ having

$$g(x) = a_i \quad \text{for } i/n \leq x < (i + 1)/n \ (i = 0, 1, \ldots, n-1),$$

i.e., for $[nx] = i$. Then $|g|_1 = n^{-1}|Q|_1$, so that $f = g \ast g$ has

$$|f|_1 = n^{-2}|Q^2|_1 = n^{-2}|P|_1. \tag{2.3}$$

Let $x$ be given. The interval $I = [0,1)$ is the disjoint union of the intervals (possibly empty) $I_{i,j}(x)$ ($i = 0, 1, \ldots, n-1; j \in \mathbb{Z}$) consisting of numbers $y$ with

$$[ny] = i, \quad [n(x-y)] = j - i.$$

When $y \in I_{i,j}(x)$ and $0 \leq i' < n$, then $y + (i' - i)/n \in I_{i',j}(x)$. Therefore $I_{i,j}(x)$ has length independent of $i$; denote this length by $L_j(x)$. Clearly $L_j(x) = 0$ unless $j = [nx]$ or $[nx - 1]$. We have

$$1 = \sum_{i=0}^{n-1} \sum_j L_j(x) = n \sum_j L_j(x). \tag{2.4}$$

For $y \in I_{i,j}(x)$ with $0 \leq i < n$,

$$g(y)g(x-y) = \begin{cases} a_ia_{j-i} & \text{when } j-n < i \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\int_{I_{i,j}(x)} g(y)g(x-y) \, dy = \begin{cases} a_ia_{j-i} & \text{when } j-n < i \leq j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Now

$$\sum_{i=0}^j a_i a_{j-i} = b_j \leq |P|_\infty,$$

where $b_j$ is the coefficient of $X^j$ in $P$. Taking the sum of (2.5) over $i = 0, 1, \ldots, n-1$ and $j \in \mathbb{Z}$, and observing (2.4), we obtain

$$f(x) = \int g(y)g(x-y) \, dy \leq |P|_\infty \sum_j L_j(x) = |P|_\infty/n.$$

Therefore $|f|_\infty \leq |P|_\infty/n$, so that in conjunction with (2.3),

$$n^{-1}|P|_1/|P|_\infty \leq |f|_1/|f|_\infty \leq B.$$

Assertion (ii) follows.
3. Assertion (iii) of Theorem 1. Pick \( f \in \mathcal{F} \) with \(|f_1|/|f|_\infty \) close to \( B \). We may suppose that \(|f|_\infty = 1 \) and \(|f_1| \) is close to \( B \), in particular that \(|f_1| \geq 1 \). Say \( f = g * g \). Then for \( r < s \),

\[
(3.1) \quad \left( \int_r^s g(x) \, dx \right)^2 \leq \iint_{2r \leq x+y \leq 2s} g(x)g(y) \, dx \, dy
\]

\[
= \int_{2r}^{2s} dz \int_{2r}^{z} g(x)g(z-y) \, dy = \int_{2r}^{2s} f(z) \, dz \leq 2(s-r).
\]

Setting \( G(y) = \int_0^y g(y) \, dy \), so that \( G(y) \leq \sqrt{2y} \), and using partial integration, we obtain

\[
(3.2) \quad \int_0^\delta (\delta - x)g(x) \, dx = \int_0^\delta G(y) \, dy \leq \int_0^\delta (2y)^{1/2} \, dy < \delta^{3/2}.
\]

Similarly,

\[
\int_{1-\delta}^1 (\delta - (1-x))g(x) \, dx < \delta^{3/2}.
\]

With \( c \in \frac{1}{2} \mathbb{Z} \) in \( 1 \leq c \leq (n-1)/2 \) to be determined later, set

\[
a_i = \frac{n}{2c} \int_{(i+1/2-c)/n}^{(i+1/2+c)/n} g(x) \, dx \quad (0 \leq i < n)
\]

and

\[
Q(X) = \sum_{i=0}^{n-1} a_i X^i.
\]

