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Rational points on certain quintic hypersurfaces

by

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1. Introduction. In this paper we are interested in the existence of integer and rational points on the hypersurface given by the equation

$$\mathcal{V}_f: \ f(p)+f(q)=f(r)+f(s),$$

where $f\in\mathbb{Q}[X]$ and $\deg f=5$. We assume that for each pair $a,b\in\mathbb{Q}\setminus\{0\}$ we have $f(ax+b)\neq cx^5+d$ for any $c,d\in\mathbb{Q}$. This assumption guarantees that $\mathcal{V}_f$ is an affine algebraic variety of dimension three. The set of rational points on $\mathcal{V}_f$ will be denoted by $\mathcal{V}_f(\mathbb{Q})$. In other words,

$$\mathcal{V}_f(\mathbb{Q})=\{(p,q,r,s)\in\mathbb{Q}^4: f(p)+f(q)=f(r)+f(s)\}.$$

Similarly, $\mathcal{V}_f(\mathbb{Z})$ denotes the set of integer points on $\mathcal{V}_f$, so $\mathcal{V}_f(\mathbb{Z})=\mathcal{V}_f(\mathbb{Q})\cap\mathbb{Z}^4$.

We say that the point $P=(p,q,r,s)\in\mathcal{V}_f$ is *nontrivial* if $\{p,q\}\cap\{r,s\}=\emptyset$ and $\{f(p),f(q)\}\cap\{f(r),f(s)\}=\emptyset$. We denote by T_f the set of trivial rational points on $\mathcal{V}_f$. Note that each singular point is trivial, and the number of singular points (rational or not) is finite. In the following, a *rational point* will mean a nontrivial rational point.

The problem of the existence of integer points on $\mathcal{V}_f$ was investigated in the interesting work of Browning [1], who showed that

$$M(f;B)\ll_{\varepsilon,f} B^{1+\varepsilon}(B^{1/3}+B^{2/\sqrt{5}+1/4})$$

for each $\varepsilon>0$; here $M(f;B)$ is the number of solutions $(p,q,r,s)\in\mathbb{Z}^4$ of the equation which defines $\mathcal{V}_f$ with $0<p,q,r,s\le B$ and $\{p,q\}\cap\{r,s\}=\emptyset$. The above estimate shows that the set of positive integer points on $\mathcal{V}_f$ is rather "thin". To the author's knowledge no example is known of a polynomial f of degree five with $\mathcal{V}_f(\mathbb{Z})\setminus T_f$ infinite. Moreover, we have been unable to find in the literature any example of a polynomial f of degree five which gives a positive answer to the following:

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Question 1.1. *Let $N > 1$ be given. Is it possible to construct a polynomial f of degree five such that $\sharp(\mathcal{V}_f(\mathbb{Z}) \setminus T_f) > N$?*

It is clear that the question of existence of a polynomial f of degree five with $\mathcal{V}_f(\mathbb{Q})$ infinite should be easier to tackle. So, it is natural to ask the following:

Question 1.2. *For which polynomials f of degree five the set $\mathcal{V}_f(\mathbb{Q})$ is infinite?*

It seems that these questions have not been considered before. It is also clear that in the case of Question 1.2 we can only consider polynomials of the form $f(X) = X^5+aX^3+bX^2+cX$, where $a, b, c \in \mathbb{Z}$ and at least one of a, b, c is nonzero. We will see that if $b \neq 0$, then the diophantine equation $f(p) + f(q) = f(r) + f(s)$ has a rational two-parameter solution (Theorem 2.1). In geometrical terms this means that there is a unirational surface contained in $\mathcal{V}_f$. From this we can deduce easily that the answer to Question 1.1 is positive. Moreover, we will prove that for any polynomial f of degree five there exists a $\mathbb{Q}(i)$-rational surface contained in $\mathcal{V}_f$ (Theorem 2.5).

2. Construction of rational points on $\mathcal{V}_f$. Let $f \in \mathbb{Q}[X]$ with $\deg f = 5$. In this section we will construct parametric solutions of the equation defining the hypersurface

$$\mathcal{V}_f : \ f(p) + f(q) = f(r) + f(s).$$

Since we are interested in rational solutions, we can assume without loss of generality that $f(X) = X^5 + aX^3 + bX^2 + cX$, $a, b, c \in \mathbb{Z}$ and at least one of a, b, c is nonzero.

