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On the independence of $\sigma(\phi(n))$ and $\phi(\sigma(n))$

by

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1. Introduction. In [3], two arithmetic functions $f(n)$ and $g(n)$ are called *independent* if for every positive integer k and any two permutations $i_1, \ldots, i_k$ and $j_1, \ldots, j_k$ of $\{1, \ldots, k\}$ there exist infinitely many positive integers n such that

$$f(n+i_1) < f(n+i_2) < \cdots < f(n+i_k)$$

and

$$g(n+j_1) > g(n+j_2) > \cdots > g(n+j_k).$$

In [3], it was shown that the number of distinct prime factors $\omega(n)$ of n and the number of divisors $\tau(n)$ of n are independent. In [2], it was shown that the Euler function $\phi(n)$ and the Carmichael function $\lambda(n)$ are independent. We recall that $\phi(n)$ is the cardinality of the group of invertible elements modulo n, while $\lambda(n)$ is the exponent (maximal order of elements) of this group. In this paper, we prove that the compositions $\sigma(\phi(n))$ and $\phi(\sigma(n))$ are independent in the above sense.

Theorem 1. *The functions $\sigma(\phi(n))$ and $\phi(\sigma(n))$ are independent.*

We recall that the compositions $\sigma \circ \phi$ and $\phi \circ \sigma$ have already been investigated in a series of papers [7], [1], [6]. A similar method can be used to prove that $\phi \circ \phi$ and $\sigma \circ \sigma$ are independent. We do not enter into details. On the other hand, it would be interesting to find an effective version of the above theorem, namely to find explicit sequences of n's tending to infinity with k for which the two inequalities

$$\sigma(\phi(n+i_1)) < \sigma(\phi(n+i_2)) < \cdots < \sigma(\phi(n+i_k)) \tag{1}$$

and

$$\phi(\sigma(n+j_1)) > \phi(\sigma(n+j_2)) > \cdots > \phi(\sigma(n+j_k)) \tag{2}$$

hold. We leave this as a challenge to the reader.

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Throughout this paper, we use the Vinogradov symbols $\gg$ and $\ll$ as well as the Landau symbol O with their usual meanings. All the implied constants depend at most on the number k. Sometimes we will emphasize this dependence by writing O_k, etc.

2. The proof. We assume that $k \geq 2$, as otherwise there is nothing to prove. We take $K = k!^2$ and let $\lambda_1, \ldots, \lambda_k$ be k distinct primes $> k^2$. We put $\Lambda = \prod_{i=1}^k \lambda_i$, $N = K\Lambda$ and let $n_0 \pmod N$ be the solution of the system of congruences

$$n \equiv 0 \pmod K, \qquad n + i \equiv \lambda_i \pmod{\lambda_i^2} \qquad \text{for all } i = 1, \ldots, k.$$

The congruence class $n_0 \pmod N$ exists by the Chinese Remainder Lemma. We assume that $n_0 > 0$ is the smallest positive integer in its congruence class modulo N.

If $n \equiv n_0 \pmod N$, then there exists a positive integer m such that $n = Nm + n_0$. Furthermore,

$$n+i = (Nm+n_0)+i = i\lambda_i\left(\frac{N}{i\lambda_i}m + \frac{n_0+i}{i\lambda_i}\right) := i\lambda_i(a_im+b_i), \quad i = 1, \ldots, k.$$

For a positive integer m we write $p(m)$ for the smallest prime factor of m.

We let $x > N$, u a positive integer depending on k to be determined later and put

$$\mathcal{E}(x) = \{1 \leq m \leq x : p(a_im + b_i) > x^{1/u} \text{ for all } i = 1, \ldots, k\}. \tag{3}$$

Lemma 1. *There exist constants $u_0 \geq 3$, c_1, c_2, x_0 depending on k such that if $x > x_0$, $u > u_0$ and $x^{1/u} > N^2$, then*

$$\#\mathcal{E}(x) = c(x)\,\frac{xu^k}{(\log x)^k} \qquad \textit{for some } c(x) \in [c_1, c_2].$$

