On totally real Hilbert–Speiser fields of type $C_p$

by

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1. Introduction. Let $L/K$ be a finite Galois extension of number fields with Galois group $G$. Then by a theorem of Noether it is well known that the ring of integers $O_L$ is a projective module over the group ring $O_KG$ if and only if $L/K$ is tamely ramified. If $O_L$ is in fact free (necessarily of rank 1) over $O_KG$, then $L/K$ is said to have a normal integral basis.

A number field $K$ is called a Hilbert–Speiser field if every finite abelian tamely ramified extension $L/K$ has a normal integral basis. The celebrated Hilbert–Speiser Theorem says that $\mathbb{Q}$ is such a field, and the main result of [GRRS99] is that $\mathbb{Q}$ is in fact the only such field. By fixing a finite abelian group $G$ one can consider a finer problem: given a number field $K$, does every tame $G$-Galois extension $L/K$ have a normal integral basis? If so, $K$ is said to be a Hilbert–Speiser field of type $G$. The simplest case to consider is when $G = C_p$, the cyclic group of prime order $p$. This has been studied, for instance, in [Car03], [Car04], [Her05], [Ich02], [Ich04], [Ich07a], [Ich07b] and [IST07]. We continue the investigation of this case by establishing the following result, the proof of which is based on a detailed analysis of locally free class groups and ramification indices.

**Theorem 1.1.** Let $K$ be a totally real number field and let $p \geq 5$ be prime. Suppose that $K/\mathbb{Q}$ is ramified at $p$. If $p = 5$ and $[K(\zeta_5) : K] = 2$, assume further that there exists a prime $p$ of $K$ above $p$ such that the ramification index of $p$ in $K/\mathbb{Q}$ is at least 3. Then $K$ is not Hilbert–Speiser of type $C_p$.

**Remark 1.2.** Some extra conditions in the case $p = 5$ and $[K(\zeta_5) : K] = 2$ are required because, for example, as noted in [Ich07a, Remark 1], $K = \mathbb{Q}(\sqrt{5})$ is in fact Hilbert–Speiser of type $C_5$.

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Theorem 1.1 can be seen as an analogue of the following result of Herreng (see [Her05, §3]). The authors are grateful to Nigel P. Byott for pointing out that the original hypothesis that $K/\mathbb{Q}$ is Galois can be weakened as below.

**Theorem 1.3 (Herreng).** Let $K$ be a totally imaginary number field and let $p$ be an odd prime. Suppose that every prime $\mathfrak{p}$ of $K$ above $p$ is ramified. If

- (a) $p > [K : \mathbb{Q}]$, or
- (b) $p \geq 5$ and $\zeta_p \in K$, or
- (c) $p \geq 7$ and the ramification index in $K/\mathbb{Q}$ of every prime $\mathfrak{p}$ of $K$ above $p$ is at least 3,

then $K$ is not Hilbert–Speiser of type $C_p$.

Combining Theorems 1.1 and 1.3(a) we immediately obtain the following result, which in many (but not all) respects is a significant sharpening of [Ich07a, Theorems 1 and 2].

**Theorem 1.4.** Let $K$ be a Hilbert–Speiser field of type $C_p$ for some odd prime $p$. If either

- (a) $K$ is totally real and $p \geq 7$, or
- (b) $K$ is totally imaginary and $p > [K : \mathbb{Q}]$,

then $K \cap \mathbb{Q}(\zeta_p^n) = \mathbb{Q}$ for all $n \geq 1$.

2. Realizable classes. We briefly recall the work of McCulloh on realizable classes in the special case of cyclic extensions of prime degree (see [McC83] for further details).

Let $K$ be a number field and let $p$ be a prime. Let $\Delta \simeq (\mathbb{Z}/p\mathbb{Z})^\times$ be the group of automorphisms of $C_p$. Then the locally free class group $\text{Cl}(\mathcal{O}_KC_p)$ is a $\Delta$-module. As $L/K$ varies over all tame $C_p$-Galois extensions of $K$, the class $(\mathcal{O}_L)$ of $\mathcal{O}_L$ varies over a subset $R(\mathcal{O}_KC_p)$ of $\text{Cl}(\mathcal{O}_KC_p)$. This subset is in fact a subgroup which can be described explicitly.