Then

\[
|Q|_1 = \sum_{i=0}^{n-1} a_i = \frac{n}{2c} \int_0^1 \nu(x)g(x) \, dx
\]

where \( \nu(x) \) is the number of integers \( i \), \( 0 \leq i < n \), having \( (i+1/2-c)/n \leq x \leq (i+1/2+c)/n \). Then \( \nu(x) \) is the number of integers \( i \) having

\[
\max(0, nx - 1/2 - c) \leq i \leq \min(n - 1, nx - 1/2 + c).
\]

When \( (c+1/2)/n \leq x \leq 1 - (c+1/2)/n \), this becomes the interval \( nx - 1/2 - c \leq i \leq nx - 1/2 + c \), so that \( \nu(x) \geq 2c \), as \( c \in \frac{1}{2} \mathbb{Z} \). When \( x < (c+1/2)/n \), the interval becomes \( 0 \leq i \leq nx - 1/2 + c \), and \( \nu(x) \geq nx + c - 1/2 = 2c - (c + 1/2 - nx) \). On the other hand when \( x > 1 - (c+1/2)/n \), then \( \nu(x) \geq 2c - (c+1/2 - n(1-x)) \). Therefore
\[ |Q|_1 \geq n \int_0^1 g(x) \, dx - \frac{n}{2c} \int_0^{(c+1/2)/n} (c + 1/2 - nx)g(x) \, dx \]
\[ - \frac{n}{2c} \int_1^{1-(c+1/2)/n} (c + 1/2 - (1-x))g(x) \, dx. \]

Applying (3.2) with \( \delta = (c + 1/2)/n \) we obtain
\[ \frac{n}{2c} \int_0^{(c+1/2)/n} (c + 1/2 - nx)g(x) \, dx \]
\[ < \frac{n^2}{2c} ( (c + 1/2)/n )^{3/2} < n((c + 1/2)/n)^{1/2}. \]

The same bound applies to the last term on the right hand side of (3.3), so that
\[ |Q|_1 \geq n|g|_1(1 - 2((c + 1/2)/n)^{1/2}/|g|_1). \]

Here \( |g|_1 \geq 1 \) since \( |f|_1 \geq 1. \)

The polynomial \( P = Q^2 \) lies in \( \mathcal{P}(n) \) and has
\[ |P|_1 \geq n^2|f|_1(1 - 4((c + 1/2)/n)^{1/2}). \]

The coefficients of \( P \) are
\[ b_l = \sum_{i+j=l} a_i a_j = \left( \frac{n}{2c} \right)^2 \int \int g(x)g(y) \, dx \, dy. \]

Setting \( z = x + y \), so that \( (l + 1 - 2c)/n \leq z \leq (l + 1 + 2c)/n \), we obtain
\[ b_l = \left( \frac{n}{2c} \right)^2 \int \int \mu(z, x)g(x)g(z - x) \, dx \]

where \( \mu(z, x) \) is the number of integers \( i \) in \( 0 \leq i \leq n-1 \) with \( (i+1/2-c)/n \leq x \leq (i+1/2+c)/n \) and \( (l-i+1/2-c)/n \leq z - x \leq (l-i+1/2+c)/n \). Thus \( h = i - nx + 1/2 \) lies in the range
\[ \max(-c, -c + l + 1 - nz) \leq h \leq \min(c, c + l + 1 - nz), \]
and \( \mu(z, x) \leq \lambda(z) \), which is the length of the “interval” (possibly empty)
\[ -c - 1/2 + \max(0, l + 1 - nz) \leq h \leq c + 1/2 + \min(0, l + 1 - nz). \]
Therefore
\[ b_l \leq \frac{n}{2c} \int dz \lambda(z) \int g(x) g(z-x) \, dx \]
\[ = \frac{n}{2c} \int dz \lambda(z) f(z) \leq \left( \frac{n}{2c} \right)^2 \int \lambda(z) \, dz. \]