Proof. This follows easily from the Fundamental Lemma of the Combinatorial Sieve. Indeed, let

$$F_i(m) = a_im + b_i \qquad \text{for } i = 1, \ldots, k,$$

and put

$$F(m) = \prod_{i=1}^k F_i(m).$$

For a prime number p put $\rho(p) = \#\{1 \leq m \leq p-1 : F(m) \equiv 0 \pmod p\}$. Note that, if $p \mid N$, then $p \mid a_i$ for all $i = 1, \ldots, k$ but $p \nmid b_i$ for any $i = 1, \ldots, k$. Thus, $\rho(p) = 0$ for such primes p. On the other hand, if $p \nmid N$, then $p \nmid a_i$ for any $i = 1, \ldots, k$. In particular, $m \equiv b_ia_i^{-1} \pmod p$ is a solution of $F(m) \equiv 0 \pmod m$. Let us show that these solutions are distinct. If not, there are $i \neq j$

such that

$$p \mid a_i b_j - a_j b_i = \frac{1}{ij\lambda_i\lambda_j}\,(N(n_0+j) - N(n_0+i)) = \frac{N(j-i)}{ij\lambda_i\lambda_j} \mid N^2,$$

therefore $p \mid N$, which is a contradiction. Thus, $\rho(p) = k$ for all $p \nmid N$. The version of the Fundamental Lemma of the Combinatorial Sieve appearing on page 85 in [5] (Theorem 2.6) together with the remarks on the bottom of page 86 there show that for $x^{1/u} > N^2$, we have

$$\begin{aligned}(4)\quad & \#\mathcal{E}(x)\\ &= x \prod_{p \le x^{1/u}} \left(1 - \frac{\rho(p)}{p}\right)\{1 + O_k(\exp(-u(\log u - \log\log 3u - k - 2)))\}\\ &\quad + O_k\left(\frac{x}{\exp(\sqrt{\log x})}\right).\end{aligned}$$

Since $x^{1/u} > N^2$, we have

$$\prod_{p \le x^{1/u}} \left(1 - \frac{\rho(p)}{p}\right) = \prod_{k < p \le x^{1/u}} \left(1 - \frac{k}{p}\right) \prod_{i=1}^{k} \left(1 - \frac{k}{\lambda_i}\right)^{-1}.$$

Note that for the second product above, since $\lambda_i > k^2$ for $i = 1, \ldots, k$, we have

$$(5)\qquad 1 < \prod_{i=1}^{k} \left(1 - \frac{k}{\lambda_i}\right)^{-1} < \left(1 - \frac{1}{k}\right)^{-k} \le 4,$$

whereas for the first, we have

$$\begin{aligned}(6)\quad \prod_{k<p<x^{1/u}} \left(1 - \frac{k}{p}\right) &= \exp\Bigg(\sum_{k<p<x^{1/u}} \log\left(1 - \frac{k}{p}\right)\Bigg)\\ &= \exp\Bigg(-\sum_{k<p<x^{1/u}} \frac{k}{p} + O\Bigg(\sum_{p>k} \frac{k}{p^2}\Bigg)\Bigg)\\ &= \exp\left(-k(\log\log x^{1/u} - \log\log k) + O\left(\frac{k}{\log k}\right)\right)\\ &= \exp(k \log u - k \log\log x + O(k \log\log k))\\ &= \frac{u^k}{(\log x)^k} \exp(O(k \log\log k)).\end{aligned}$$

We now choose $u_0 := u_0(k) \ge 3$ such that for $u > u_0$,

$$1 + O_k(-u(\log u - \log\log(3u) - k - 2)) \in [1/2, 2].$$

Next, we note that by inequalities (5), (6) and estimate (4),

$$\#\mathcal{E}(x) = c_1(x)\,\frac{xu^k}{(\log x)^k} + O_k\biggl(\frac{x}{\exp(\sqrt{\log x})}\biggr),$$

where $c_1(x) \in [\alpha, \beta]$, with $\alpha < \beta$ some positive constants depending on k. Taking now x_0 sufficiently large with respect to k, since $u \geq 3$, it follows that the desired estimate holds with $c_1 = \alpha/2$ and $c_2 = 2\beta$. ∎

We now shrink $\mathcal{E}(x)$ by removing some of its elements. Recall that for a positive integer m its Möbius function is $\mu(m) = (-1)^{\omega(m)}$ if m is squarefree and $\mu(m) = 0$ otherwise. Let

$$\mathcal{E}_1(x) = \{m \in \mathcal{E}(x) : \mu(F_i(m)) = 0 \text{ for some } i = 1, \ldots, k\},$$
$$\mathcal{E}_2(x) = \{m \in \mathcal{E}(x) : F_i(m) \text{ is prime for some } i = 1, \ldots, k\}.$$

Next we give upper bounds for the cardinalities of $\mathcal{E}_i(x)$ for $i = 1, 2$.