Let $\text{Cl}(\mathcal{O}_K)$ denote the ideal class group of $K$ and let $\text{Cl}'(\mathcal{O}_KC_p)$ be the kernel of the map $\text{Cl}(\mathcal{O}_KC_p) \to \text{Cl}(\mathcal{O}_K)$ induced by augmentation. Let $J$ be the Stickelberger ideal in $\mathbb{Z}\Delta$ (the definition of $J$ will be given later). The key result of relevance to the present paper is that $R(\mathcal{O}_KC_p)$ is the subgroup $\text{Cl}'(\mathcal{O}_KC_p)^J$ of $\text{Cl}(\mathcal{O}_KC_p)$ where $\text{Cl}'(\mathcal{O}_KC_p)^J = \{c^\alpha : c \in \text{Cl}'(\mathcal{O}_KC_p), \alpha \in J\}$.

3. The proof of Theorem 1.1. Let $K$ be a totally real number field and let $p \geq 5$ be prime. Let $\mathfrak{p}$ be some prime of $K$ above $p$ and let $e$ denote the ramification index of $\mathfrak{p}$ in $K/\mathbb{Q}$. We will assume that $p$ is ramified in $K/\mathbb{Q}$ and so $\mathfrak{p}$ can be chosen such that $e \geq 2$. Under these hypotheses we
shall show that $K$ is not Hilbert–Speiser of type $C_\rho$ (note that in the case $p = 5$ and $[K(\zeta_p) : K] = 2$ we shall have to assume that $e \geq 3$).

The basic idea of the proof will be to construct certain $\mathcal{O}_K$-algebras $\Gamma$ and $S$ such that $\Gamma \subseteq S$ with $S/\Gamma \simeq \mathcal{O}_K/\mathfrak{p}$ as $\mathcal{O}_K$-modules. Together, $S$ and $\Gamma$ will be used to construct a non-trivial subgroup of the realizable classes $R(\mathcal{O}_K \mathcal{C}_p) = \mathcal{C}(\mathcal{O}_K \mathcal{C}_p)^J$ described in Section 2, thereby giving the desired result. At all primes $q \neq \mathfrak{p}$ of $K$ the completions $S_q$ and $\Gamma_q$ will be equal, so the essential part of the argument will be local at $\mathfrak{p}$.

Let $\phi_p(z)$ be the $p$th cyclotomic polynomial. Then $\Gamma := \mathcal{O}_K[z]/(\phi_p(z))$ is an $\mathcal{O}_K$-algebra, but is a domain if and only if $[K(\zeta_p) : K] = p - 1$. The group $\Delta := (\mathbb{Z}/p\mathbb{Z})^\times$ acts on $\Gamma$ in the following way: to each $\tilde{a} \in \Delta$ we associate an automorphism $\sigma_a$ of $\Gamma$ defined by $\sigma_a(z) = z^a$, where the image of $z$ in $\Gamma$ is again written $z$. Let $\omega : \Delta \to \mathbb{Z}_p^\times$ be the Teichmüller character, so that $\omega(\sigma_a) = \tilde{a}$ where $\tilde{a}p^{-1} = 1$, and $\tilde{a} \equiv a \pmod{p}$.

There exists an element $\lambda$ such that $\mathbb{Z}_p[\zeta_p] = \mathbb{Z}_p[\lambda]$ with $\lambda^{p-1} = -p$, $\lambda \equiv 1 - \zeta_p \pmod{(1 - \zeta_p)^2}$ (see, for example, [Lan90, Chapter 14, Lemma 3.1]). Furthermore, $\Delta$ acts on $\lambda^i \mathbb{Z}_p$ through the character $\omega^i$ with $i \in \{0, \ldots, p-2\}$. Note that $\Gamma_p = \mathcal{O}_K \mathcal{C}_p \otimes \mathbb{Z}_p[\zeta_p]$, for which $\{1, \lambda, \lambda^2, \ldots, \lambda^{p-2}\}$ is an $\mathcal{O}_K$-basis. Let $\pi$ denote a parameter of $\mathcal{O}_K$ and define the element $x := (1/\pi) \otimes \lambda^{p-2}$ in $K_p \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_p) = K_p \Gamma$ (we will abuse notation and write $x = \lambda^{p-2}/\pi$).