But \( \int \lambda(z) \, dz \) is the area of the domain in the \((h, z)\)-plane given by (3.5). Here \( h \) is contained in an interval of length \( 2c + 1 \), and given \( h \), the variable \( z \) lies in an interval of length \( (2c + 1)/n \), so that
\[ b_l \leq \left( \frac{n}{2c} \right)^2 (2c + 1)^2/n = n \left( 1 + \frac{1}{2c} \right)^2. \]

Therefore \( |P|_{\infty} \leq n(1 + 1/(2c))^2 \), and by (3.4),
\[ A(n) \geq \frac{1}{n} |P|_{1}/|P|_{\infty} \geq |f|_{1} \left( 1 - 4 \left( \left( c + \frac{1}{2} \right)/n \right)^{1/2} \right)/\left( 1 + \frac{1}{2c} \right)^2. \]

We now pick \( c \in \frac{1}{2} \mathbb{Z} \) with \( n^{1/3} - 1 \leq c < n^{1/3} - 1/2 \). When \( n \geq 8 \), which we may clearly suppose in proving assertion (iii), then \( 1 \leq n^{1/3}/2 \leq c < (n - 1)/2 \). Since \( f \) may be chosen with \( |f|_{1} \) arbitrarily close to \( B \),
\[ A(n) \geq B(1 - 4n^{-1/3})/(1 + n^{-1/3})^2 > B(1 - 6n^{-1/3}). \]

4. The lower bound in Theorem 2. Set \( f = g * g \) where \( g(x) = x^{-1/2} \) in \( 0 < x < 1 \), and \( g(x) = 0 \) otherwise. Then \( f \in \mathcal{F} \), and \( |f|_{1} = |g|_{1}^2 = 4 \). For \( 0 < z \leq 2 \),
\[ f(z) = \int (z-x)^{-1/2} x^{-1/2} \, dx, \]
with the range of integration \( \max(0, z - 1) \leq x \leq \min(1, z) \). Setting \( x = y^2 z \) we obtain
\[ f(z) = 2 \int \frac{dy}{(1 - y^2)^{1/2}}, \]
the integration being over \( y \geq 0 \) with \( 1 - 1/z \leq y^2 \leq \min(1/z, 1) \). When \( 0 < z \leq 1 \), this range is \( 0 \leq y \leq 1 \), so that \( f(z) = \pi \), whereas in \( 1 < z \leq 2 \) the range is smaller, and \( f(z) < \pi \). We may conclude that \( |f|_{\infty} = \pi \), and \( B \geq |f|_{1}/|f|_{\infty} = 4/\pi \).

5. The upper bound \( B \leq 7/4 \). The upper bound of Theorem 2 will be established in three stages. Here we will show that \( B \leq 7/4 = 1.75 \), and in the following stages we will prove that \( B \leq 7/4 - 1/80 = 1.7375 \), then that \( B \leq 1.7373 \).

Our problem is invariant under translations. To exhibit symmetry, we therefore redefine \( \mathcal{F} \) to consist of functions \( f = g * g \) with \( g \) nonzero, non-negative and integrable, with support in \([-1/2, 1/2] \), so that \( f \) has support
in $[-1, 1]$. We will suppose throughout that $f \in \mathcal{F}$ with $|f|_\infty = 1$, and we will give upper bounds for $|f|_1$.

Lemma 1.

$$\int_{1/2}^1 f(z)f(-z)\,dz \leq 1/4.$$ 

As a consequence of this lemma,

$$|f|_1 = \int_{-1}^1 f(z)\,dz = \int_0^1 (f(z) + f(-z))\,dz \leq 1 + \int_{1/2}^1 (f(z) + f(-z))\,dz$$

$$\leq 1 + \int_{1/2}^1 (1 + f(z)f(-z))\,dz \leq \frac{3}{2} + \frac{1}{4} = \frac{7}{4},$$

so that indeed $B \leq 7/4$.

Proof of Lemma 1.