Lemma 2. *We keep the notations and hypothesis of Lemma* 1. *Then*

(i) *There exists a positive constant c_3 depending on k such that*

$$\#\mathcal{E}_1(x) \leq c_3 x^{1-1/u}.$$

(ii) *There exists a positive constant c_4 depending on k such that*

$$\#\mathcal{E}_2(x) \leq c_4\,\frac{xu^{k-1}}{(\log x)^k}.$$

Proof. (i) Assume that $m \in \mathcal{E}(x)$ and $p^2 \mid F_i(m)$ for some $i = 1, \ldots, k$. Fix i and p. Then $m \equiv -b_i a_i^{-1} \pmod{p^2}$. Since $m \leq x$, the number of such m is $\leq \lfloor x/p^2 \rfloor + 1 \leq x/p^2 + 1$. Note that since $x^{1/u} > N^2$, we get $a_i m + b_i \leq Nm + n_0 + k \leq 3Nm \leq 3x^{1+1/2u} < x^{1+1/u}$, and since $p^2 \mid a_i m + b_i$, we obtain $p \leq x^{1/2+1/2u} < x^{2/3}$. Thus, summing up over all possible values of $i \in \{1, \ldots, k\}$ and $p \in (x^{1/u}, x^{2/3})$ yields

$$\#\mathcal{E}_1(x) \leq kx \sum_{p > x^{1/u}} \frac{1}{p^2} + k\pi(x^{2/3}) \leq c_3 x^{1-1/u},$$

where $c_3 > 0$ is some constant depending on k, as desired.

(ii) This follows easily from Brun's sieve. Indeed, let us fix $i \in \{1, \ldots, k\}$ and estimate the cardinality of the subset of $m \in \mathcal{E}(x)$ such that $F_i(m)$ is prime. This set is contained in the set of those $m \leq x$ such that either $m < x^{1/2}$, or

$$p(F_i(m)) > x^{1/2} \quad \text{and} \quad p(F_j(m)) > x^{1/u} \qquad \text{for all } j \neq i \in \{1, \ldots, k\}.$$

This last problem is a sieving problem. In this case, $\rho(p) = \#\{0 \leq m \leq p-1 : F(m) \equiv 0 \pmod p\}$ equals 0 if $p \mid N$, k if $p \nmid N$ and $p < x^{1/u}$, and 1 if $x^{1/u} \leq x \leq x^{1/2}$. By the Brun sieve (see Theorem 2.4 on page 76 in [5]), the number of such $m \in [x^{1/2}, x]$ is

$$\ll x \prod_{\substack{k<p<x^{1/u} \\ p\nmid N}} \left(1-\frac{k}{p}\right) \prod_{x^{1/u}<p<x^{1/2}} \left(1-\frac{1}{p}\right), \tag{7}$$

where the implied constant is absolute. The proof of Lemma 1 shows that the first product is $O_k(u^k/(\log x)^k)$. Mertens' formula shows that the second product is

$$\frac{\log(x^{1/u})}{\log x}\left(1+O\left(\frac{1}{\log(x^{1/u})}\right)\right) = u^{-1}\left(1+O\left(\frac{u}{\log x}\right)\right). \tag{8}$$

Thus,

$$\#\{m \in \mathcal{E}(x) : F_i(m) \text{ is prime}\} \le \sqrt{x} + O_k\left(\frac{xu^{k-1}}{(\log x)^k}\right).$$

Summing up the above inequality over k implies the desired estimate. ∎

It follows from Lemmas 1 and 2 that the set of $m \in \mathcal{E}(x)$ such that $F_i(m)$ is squarefree for each $i = 1, \ldots, k$ and none is a prime, denoted by $\mathcal{E}_3(x)$, has cardinality

$$\ge c_1 \frac{u^k x}{(\log x)^k} - c_3 x^{1-1/u} - \frac{c_4 x u^{k-1}}{(\log cx)^k},$$

where c_1, c_3 and c_4 depend only on k. We see therefore that there exists $u_1 \ge \max\{3, 2c_4/c_1\}$ and x_1 depending on k such that for $u = u_1$ and $x \ge x_1$, the above number is $\ge (c_1/2)u_1^k x/(\log x)^k$. From now on, we shall assume that $u = u_1$ and that $x \ge x_1$, so that

$$\#\mathcal{E}_3(x) \ge c_5 \frac{x^k}{(\log x)^k},$$

where we put $c_5 = c_1/2$.