**Lemma 3.1.** We have $x^2, x^3, \lambda x, \pi x \in \Gamma_p$.

**Proof.** Since $e \geq 2$, we have $p/\pi^2 \in \mathcal{O}_K$. Hence

$$x^2 = \frac{\lambda^{2p-4}}{\pi^2} = \frac{-p\lambda^{p-3}}{\pi^2} \quad \text{and} \quad x^3 = \frac{(-p)^2\lambda^{p-4}}{\pi^3}$$

are both in $\Gamma_p$ (we have used $p \geq 5$ here). Furthermore, it is clear that

$$\lambda x = \frac{\lambda^{p-1}}{\pi} = \frac{-p}{\pi} \quad \text{and} \quad \pi x = \lambda^{p-2}$$

are both in $\Gamma_p$. ■

We shall now consider three cases, the first two of which overlap.

**3.1. The case $[K(\zeta_p) : K] > 2$.** A consequence of Lemma 3.1 is that the $\mathcal{O}_K$-module $T := \Gamma_p + x\mathcal{O}_K$ is in fact an $\mathcal{O}_K$-algebra. Furthermore, we have

$$\pi T = \pi \Gamma_p + \lambda^{p-2}\mathcal{O}_K \subseteq \Gamma_p \subseteq T$$

since $\lambda^{p-2}$ is part of an $\mathcal{O}_K$-basis of $\Gamma_p$. We now let $S$ be the $\mathcal{O}_K$-order defined by

$$S_q = \Gamma_q \quad (q \neq \mathfrak{p}), \quad S_{\mathfrak{p}} = T.$$ 

We find that $\Gamma \subseteq S$ and $\pi S \subseteq \Gamma$ (note that we have abused notation in the obvious way here). Furthermore, the ring $\bar{S} := S/\pi S$ is isomorphic to
Let $P$ be the image of $\Gamma$ under the canonical map $S \to \tilde{S}$. We have a Milnor square

$$
\begin{array}{ccc}
\Gamma & \hookrightarrow & S \\
\downarrow & & \downarrow \\
\tilde{\Gamma} & \hookrightarrow & \tilde{S}
\end{array}
$$

where the horizontal arrows are the natural inclusions and the vertical arrows are the natural projections (note that this is a special case of a fiber product). Note that we have $\delta(\lambda^{p-2}) \equiv \omega^{p-2}(\delta)\lambda^{p-2}$ modulo $p\mathbb{Z}_p[\zeta_p]$ for every $\delta \in \Delta$, so $\delta(x) \in x + \Gamma \subset S$. Hence $\Delta$ acts on $S$ and so acts on each of the rings in the Milnor square. By [CR87, p. 242] we have the following exact sequence

$$
K_1(S) \times K_1(\tilde{\Gamma}) \to K_1(\tilde{S}) \to \text{Cl}(\Gamma) \to \text{Cl}(S) \to 0.
$$

As all the rings above are commutative, this becomes

$$
S^\times \times \tilde{\Gamma}^\times \to \tilde{S}^\times \to \text{Cl}(\Gamma) \to \text{Cl}(S) \to 0.
$$

Hence we have an embedding of $\Delta$-modules

$$
N := \frac{\tilde{S}^\times}{\tilde{\Gamma}^\times \cdot \text{im}(S^\times)} \hookrightarrow \text{Cl}(\Gamma),
$$

where $\text{im}(S^\times)$ is the image of $S^\times$ under the map $S \to \tilde{S}$.