(5.1) \[ f(z) = (g \ast g)(z) = \int g(x)g(z-x)\,dx = 2 \int_{x+y=z} g(x)g(y)\,dx. \]

(It is to exhibit symmetry that we write $y$ for $z-x$.) Similarly

(5.2) \[ f(-z) = 2 \int_{u+v=-z}^{u \leq v} g(u)g(v)\,du. \]

Here $x, y, u, v$ may be restricted to lie in $[1/2, -1/2]$. When $\delta \geq 0$ and $z \geq 1/2 - \delta$, then $x = z - y \geq 1/2 - \delta - 1/2 = -\delta$, also $v = -u - z \leq 1/2 - 1/2 + \delta = \delta$, so that

$$u \leq v \leq \delta, \quad -\delta \leq x \leq y.$$

We obtain

$$\int_{1/2-\delta}^1 f(z)f(-z)\,dz \leq 4 \int_{1/2-\delta}^1 dz \int_{-\delta \leq x \leq y}^{u \leq v \leq \delta} g(x)g(y)g(u)g(v)\,dx\,du.$$

In this integral $u \leq -z/2 \leq -1/4 + \delta/2$, and $y \geq z/2 \geq 1/4 - \delta/2$. Setting $w = u + y = -x - v$ we have $w \leq u + 1/2 \leq 1/4 + \delta/2$, and in fact $|w| \leq 1/4 + \delta/2$. Replacing the variables $x, u, z$ in the above integral by
x, y = z - x, w = u + z - x, we obtain the bound

\[
(5.3) \quad 4 \int_{-1/4-\delta/2}^{1/4+\delta/2} dw \int_{x+y\geq 1/2-\delta}^{y+u=w} g(x)g(y)g(u)g(v) \, dx \, dy.
\]

Let us now take \( \delta = 0. \) In this case

\[
\int_{1/2}^{1} f(z)f(-z) \, dz \leq 4 \int_{-1/4}^{1/4} dw \int_{u\leq v\leq 0\leq x\leq y}^{x+v=-w} g(x)g(y)g(u)g(v) \, dx \, dy.
\]

Interchanging the rôles of the variables \( x, y, \) and as a result those of \( u, v, \) and replacing \( w \) by \(-w,\) we get an integral as before, except that the region \( u \leq v \leq 0 \leq x \leq y \) is replaced by the region \( v \leq u \leq 0 \leq y \leq x. \) These regions are essentially disjoint, and are contained in \( u \leq 0 \leq y, \ v \leq 0 \leq x. \) We therefore obtain

\[
\leq 2 \int_{-1/4}^{1/4} dw \left( \int_{v\leq 0\leq x}^{x+v=-w} g(x)g(v) \, dx \right) \left( \int_{u\leq 0\leq y}^{y+u=w} g(y)g(u) \, dy \right)
\]

\[
= 2 \int_{-1/4}^{1/4} dw \tilde{f}(w)\tilde{f}(-w)
\]

with

\[
(5.4) \quad \tilde{f}(w) = \int_{y+u=w}^{y\leq 0\leq w} g(y)g(u) \, dy.
\]

Thus

\[
(5.5) \quad \int_{1/2}^{1} f(z)f(-z) \, dz \leq 4 \int_{0}^{1/4} \tilde{f}(w)\tilde{f}(-w) \, dw.
\]

It is clear from (5.1) and (5.4) that \( \tilde{f}(w) \leq f(w)/2 \leq 1/2, \) so that we obtain \( \leq 1/4, \) and Lemma 1 follows. 

6. The upper bound \( B \leq 1.7375. \) With \( f = g * g \) as above, and \( \varepsilon = \pm 1, \) set

\[
I_{\varepsilon} = \int_{0}^{1/8} g(\varepsilon x) \, dx, \quad J_{\varepsilon} = \int_{\varepsilon y \geq 0, \varepsilon u > 0}^{\varepsilon (y+u) \leq 1/4} g(y)g(u) \, dy \, du.
\]
Lemma 2. (i) $\int_{1/2}^{1} f(z) f(-z) \, dz \leq 1/4 - J_\varepsilon$.