Now let $\delta > 0$ be some parameter depending on k, to be fixed later. We need the following result concerning the distribution of primes p such that either $p-1$ has a large divisor sum, or $p+1$ has a small Euler function.

Lemma 3.

(i) *Let $\delta > 1$ and $\mathcal{P}_\delta = \{p : \sigma(p-1) > \delta(p-1)\}$. Then*

$$\mathcal{P}_\delta(x) := \#(\mathcal{P}_\delta \cap [1, x]) = O(\pi(x)/\log\delta).$$

(ii) *Let $\delta > 1$ and $\mathcal{Q}_\delta = \{p : \phi(p+1) < \delta^{-1}(p+1)\}$. Then*

$$\mathcal{Q}_\delta(x) := \#(\mathcal{Q}_\delta \cap [1, x]) = O(\pi(x)/\log\delta).$$

Proof. (i) We put

$$h(n) = \sum_{p\mid n} \frac{1}{p-1}.$$

Observe that for $p \in \mathcal{P}_\delta$ we have

$$\delta < \frac{\sigma(p-1)}{p-1} < \frac{p-1}{\phi(p-1)} = \prod_{q \mid p-1}\left(1+\frac{1}{q-1}\right) < \exp(h(p-1)),$$

therefore $h(p-1) > \log\delta$. Next, put

$$S(x) = \sum_{p\le x} h(p-1) = \sum_{p\le x}\sum_{q\mid p-1}\frac{1}{q-1}.$$

We interchange the order of summation above and get

$$S(x) = \sum_{q\le x}\frac{1}{q-1}\sum_{\substack{p\le x\\ p\equiv 1 \,(\mathrm{mod}\, q)}} 1 = \sum_{q\le x}\frac{\pi(x;1,q)}{q-1},$$

where, as usual, we write $\pi(x;a,b)$ for the number of primes $p \le x$ in the arithmetic progression $p \equiv a \pmod b$. We split the above sum at $x^{1/2}$. In the lower range, we use the Montgomery–Vaughan [8] estimate

$$\pi(x;1,q) \le \frac{2x}{\log(x/q)} \le \frac{4x}{\log x},$$

while in the upper range we use the trivial inequality $\pi(x;1,q) \le x/q$. We get

$$S(x) \le \sum_{q\le x^{1/2}}\frac{4x}{q(q-1)\log x} + x\sum_{x^{1/2}<q<x}\frac{1}{q(q-1)} \ll \frac{x}{\log x} + x^{1/2} \ll \pi(x).$$

On the other hand, it is clear that $S(x) \ge \#\mathcal{P}_\delta(x)\log\delta$. Hence,

$$\#\mathcal{P}_\delta(x)\log(\delta) \ll \pi(x),$$

which implies the desired estimate.

(ii) Since $\phi(m)/m > (6/\pi^2)m/\sigma(m)$, it follows that if $\phi(p+1)/(p+1) < \delta^{-1}$, then $\sigma(p+1)/(p+1) > 6\delta/\pi^2$. Now the conclusion follows by typographical changes from the preceding argument (say, replace $p-1$ above by $p+1$). ∎

We are now ready to remove some more elements from $\mathcal{E}(x)$.