For every $\Delta$-module $X$, let $X^-$ and $X^{\omega^{-1}}$ denote the minus part and the $\omega^{-1}$-part of $\mathbb{Z}_p \otimes_{\mathbb{Z}} X$, respectively. Then $X^{\omega^{-1}} \subseteq X^-$. We will show that $N^{\omega^{-1}}$ contains a submodule $M$ of order $p$. Note that by the definition of $x$ and the action of $\Delta$, we have $x \in S^{\omega^{-1}}$. We define $\bar{x} \in \tilde{S}$ to be the image of $x \in T$ under the natural projection $T \to T/\pi T \simeq \tilde{S}$ and note that $\bar{x} \in \tilde{S}^{\omega^{-1}}$.

Let $[\exp](z) := \sum_{i=0}^{p-1} (1/!)z^i$ denote the truncated exponential series. Whenever the ideal $(a, b)$ generated by $a$ and $b$ satisfies $(a, b)^p = 0$, we have $[\exp](a + b) = [\exp](a) \cdot [\exp](b)$ (see the proof of [CGM+98, $p$-elementary group schemes—constructions and Raynaud’s theory, Remark 1.1]). Let $y := [\exp](\bar{x}) \in \tilde{S}$. Since $y^p = [\exp](p\bar{x}) = [\exp](0) = 1$, we have $y \in \tilde{S}^\times$. We note that $y \notin \tilde{\Gamma}$ (the summand with $i = 1$ is $\tilde{x}$ and hence plainly outside $\tilde{\Gamma}$, and all other summands are in $\tilde{\Gamma}$ by Lemma 3.1). Moreover, as $[\exp]$ is compatible with the $\Delta$-action, we have $y \in (\tilde{S}^\times)^{\omega^{-1}} = \tilde{S}^\times \cap \tilde{S}^{\omega^{-1}}$.

**Lemma 3.2.** We have $(\tilde{\Gamma}^\times \cdot \text{im}(S^\times))^{\omega^{-1}} = (\tilde{\Gamma}^\times)^{\omega^{-1}}$.

**Proof.** Let $\mathcal{M}$ denote the maximal order in $KS = K\Gamma$. Then $\mathcal{M} = \text{ind}_{\Delta_0}^\Delta \mathcal{O}_{K(\zeta_p)}$ with $\Delta_0 = \text{Gal}(K(\zeta_p)/K)$. We consider $S^{\times -} \subseteq \mathcal{M}^{\times -}$; since $K$ is totally real, complex conjugation $j \in \Delta_0$ acts on each factor of $\mathcal{M}$ separately, and we see that $\mathcal{O}_{K(\zeta_p)}^{\times -}$ is the multiplicative group of roots of unity $\langle \zeta_{pf} \rangle$ for some $f \geq 1$ (see [Was97, Theorem 4.12]). Hence $\mathcal{M}^{\times -} = \text{ind}_{\Delta_0}^\Delta \langle \zeta_{pf} \rangle$. 

\[ S_p/\pi S_p = T/\pi T. \]
Suppose that \( f = 1 \). Then \( \Delta_0 \) acts on \( \zeta_p \) via \( \omega|_{\Delta_0} \), and from the Frobenius reciprocity theorem one deduces that \( \text{ind}_{\Delta_0}^\Delta \langle \zeta_p \rangle \) has non-trivial \( \omega^{-1} \)-part if and only if \( \omega^{-1}|_{\Delta_0} = \omega|_{\Delta_0} \), that is, if and only if \( \omega^2 \) is trivial on \( \Delta_0 \). But this is not the case since \([K(\zeta_p) : K] = |\Delta_0| > 2\) by hypothesis. Now suppose \( f > 1 \). Then considering the short exact sequence

\[
1 \rightarrow \text{ind}_{\Delta_0}^\Delta \langle \zeta_{pf^{-1}} \rangle \rightarrow \text{ind}_{\Delta_0}^\Delta \langle \zeta_{pf} \rangle \rightarrow \text{ind}_{\Delta_0}^\Delta \langle \zeta_p \rangle \rightarrow 1,
\]

we see that the middle term has trivial \( \omega^{-1} \)-part if and only if the same is true of both the outer terms. It now follows by induction on \( f \) that \( \mathcal{M}^{x} = \text{ind}_{\Delta_0}^\Delta \langle \zeta_{pf} \rangle \) has trivial \( \omega^{-1} \)-part. Hence \((S^x)^{-1} \) is trivial, and the lemma is proved.