Our aim is to prove the following theorem.

Theorem 2.1. *Let $f(X) = X^5 + aX^3 + bX^2 + cX \in \mathbb{Z}[X]$, where $b \neq 0$. Then there exists a $\mathbb{Q}$-unirational elliptic surface $\mathcal{E}_f$ such that $\mathcal{E}_f(\mathbb{Q}) \subset \mathcal{V}_f(\mathbb{Q})$. In particular, the set $\mathcal{V}_f(\mathbb{Q})$ is infinite.*

Proof. In the equation defining $\mathcal{V}_f$ we make a (noninvertible) substitution

$$(1) \qquad p = x, \quad q = y - x, \quad r = z, \quad s = y - z.$$

As a result,

$$f(x) + f(y - x) - f(z) - f(y - z) = (x - z)(x - y + z)G(x, y, z),$$

where $G(x, y, z) = 2b + 3ay + 5x^2y - 5xy^2 + 5y^3 - 5y^2z + 5yz^2$. From the geometric point of view this substitution amounts to intersecting the hypersurface $\mathcal{V}_f$ with the hyperplane $L : p+q = r+s$ ((1) gives a parametrization of L).

Note that the equation $G(x, y, z) = 0$ has a solution in rational numbers if and only if the discriminant of the polynomial G with respect to z is the square of a rational number, say v. Thus, we are interested in the rational points on the surface

$$\mathcal{S}:\ v^2 = -5y(15y^3 + 20xy(x-y) + 12ay + 8b) =: \Delta(x, y).$$

If we make a change of variables

$$(x, y, w) = \left(-\frac{5b(t+1)}{X+5a}, -\frac{10b}{X+5a}, \frac{20bY}{(X+5a)^2}\right),$$

with the inverse

$$(X, t, Y) = \left(-\frac{5(2b+ay)}{y}, \frac{2x-y}{y}, \frac{5bw}{y^2}\right)$$

the surface $\mathcal{S}$ is transformed to

$$\mathcal{E}:\ Y^2 = X^3 - 75a^2X - 125(5bt^2 + 10b^2 + 2a^3).$$

Note that the surface $\mathcal{E}$ is of degree three and contains a rational curve at infinity $[X : Y : t : Z] = [0 : 1 : t : 0]$, so the Segre theorem shows that $\mathcal{E}$ is unirational. This implies the existence of a two-parameter solution of the equation defining $\mathcal{E}$. For the convenience of the reader we will show how this solution can be constructed.

Set $F(X, Y, t) = Y^2 - (X^3 - 75a^2X - 125(5bt^2 + 10b^2 + 2a^3))$. We use the method of undetermined coefficients to find a two-parameter solution of $F(X, Y, t) = 0$. Let u, v be parameters and set

$$X = T^2 + 10uT + p, \quad Y = T^3 + qT^2 + rT, \quad t = (v/5b)T^2 + s. \tag{2}$$

We want to find $p, q, r, s, T \in \mathbb{Q}(u, v)$ such that the equation $F(X, Y, t) = 0$ is satisfied identically. For the quantities given by (2) we have

$$F(X, Y, t) = a_0 + a_1T + a_2T^2 + a_3T^3 + a_4T^4 + a_5T^5,$$

where

$$\begin{aligned} a_0 &= 250a^3 + 1250b^2 + 75a^2p - p^3 + 625b^2s^2, & a_1 &= 30(5a-p)(5a+p)u, \\ a_2 &= 75a^2 - 3p^2 + r^2 - 300pu^2 + 250bsv, & a_3 &= 2(qr - 30pu - 500u^3), \\ a_4 &= -3p + q^2 + 2r - 300u^2 + 25v^2, & a_5 &= 2(q - 15u). \end{aligned}$$

The system of equations $a_2 = a_3 = a_4 = a_5 = 0$ has exactly one solution in $\mathbb{Q}(u, v)$ given by

$$\begin{aligned} p &= 25(u^2 - 3v^2)/3, & q &= 15u, \\ r &= 50(u^2 - v^2), & s &= (25u^4 - 450u^2v^2 - 75v^4 - 9a^2)/30bv. \end{aligned} \tag{3}$$

If p, q, r, s are given by (3) then $F(T^2+10uT+p, T^3+qT^2+rT, (v/5b)T^2+s) \in \mathbb{Q}(u, v)[T]$ and $\deg_T F = 1$. So this polynomial has a root in the field

$\mathbb{Q}(u,v)$ given by

$$T = -\frac{250a^3 + 1250b^2 + 75a^2p - p^3 + 625b^2s^2}{30(5a-p)(5a+p)u}.$$

Putting the calculated values p, q, r, s, T into (2) we get the desired solutions depending on two parameters u, v. ∎

REMARK 2.2. The same method was used by Whitehead [4] to prove the unirationality of the surface $z^2 = h(x,y)$, where $h \in \mathbb{Q}[x,y]$ has degree three. This theorem can also be found in [3, p. 85].