Lemma 4. *Let $\delta > 1$. There exists a constant c_6 which depends only on k such that if*

$$\mathcal{E}_4(x) := \{m \in \mathcal{E}(x) : p \mid F(m) \text{ for some } p \in \mathcal{P}_\delta \cup \mathcal{Q}_\delta\},$$

then

$$\#\mathcal{E}_4(x) \le c_6\,\frac{x^k}{(\log\delta)(\log x)^k}.$$

Proof. We fix $i = 1,\dots,k$ and assume that $m \in \mathcal{E}_3(x)$ and that $p \mid F_i(m)$ for some $p \in \mathcal{P}_\delta \cup \mathcal{Q}_\delta$. We fix p. The congruence $a_i m + b_i \equiv 0 \pmod p$ puts m in the congruence class $-b_i a_i^{-1}$ modulo p. Let m_p be the smallest positive integer m in this congruence class and write $m = pl + m_p$. Note that $a_i m + b_i < 3Nm < 3x^{1+1/2u}$ and $\omega(a_i m + b_i) \ge 2$. Hence, $(a_i m + b_i)/p$ is divisible by a prime $q > x^{1/u}$, giving $3x^{1+1/2u}/p > x^{1/u}$, therefore $x/p > x^{1/2u}/3 > x^{1/4u}$, where the last inequality holds because $x^{1/2u} > N > 3^2$. Observe that since $m \le x$, we get $l < x/p$. Hence, we are led to count the number of nonnegative integers $l \le x/p$ such that $a_i l + (b_i + m_p)/p$ and $F_j(m) = a_i pl + (a_i m_p + b_i)$ for $j \ne i$ are k linear forms in l all free of factors $< x^{1/u}$. By the Brun sieve, the number of such l's is

$$\ll \frac{x}{p(\log(x/p))^k} \ll \frac{x}{p(\log(x^{1/4u}))^k} \ll \frac{x(4u)^k}{(\log x)^k}.$$

Since u depends on k, we conclude that the above number is $\ll_k x/p(\log x)^k$. Summing up over $i = 1, \dots, k$ and then over $p \in \mathcal{P}_\delta \cup \mathcal{Q}_\delta$, we get

$$\#\mathcal{E}_4(x) \ll_k \frac{x}{(\log x)^k} \sum_{\substack{x^{1/u} < p < x^{1+1/u} \\ p \in \mathcal{P}_\delta \cup \mathcal{Q}_\delta}} \frac{1}{p}. \tag{9}$$

Lemma 3 and Abel's summation formula show that uniformly in $3 \le y < z$ we have

$$\begin{aligned} \sum_{\substack{y<p<z \\ p \in \mathcal{P}_\delta \cup \mathcal{Q}_\delta}} \frac{1}{p} &\ll \frac{\#\mathcal{P}_\delta(y) + \#\mathcal{Q}_\delta(y)}{y} + \int_y^z \frac{d(\mathcal{P}_\delta \cup \mathcal{Q}_\delta)(t)}{t} \\ &\ll \frac{\pi(y)}{(\log \delta)y} + \int_y^z \frac{d\pi(t)}{(\log \delta)t} \ll \frac{1}{\log \delta}\Big(1 + \int_y^z \frac{dt}{t \log t}\Big) \\ &\le \frac{1}{\log \delta}\,(1 + \log\log z - \log\log y). \end{aligned}$$

With $y = x^{1/u}$, $z = x^{1+1/u}$, we get

$$\begin{aligned} \sum_{\substack{x^{1/u} < p < x^{1+1/u} \\ p \in \mathcal{P}_\delta \cup \mathcal{Q}_\delta}} \frac{1}{p} &\ll \frac{1}{\log \delta}\,(1 + \log\log(x^{1+1/u}) - \log\log x^{1/u}) \\ &= \frac{1}{\log \delta}(1 + \log(u+1)) \ll_k \frac{1}{\log \delta}, \end{aligned}$$

where the last inequality follows from the fact that u is fixed in terms of k. The above estimate together with estimate (9) completes the proof of the lemma. ∎

Lemma 4 shows that if we choose $\delta = \delta_1 = \exp(2c_6/c_5)$, then the set $\mathcal{E}_5(x)$ of $m \in \mathcal{E}_3(x)$ such that no prime factor of $F_i(m)$ for $i = 1, \dots, k$ is in $\mathcal{P}_\delta \cup \mathcal{Q}_\delta$, has cardinality

$$\ge c_7 \frac{x}{(\log x)^k}, \quad \text{where} \quad c_7 = c_5/2.$$

From now on, we work with this value of δ.