Let \( \bar{y} \) denote the projection of \( y \) to \( N \). If \( \bar{y} \) were trivial in \( N \), then \( y \) would have to be in \((\bar{T} \times \im(S^x))^{\omega^{-1}} = (\bar{T}^{x})^{\omega^{-1}}\). However, we have already noted that \( y \) is not even in \( \bar{T} \). Hence \( M := \langle \bar{y} \rangle \) is a non-trivial \( \Delta \)-submodule of \( N \) with \( M^{\omega^{-1}} = M \).

**3.2. The case \( e \geq 4 \).** Let \( x_1 = x = \lambda^{p-2}/\pi = (1/\pi) \otimes \lambda^{p-2} \) be as above and define \( x_2 = \lambda^{p-2}/\pi^2 = (1/\pi^2) \otimes \lambda^{p-2} \).  

**Lemma 3.3.** We have \( x_2^2, x_2^3, \lambda x_2, \pi^2 x_2, x_1 x_2, x_1^2 x_2, x_1 x_2^2 \in \Gamma_p \).

**Proof.** We use the assumption that \( p > 5 \) without further mention. Since \( e \geq 4 \), we have \( p/\pi^4 \in \mathcal{O}_{K_p} \). Hence

\[
x_2^2 = \frac{\lambda^{2p-4}}{\pi^4} = \frac{-p\lambda^{p-3}}{\pi^4} \quad \text{and} \quad x_2^3 = \frac{p^2\lambda^{p-4}}{\pi^6} \quad \text{are both in } \Gamma_p.
\]

Furthermore, it is clear that

\[
\lambda x_2 = \frac{\lambda^{p-1}}{\pi^2} = \frac{-p}{\pi^2} \quad \text{and} \quad \pi^2 x_2 = \lambda^{p-2}
\]

are both in \( \Gamma_p \). Finally,

\[
x_1 x_2 = \frac{\lambda^{2p-4}}{\pi^3} = \frac{(-p)\lambda^{p-3}}{\pi^3}, \quad x_1^2 x_2 = \frac{\lambda^{3p-6}}{\pi^4} = \frac{(-p)^2\lambda^{p-4}}{\pi^4}, \quad \text{and}
\]

\[
x_1 x_2^2 = \frac{\lambda^{3p-6}}{\pi^5} = \frac{(-p)^2\lambda^{p-4}}{\pi^5}
\]

are all in \( \Gamma_p \).  

A consequence of Lemmas 3.1 and 3.3 is that the \( \mathcal{O}_{K_p} \)-module \( T := \Gamma_p + x_1 \mathcal{O}_{K_p} + x_2 \mathcal{O}_{K_p} \) is in fact an \( \mathcal{O}_{K_p} \)-algebra. Furthermore, we have

\[
\pi^2 T = \pi^2 \Gamma_p + \pi \lambda^{p-2} \mathcal{O}_{K_p} + \lambda^{p-2} \mathcal{O}_{K_p} = \pi^2 \Gamma_p + \lambda^{p-2} \mathcal{O}_{K_p} \subseteq \Gamma_p \subseteq T
\]

since \( \lambda^{p-2} \) is part of an \( \mathcal{O}_{K_p} \)-basis of \( \Gamma_p \). We now let \( S \) be the \( \mathcal{O}_K \)-order defined by

\[
S_q = \Gamma_q \quad (q \neq p), \quad S_p = T.
\]
Then $\Gamma \subseteq S$ and $\pi^2 S \subseteq \Gamma$. The same argument as in the previous case gives an embedding of $\Delta$-modules

$$N := \frac{\tilde{S}^x}{\tilde{\Gamma}^x \cdot \text{im}(S^x)} \hookrightarrow \text{Cl}(\Gamma).$$

By definition of $x_1, x_2$ and the action of $\Delta$, we have $x_1, x_2 \in S^{\omega^{-1}}$. Let $y_1 = [\exp](\tilde{x}_1), y_2 = [\exp](\tilde{x}_2) \in \tilde{S} = S/\pi^2 S$. As in the previous case, both $y_1, y_2$ are elements of order $p$ in $(\tilde{S}^x)^{\omega^{-1}}$.