(ii) For $0 \leq \delta \leq 1/6$,

$$\int_{1/2-\delta}^{1} f(z) f(-z) \, dz \leq \frac{1}{4} + \frac{\delta}{2} + \left( \int_{-\delta}^{\delta} g(x) \, dx \right)^2.$$

As a consequence,

$$|f|_1 = \int_{0}^{1/2} (f(z) + f(-z)) \, dz = \int_{0}^{1/2-\delta} f(z) f(-z) \, dz + \int_{1/2-\delta}^{1} f(z) f(-z) \, dz \leq 1 - 2\delta + \int_{1/2-\delta}^{1} (1 + f(z) f(-z)) \, dz$$

$$\leq \frac{3}{2} - \delta + \int_{1/2-\delta}^{1} f(z) f(-z) \, dz \leq \frac{7}{4} - \frac{\delta}{2} + \left( \int_{-\delta}^{\delta} g(x) \, dx \right)^2.$$

Setting $\delta = 1/8$ we obtain

$$|f|_1 \leq \frac{27}{16} + (I_1 + I_{-1})^2 \leq \frac{27}{16} + 4M^2$$

with $M = \max(I_1, I_{-1})$. On the other hand by (i),

$$|f|_1 \leq \frac{3}{2} + \int_{1/2}^{1} f(z) f(-z) \, dz \leq \frac{7}{4} - \max_{\varepsilon = \pm 1} J_\varepsilon \leq \frac{7}{4} - M^2.$$

In conjunction with (6.2) this gives $|f|_1 \leq 7/4 - 1/80 = 1.7375$, so that indeed $B \leq 1.7375$.

Proof of Lemma 2. When $w > 0$, we cannot have $y + u = w$ and $u \leq y < 0$. Therefore $\tilde{f}(w)$ as given by (5.4) is

$$\tilde{f}(w) = \int_{y+u=w}^{u \leq y} g(y)g(u) \, dy - \int_{y+u=w}^{0 \leq u \leq y} g(y)g(u) \, dy = \frac{1}{2} f(w) - \frac{1}{2} \tilde{f}(w)$$

with

$$\tilde{f}(w) = \int_{y+u=w}^{y,u \geq 0} g(y)g(u) \, dy.$$

Now (5.5) yields

$$\int_{1/2}^{1} f(z) f(-z) \, dz \leq \int_{0}^{1/4} (f(w) - \tilde{f}(w)) f(-w) \, dw \leq \int_{0}^{1/4} (1 - \tilde{f}(w)) \, dw.$$
The bound $1/4 - J_{-1}$ is obtained similarly, so that assertion (i) is established.

We will now suppose $\delta > 0$, and we return to the bound (5.3). We first deal with the part where $v \leq x$ in the integral, so that

$$u \leq v \leq x.$$

After interchanging the rôles of $x$ and $y$, and of $u$ and $v$, and replacing $w$ by $-w$, the integrand will be the same, but now

$$v \leq u \leq y \leq x.$$

The interiors of the domains (6.4), (6.5) are disjoint, and are contained in the region with $v \leq x$ and $u \leq y$, so that this part of (5.3) is

$$\leq 2 \int_{-1/4 - \delta/2}^{1/4 + \delta/2} dw \left( \int_{x+v=-w}^{x} g(x)g(v) \, dx \right) \left( \int_{y+u=w}^{1/4 + \delta/2} g(y)g(u) \, dy \right)$$

$$= \frac{1}{2} \int_{-1/4 - \delta/2}^{1/4 + \delta/2} dw \, f(-w) \, f(w) = \int_{0}^{1/4 + \delta/2} f(w) \, f(-w) \, dw \leq 1/4 + \delta/2.$$