Let x be large and $m \in \mathcal{E}_5(x)$. With $n = Nm + n_0$, we have

$$n + i = i\lambda_i(a_i m + b_i) = i\lambda_i p_1^{(i)} \cdots p_{s_i}^{(i)},$$

where since $a_i m + b_i < 3Nm < x^{1+1/u}$ but each $p_l^{(j)} > x^{1/u}$, we have $s_i \le u$. Thus,

$$\phi(n+i) = \phi(i)(\lambda_i - 1)\prod_{l=1}^{s_i}(p_l^{(i)} - 1), \qquad \sigma(n+i) = \sigma(i)(\lambda_i + 1)\prod_{l=1}^{s_i}(p_l^{(i)} + 1).$$

Observe that since $a\sigma(b) \le \sigma(ab) \le \sigma(a)\sigma(b)$, we have

$$\sigma(\lambda_i - 1)\phi(i)\prod_{l=1}^{s_i}(p_l^{(i)} - 1) \le \sigma(\phi(n+i)) \le \sigma(\lambda_i - 1)\sigma(\phi(i))\prod_{l=1}^{s_i}\sigma(p_l^{(i)} - 1)$$

for all $i = 1, \ldots, k$. Thus,

$$\begin{aligned}\frac{\sigma(\lambda_i - 1)\phi(i)}{\lambda_i i}\prod_{l=1}^{s_i}\left(1 - \frac{1}{p_l^{(i)}}\right) &\le \frac{\sigma(\phi(n+i))}{n+i} \\ &\le \frac{\sigma(\lambda_i - 1)\sigma(\phi(i))}{\lambda_i i}\prod_{l=1}^{s_i}\frac{\sigma(p_l^{(i)} - 1)}{p_l^{(i)}}\end{aligned}$$

for all $i = 1, \ldots, k$. We put $c_i = \phi(i)/i$ and $d_i = \sigma(\phi(i))$ for $i = 1, \ldots, k$. Since $p_i^{(1)}, \ldots, p_i^{(s_i)}$ are all primes $> x^{1/u}$ and not in $\mathcal{P}_\delta$, the above inequality yields

$$c_i\,\frac{\sigma(\lambda_i - 1)}{\lambda_i}\left(1 - \frac{1}{x^{1/u}}\right)^u \le \frac{\sigma(\phi(n+i))}{n+i} \le d_i\,\frac{\sigma(\lambda_i - 1)}{\lambda_i}\,\delta^u. \tag{10}$$

A similar argument based on the inequality $\phi(a)\phi(b) \le \phi(ab) \le \phi(a)b$ shows that

$$e_i\,\frac{\phi(\lambda_i + 1)}{\lambda_i}\,\delta^{-u} \le \frac{\phi(\sigma(n+i))}{n+i} \le f_i\,\frac{\phi(\lambda_i + 1)}{\lambda_i}\left(1 + \frac{1}{x^{1/u}}\right)^u, \tag{11}$$

where $e_i = \phi(\sigma(i))/i$ and $f_i = \sigma(i)/i$ for all $i = 1, \ldots, k$.

The above inequalities show that it suffices to construct primes $\lambda_1, \ldots, \lambda_k$ such that

$$d_{i_l}\,\frac{\sigma(\lambda_{i_l} - 1)}{\lambda_{i_l}}\,\delta^u < c_{i_{l+1}}\,\frac{\sigma(\lambda_{i_{l+1}} - 1)}{\lambda_{i_{l+1}}} \tag{12}$$

for all $l = 1, \ldots, k-1$, and also

$$e_{j_l}\,\frac{\phi(\lambda_{j_l} + 1)}{\lambda_{j_l}}\,\delta^{-u} > f_{j_{l+1}}\,\frac{\phi(\lambda_{j_{l+1}} + 1)}{j_{l+1}} \tag{13}$$

for all $l = 1, \ldots, k-1$. Indeed, once these inequalities hold, for large x inequalities (10) and (11) will show that (1) and (2) are satisfied for $n \in \mathcal{E}_5(x)$.