**Lemma 3.4.** $(S^x)^{\omega^{-1}}$ is cyclic.

**Proof.** Let $\mathcal{M}$ denote the maximal order in $KS = K\Gamma$. As $S \subseteq \mathcal{M}$, we have $(S^x)^{\omega^{-1}} \subseteq (\mathcal{M}^x)^{\omega^{-1}}$, and so it suffices to show that $(\mathcal{M}^x)^{\omega^{-1}}$ is a cyclic group.

By the same argument as for Lemma 3.2, we obtain $\mathcal{M}^x = \text{ind}_{\Delta_0}^A \langle \zeta_p \rangle$. Furthermore, as $\omega^{-1}$ is an odd character, we have $(\mathcal{M}^x)^{\omega^{-1}} \subseteq \mathcal{M}^x$. Now $\langle \zeta_p \rangle$ is trivially cyclic as a $\mathbb{Z}_p[\Delta_0]$-module; hence

$$\mathcal{M}^x = \text{ind}_{\Delta_0}^A \langle \zeta_p \rangle = \mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p[\Delta_0]} \langle \zeta_p \rangle$$

is cyclic as a $\mathbb{Z}_p[\Delta]$-module. Thus $(\mathcal{M}^x)^{\omega^{-1}} = \mathbb{Z}_p(\omega^{-1}) \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{M}^x$ is cyclic as a $\mathbb{Z}_p(\omega^{-1})$-module, where $\mathbb{Z}_p(\omega^{-1})$ is the ring extension of $\mathbb{Z}_p$ obtained by adjoining the image of $\omega^{-1}$. However, $\omega^{-1}$ takes its values in $\mathbb{Z}_p^x$, and so $\mathbb{Z}_p(\omega^{-1}) = \mathbb{Z}_p$. Therefore $(\mathcal{M}^x)^{\omega^{-1}}$ is cyclic as a $\mathbb{Z}_p$-module, and hence is cyclic as a group. ■

Let $\tilde{y}_1, \tilde{y}_2 \in (\tilde{S}^x/\tilde{\Gamma}^x)^{\omega^{-1}}$ be the images of $y_1, y_2$ under the natural projection. Since $y_1, y_2 \notin \tilde{\Gamma}$ (the summand with $i = 1$ is outside $\tilde{\Gamma}$ and all others are in $\tilde{\Gamma}$ by Lemmas 3.1 and 3.3), $\tilde{y}_1, \tilde{y}_2$ are also each of order $p$. Suppose that $\tilde{y}_1^{k_1} \tilde{y}_2^{k_2}$ is trivial for some $k_1, k_2 \in \{1, \ldots, p-1\}$. This would mean that $[\exp](k_1 \tilde{x}_1 + k_2 \tilde{x}_2)$ is in $\tilde{\Gamma}$. By virtue of Lemma 3.3, we have $[\exp](k_1 \tilde{x}_1 + k_2 \tilde{x}_2) \equiv 1 + k_1 \tilde{x}_1 + k_2 \tilde{x}_2$ modulo $\tilde{\Gamma}$. Therefore we would obtain $k_1 \tilde{x}_1 + k_2 \tilde{x}_2 \in \tilde{\Gamma}$ and so $k_1 x_1 + k_2 x_2 \in \Gamma_p$, which is impossible. Hence the subgroup $\langle \tilde{y}_1, \tilde{y}_2 \rangle \simeq \langle y_1, y_2 \rangle \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ is non-cyclic. Let $\tilde{y}_1, \tilde{y}_2$ be the projections of $\tilde{y}_1, \tilde{y}_2$ to $N$ and let $M := \langle \tilde{y}_1, \tilde{y}_2 \rangle \subseteq N^{\omega^{-1}}$. Note that $M$ is non-trivial because $(\text{im}(S^x))^{\omega^{-1}}$ is cyclic by Lemma 3.4, but $\langle \tilde{y}_1, \tilde{y}_2 \rangle$ is non-cyclic. Hence $M$ is a non-trivial $\Delta$-submodule of $N$ with $M^{\omega^{-1}} = M$.