It remains for us to deal with the part of (5.3) where $x \leq v$ in the integral, so that $-\delta \leq x \leq v \leq \delta$. This part is

$$\leq 4 \int dw \int_{-\delta \leq x \leq v \leq \delta} g(x)g(v) \, dx \int_{y+u=w}^{1/2 - \delta - x} g(y)g(u) \, dy.$$

When $0 < \delta \leq 1/6$, then $y \geq 1/2 - 2\delta \geq \delta \geq u$, and the last integral is

$$\leq \int_{y+u=w}^{1/2 - \delta - x} g(y)g(u) \, dy = f(w)/2 \leq 1/2.$$

Therefore the part in question of (5.3) becomes

$$\leq 2 \int dw \int_{-\delta \leq x \leq v \leq \delta} g(x)g(v) \, dx = \int dw \int_{-\delta \leq x \leq v \leq \delta} g(x)g(v) \, dx = \left( \int_{-\delta}^{\delta} g(x) \, dx \right)^2.$$

Together with (6.6) this gives the asserted bound for $\int_{1/2 - \delta}^{1} f(z) f(-z) \, dz$. ■
7. The upper bound 1.7373. In fact we will show that

\[(7.1) \quad B \leq 7/4 - 1/80 - \xi < 1.7373\]

where $\xi = 0.000200513 \ldots$ is a root of the transcendental equation

$$ F(b(x)/a(x)) = 1/2, $$

where $a(x) = 1/10 - 2x$, $b(x) = (\sqrt{1/20} - x - \sqrt{1/80 + x})^2/2$, and

$$ F(x) = \sqrt{x^2 + x + \log(\sqrt{x^2 + x + \sqrt{x}})}. $$

The calculation of $\xi$ has kindly been performed by Dr. A. Pokrzywa.

We will suppose that $f \in \mathcal{F}$, $|f|_{\infty} = 1$ and

\[(7.2) \quad |f|_1 > 7/4 - 1/80 - \xi,\]

and we will reach a contradiction, thereby establishing the truth of (7.1), and hence of Theorem 2.

Retaining earlier notation we now set $a = a(\xi)$,

$$ u = I_1 + I_{-1}, \quad v = |I_1 - I_{-1}|, \quad m = \min(I_1, I_{-1}) = (u - v)/2, $$

and observe that $M = \max(I_1, I_{-1}) = (u + v)/2$. Also, $u_0, u_1$ will be the positive numbers with

$$ u_0^2 = 1/20 - \xi = a/2, \quad u_1^2 = 1/20 + 4\xi. $$

We may suppose that

\[(7.3) \quad u \geq u_0,\]

for otherwise (6.2) yields $|f|_1 \leq 27/16 + u_0^2 = 7/4 - 1/80 - \xi$, against (7.2).

We further may suppose that

\[(7.4) \quad u + v \leq u_1,\]

for otherwise (6.3) yields $|f|_1 \leq 7/4 - u_1^2/4 = 7/4 - 1/80 - \xi$, contradicting (7.2). As a consequence,

$$ 2u^2 - m^2/2 = 2u^2 - (u - v)^2/8 = 3u^2/2 + u(u + v)/2 - (u + v)^2/8 \leq 3u^2/2 + 3u(u + v)/8 \leq 15u_1^2/8 < 1/10 - 2\xi = a, $$

so that

\[(7.5) \quad 0 = 2u_0^2 - a \leq 2u^2 - a < m^2/2.\]

Lemma 3.

$$ \frac{7}{4} - |f|_1 \geq \frac{1}{4}(u^2 + v^2) + \int_{2u^2 - a}^{m^2/2} (\sqrt{(\eta + a)/2} - u) \frac{d\eta}{\sqrt{2\eta}}. $$
Proof. By (6.1) and (7.2),

$$1/80 + \xi > \delta/2 - \left( \int_{-\delta}^{\delta} g(x) \, dx \right)^2$$

for $\delta$ in $0 < \delta < 1/6$. Setting $\delta = 1/8 + \eta$ with $0 < \eta < 1/24$, this gives

$$\left( \int_{-1/8-\eta}^{1/8+\eta} g(x) \, dx \right)^2 > \eta/2 + 1/20 - \xi = (\eta + a)/2,$$

and

(7.6) \quad \quad G(\eta) := \int_{1/8}^{1/8+\eta} (g(x) + g(-x)) \, dx > \sqrt{(\eta + a)/2} - u.