But this is easy. Namely, we start by constructing inductively pairs (A_i, B_i) of integers such that $12 \mid A_i$, $B_i \equiv 2 \pmod{12}$, $\gcd(A_i, B_i) = 2$ and furthermore

$$d_{i_l} \frac{\sigma(A_{i_l})}{A_{i_l}} \delta^u < c_{i_{l+1}} \frac{\sigma(A_{i_{l+1}})}{A_{i_{l+1}}} \tag{14}$$

for all $l = 1, \ldots, k-1$ and

$$e_{j_l} \frac{\phi(B_{j_l})}{B_{j_l}} \delta^{-u} > f_{j_{l+1}} \frac{\phi(B_{j_{l+1}})}{B_{j_{l+1}}} \tag{15}$$

for all $l = 1, \ldots, k-1$. The fact that this is possible follows from a result of Erdős and Schinzel [4] (see also the recent paper of Wong [9]). Let z be larger than $\max\{A_i, B_i : i = 1, \ldots, k\}$ and let Q_i be the product of all primes $p \le z$ which do not divide $A_i B_i$. Consider the arithmetic progression $\lambda_i \equiv 1 + A_i \pmod{A_i^2}$, $\lambda_i \equiv -1 + B_i \pmod{B_i^2}$ and $\lambda_i \equiv 2 \pmod{Q_i^2}$. This progression is solvable by the Chinese Remainder Lemma and yields a residue class modulo $M = 2^2 \prod_{p \le z} p^2$ which is coprime to the modulus. Moreover, if λ_i is in the above progression modulo M, then $(\lambda_i - 1)/A_i$ and $(\lambda_i + 1)/B_i$ are both integers coprime to all primes $p < z$. We use Linnik's theorem to find such a prime $\lambda_i \le M^{O_k(1)} = e^{O_k(z)}$ (the implied constant might depend on k since we need to ensure that $\lambda_1, \ldots, \lambda_k$ end up being different primes). Since $\omega(m) \ll \log m / \log\log m$ for all positive integers m, the number of prime factors of either $(\lambda_i - 1)/A_i$ or $(\lambda_i + 1)/B_i$ is $\ll_k \log M^{O(1)} / \log\log M^{O(1)} \ll_k z/\log z$. Since the smallest such prime is $> z$, we have that

$$\begin{aligned}
\frac{\sigma(\lambda_i - 1)}{\lambda_i - 1} &= \frac{\sigma(A_i)}{A_i} \prod_{p^{\alpha_p} \,\|\, (\lambda_i - 1)/A_i} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{\alpha_p}}\right) \\
&= \frac{\sigma(A_i)}{A_i} \left(1 + O\left(\frac{1}{z}\right)\right)^{O_k(z/\log z)} = \frac{\sigma(A_i)}{A_i} \left(1 + O_k\left(\frac{1}{\log z}\right)\right),
\end{aligned}$$

and similarly

$$\frac{\phi(\lambda_i + 1)}{\lambda_i + 1} = \frac{\phi(B_i)}{B_i} \left(1 + O_k\left(\frac{1}{\log z}\right)\right).$$

Thus, the above estimates show that for z large in terms of k estimates (12) and (13) are consequences of (14) and (15).

This completes the proof of our theorem.

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References

[1] J.-M. De Koninck and F. Luca, *On the composition of the Euler function and the sum of divisors function*, Colloq. Math. 108 (2007), 31–51.
[2] N. Doyon and F. Luca, *On the local behavior of the Carmichael λ-function*, Michigan Math. J. 54 (2006), 283–300.
[3] P. Erdős, K. Győry and Z. Papp, *On some new properties of functions $\sigma(n)$, $\varphi(n)$, $d(n)$ and $\nu(n)$*, Mat. Lapok 28 (1980), 125–131.
[4] P. Erdős and A. Schinzel, *Distributions of the values of some arithmetical functions*, Acta Arith. 6 (1960/1961), 473–485.
[5] H. Halberstam and H.-E. Rickert, *Sieve Methods*, Academic Press, London, 1974.
[6] F. Luca and C. Pomerance, *On some problems of Mąkowski–Schinzel and Erdős concerning the arithmetical functions ϕ and σ*, Colloq. Math. 92 (2002), 111–130.
[7] A. Mąkowski and A. Schinzel, *On the functions $\varphi(n)$ and $\sigma(n)$*, ibid. 13 (1964), 95–99.
[8] H. L. Montgomery and R. C. Vaughan, *The large sieve*, Mathematika 20 (1973), 119–134.
[9] E. Wong, *Simultaneous approximation of reals by values of arithmetic functions*, in: Anatomy of Integers, J.-M. De Koninck et al. (eds.), CRM Proc. Lecture Notes 46, Amer. Math. Soc., 2008, 305–313.

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