**3.3. The case $[K(\zeta_p) : K] = 2$ and $e = 2$ or 3.** Note that the condition $[K(\zeta_p) : K] = 2$ implies that $(p-1)/2$ divides $e$. Hence we are reduced to considering the cases $p = 5$ and $p = 7$ (since $p \geq 11$ forces $e \geq 5$). If $p = 5$, then $e$ must be even and so in fact $e = 2$. However, this case is excluded by hypothesis. If $p = 7$, then we must have $e = 3$. In this case, we
let \( x_1 = \lambda^5/\pi \), \( x_2 = \lambda^5/\pi^2 \) and \( x_3 = \lambda^4/\pi \). It is straightforward to check that the \( \mathcal{O}_K \)-module \( T := \Gamma_p + x_1\mathcal{O}_K + x_2\mathcal{O}_K + x_3\mathcal{O}_K \) is in fact an \( \mathcal{O}_K \)-algebra. The result is then given by a slight variant of the proof of the previous case (note that \( x_1, x_2 \in S^{\omega^{-1}} \) but \( x_3 \notin S^{\omega^{-1}} \)).

3.4. The proof of Theorem 1.1. In each of the above cases, we have shown that there exists a non-trivial \( \Delta \)-submodule \( M \) of \( \text{Cl}(\Gamma) \) such that \( M^{\omega^{-1}} = M \).

**Proof of Theorem 1.1.** Recall that the Stickelberger ideal is defined to be \( J = \mathbb{Z}\Delta \cap \theta \cdot \mathbb{Z}\Delta = \text{Ann}_\Delta(\langle \zeta_p \rangle) \cdot \theta \) where \( \theta \) is the Stickelberger element \( p^{-1}\sum_{j=1}^{p-2} j\sigma_j^{-1} \). Let \( J_p \subseteq \mathbb{Z}_p\Delta \) be the \( p \)-completion of \( J \). Then \( \omega^{-1}(J_p) = \omega^{-1}(\text{Ann}_\Delta(\langle \zeta_p \rangle)) \cdot \omega^{-1}\theta \). The second factor of the last expression is the generalized Bernoulli number \( B_{1,\omega} \). Since \( p \geq 5 \), the first factor is \( \mathbb{Z}_p \). By [Was97, Corollary 5.15] we have \( B_{1,\omega} \equiv B_2/2 = 1/12 \pmod p \). Hence, \( M^J = M^{\omega^{-1}(J_p)} = M^{\mathbb{Z}_p} = M \) and therefore \( \text{Cl}(\Gamma)^J \neq 0 \).

Let \( \Sigma \) denote the sum of the elements of \( C_p \). Consider the following Milnor square:

\[
\begin{array}{ccc}
\mathcal{O}_K C_p & \xrightarrow{\alpha} & \mathcal{O}_K C_p / \mathcal{O}_K \Sigma =: \Lambda \\
\downarrow \beta & & \downarrow \gamma \\
\mathcal{O}_K & \rightarrow & \mathcal{O}_K / p\mathcal{O}_K
\end{array}
\]

where the horizontal maps are the natural projections, \( \beta \) is the augmentation map, and \( \gamma \) is the map induced by augmentation. The resulting map

\[
\text{Cl}(\mathcal{O}_K C_p) \xrightarrow{(\alpha, \beta)} \text{Cl}(\Lambda) \times \text{Cl}(\mathcal{O}_K)
\]

is surjective (see, for instance, [CR87, Corollary 49.28]). It follows immediately that

\[
\text{Cl}'(\mathcal{O}_K C_p) \rightarrow \text{Cl}(\Lambda)
\]

is surjective. However, \( \text{Cl}(\Lambda) \simeq \text{Cl}(\Gamma) \) since \( \Lambda \simeq \Gamma \), and so \( \text{Cl}(\Lambda)^J \neq 0 \). Therefore \( \text{Cl}'(\mathcal{O}_K C_p)^J = R(\mathcal{O}_K C_p) \neq 0 \), and so \( K \) is not a Hilbert–Speiser field of type \( C_p \). ■

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