On the other hand by (6.3) and (7.2), and since $m^2/2 \leq u^2/8 \leq u_1^2/8 < 1/24 < 1/8$,

$$\frac{1}{80} + \xi > \frac{1}{2} \sum_{\varepsilon = \pm 1} J_{\varepsilon} = \frac{1}{2} \left( I_1^2 + I_{-1}^2 + 2 \sum_{\varepsilon = \pm 1} \frac{1}{1/8} \int_{0}^{1/4} g(\varepsilon x) \, dx \int_{0}^{1/4-x} g(\varepsilon y) \, dy \right)$$

$$\geq \frac{1}{2} \left( \frac{u^2 + v^2}{2} + 2 \sum_{\varepsilon = \pm 1} \int_{1/8}^{1/8+m^2/2} g(\varepsilon x) \, dx \int_{0}^{1/4-x} g(\varepsilon y) \, dy \right)$$

$$= \frac{1}{4} (u^2 + v^2) + \sum_{\varepsilon = \pm 1} \int_{0}^{m^2/2} g(\varepsilon/8 + \varepsilon \eta) \, d\eta \int_{0}^{1/8-\eta} g(\varepsilon y) \, dy.$$

By (3.1) with $r = 1/8 - \eta$, $s = 1/8$,

$$\int_{0}^{1/8} g(\varepsilon y) \, dy = I_{\varepsilon} - \int_{1/8-\eta}^{1/8} g(\varepsilon y) \, dy \geq I_{\varepsilon} - \sqrt{2\eta} \geq m - \sqrt{2\eta}.$$

Thus

$$\frac{1}{80} + \xi > \frac{1}{4} (u^2 + v^2) + \sum_{\varepsilon = \pm 1} \int_{0}^{m^2/2} g(\varepsilon/8 + \varepsilon \eta)(m - \sqrt{2\eta}) \, d\eta$$

$$= \frac{1}{4} (u^2 + v^2) + \int_{0}^{m^2/2} (g(1/8 + \eta) + g(-1/8 - \eta))(m - \sqrt{2\eta}) \, d\eta.$$ 

Integrating by parts we represent the last integral as

$$\int_{0}^{m^2/2} G(\eta) \frac{d\eta}{\sqrt{2\eta}} \geq \int_{2a^2-a}^{m^2/2} G(\eta) \frac{d\eta}{\sqrt{2\eta}}.$$

Since $m^2/2 < 1/24$ we may apply (7.6) to obtain the lemma. \blacksquare
Lemma 4. In the domain of points \((u, v)\) with (7.3), (7.4), \(v \geq 0\), the function

\[ H(u, v) = \frac{1}{4} (u^2 + v^2) + \frac{\frac{1}{2} (u-v)^2}{2u^2-a} \int_{2u^2-a} \frac{\sqrt{\eta + a}/2 - u}{\sqrt{2}\eta} \, d\eta \]

satisfies \(H(u, v) \geq H(u_0, u_1 - u_0)\).

Proof.

\[ 2H(u, v) = \frac{1}{2} (u^2 + v^2) + \frac{\frac{1}{2} (u-v)^2}{2u^2-a} \sqrt{\frac{\eta + a}{\eta}} - u(u-v) + 2u\sqrt{4u^2 - 2a}. \]

Hence

\[ 2 \frac{\partial H(u, v)}{\partial v} = v + u + \left( \frac{(u-v)^2 + 8a}{(u-v)^2} \right)^{1/2} \cdot \frac{v-u}{4} \]

\[ = v + u - \frac{1}{4} \left( (u-v)^2 + 8a \right)^{1/2}. \]

We claim that this partial derivative is \(\leq 0\) in our domain. For otherwise \(16(u+v)^2 - ((u-v)^2 + 8a) > 0\), or \(15(u+v)^2 + 4uv - 8a > 0\). But \(u+v \leq u_1\) and \(4uv \leq 4u(u_1 - u) \leq 4u_0(u_1 - u_0)\) since \(u \geq u_0 > u_1/2\). Therefore \(15u_1^2 + 4u_0u_1 - 4u_0^2 - 8a > 0\). Substituting the values for \(a, u_0, u_1\) gives

\[ 4u_0u_1 \geq 1/4 - 80\xi. \]

Squaring, we get

\[ 16(1/20 + 4\xi)(1/20 - \xi) > (1/4 - 80\xi)^2, \]

which is not true. Thus our claim is proven, and

(7.7) \(H(u, v) \geq H(u, u_1 - u)\).

Next,

\[ 2H(u, u_1 - u) = -u^2 + \frac{1}{2} u_1^2 + \frac{\frac{1}{2} (2u - u_1)^2}{2u^2-a} \sqrt{\frac{\eta + a}{\eta}} - u(u-2u) + 2u\sqrt{4u^2 - 2a}, \]

so that

\[ 2 \frac{d}{du} H(u, u_1 - u) = -2u + \left( \frac{(2u - u_1)^2 + 8a}{(2u - u_1)^2} \right)^{1/2} \cdot \frac{2u - u_1}{2} \]

\[ - \left( \frac{2u^2}{2u^2-a} \right)^{1/2} \cdot 4u \]

\[ + 2(4u^2 - 2a)^{1/2} + 8u^2(4u^2 - 2a)^{-1/2} \]

\[ = -2u + \frac{1}{2} \sqrt{(2u - u_1)^2 + 8a} + 2\sqrt{4u^2 - 2a}. \]
We claim that this derivative is $\geq 0$ for $u_0 \leq u \leq u_1$. For otherwise $16u^2 \geq (2u - u_1)^2 + 8a$, so that $12u^2 + 4uu_1 - u_1^2 > 8a$. But this entails $15u_1^2 > 8a$, i.e.,

$$15(1/20 + 4\xi) > 4/5 + 16\xi,$$

which is not true. Thus our claim is correct, and

$$H(u, u_1 - u) \geq H(u_0, u_1 - u_0),$$

which together with (7.7) establishes the lemma. ■

It is now easy to arrive at the desired contradiction to (7.2). By Lemmas 3 and 4,

$$7/4 - |f|_1 \geq H(u_0, u_1 - u_0)$$

$$= \frac{1}{4}(u_0^2 + (u_1 - u_0)^2) + \int_{2u_0^2-a} \left( \frac{1}{2} \sqrt{\frac{\eta + a}{\eta}} - \frac{u_0}{\sqrt{2\eta}} \right) d\eta.$$ 

Here $2u_0^2 - a = 0$ and $\frac{1}{2}(u_0 - \frac{1}{2}u_1)^2 = b(\xi) = b$, say, and

$$\int_0^x \sqrt{\frac{\eta + a}{\eta}} d\eta = aF(x/a), \quad \int_0^x \frac{d\eta}{\sqrt{2\eta}} = \sqrt{2x}.$$ 

Therefore

$$7/4 - |f|_1 \geq \frac{1}{4}(2u_0^2 - 2u_0u_1 + u_1^2) + \frac{a}{2}F(b/a) - u_0(u_0 - u_1/2)$$

$$= -u_0^2/2 + u_1^2/4 + \frac{a}{2}F(b/a) = -\frac{1}{80} + \frac{3}{2}\xi + \frac{a}{2}F(b/a) = 1/80 + \xi,$$

contrary to (7.2). ■

**Added in proof.** Dr. Erik Bajalinov has checked that for $n \leq 26$ and $n = 31, 36, 41, 46, 51$: $A(n) < 4/\pi$, which suggests that $B = 4/\pi$.

**References**