Thanks to the theorem above we can easily prove that the answer to the Question 1.1 is positive.

COROLLARY 2.3. *For any $N \in \mathbb{N}_+$ there are infinitely many polynomials $f \in \mathbb{Z}[X]$ of degree five such that on the hypersurface $\mathcal{V}_f : f(p) + f(q) = f(r) + f(s)$ there are at least N nontrivial integer points.*

Proof. Let $f(X) = X^5 + aX^3 + bX^2 + cX$ with $b \neq 0$. From the previous theorem, the diophantine equation $f(p) + f(q) = f(r) + f(s)$ has infinitely many solutions in rational numbers. Let $(p_i/p_i', q_i/q_i', r_i/r_i', s_i/s_i')$, $i = 1, \ldots, N$, be such distinct solutions, and define

$$d = \mathrm{LCM}(p_1', q_1', r_1', s_1', \ldots, p_N', q_N', r_N', s_N').$$

If we now define $F(X) = X^5 + ad^2X^3 + bd^3X^2 + cd^4X$, then on the hypersurface $\mathcal{V}_f : F(p) + F(q) = F(r) + F(s)$ we have the points

$$(dp_i/p_i', dq_i/q_i', dr_i/r_i', ds_i/s_i')$$

for $i = 1, \ldots, N$, which are tuples of integers. ∎

This corollary gives a positive answer to Question 1.1. However, if N grows then the coefficients of the polynomial F grow too. Therefore, we can ask the following:

QUESTION 2.4. *Let $N > 1$ be given. Is it possible to construct a polynomial $f(X) = X^5 + aX^3 + bX^2 + cX$ with $\sharp(\mathcal{V}_f(\mathbb{Z}) \setminus T_f) \geq N$ and at least one nonzero coefficient a, b or c independent of N?*

As we will see, the answer to this question is also positive. First, let us go back to Question 1.2 for $f(X) = X^5 + aX^3 + cX$. Unfortunately, we are unable to prove a theorem similar to Theorem 2.1 in this case. However, we can prove the following:

THEOREM 2.5. *Let $f(X) = X^5 + aX^3 + cX \in \mathbb{Z}[X]$. If $a < 0$ and $a \not\equiv 2, 18, 34 \pmod{48}$ then the diophantine equation $f(p) + f(q) = f(r) + f(s)$ has a two-parameter rational solution.*

Proof. Set

$$(4)\qquad p=\frac{-x+y+3z}{5},\quad q=\frac{2x+y}{5},\quad r=\frac{3y}{5},\quad s=\frac{x-y+3z}{5}.$$

Then

$$f(p)+f(q)-f(r)-f(s)=\frac{6(x-y)(x+2y-3z)(x+2y+3z)F(x,y,z)}{625},$$

where $F(x,y,z,a)=x^2+2y^2+3z^2+5a$. The first three brackets in the numerator lead to trivial solutions of our equation. Thus we obtain a nontrivial solution if and only if $F(x,y,z)=0$. In particular, we must have $a<0$. For diophantine equations of degree two, the local to global principle of Hasse is true: the equation $x^2+2y^2+3z^2+5a=0$ has a solution in rational numbers if and only if it has solutions in the field $\mathbb{Q}_p$ of p-adic numbers for any given $p\in\mathbb{P}\cup\{\infty\}$, where as usual $\mathbb{Q}_\infty=\mathbb{R}$.

The theorem below gives the well-known criterion of the solvability of the diophantine equation $a_1X_1^2+a_2X_2^2+a_3X_3^2+a_4X_4^2=0$. This criterion is taken from [2].

THEOREM 2.6. *If* $f(x_1,x_2,x_3,x_4)=a_1X_1^2+a_2X_2^2+a_3X_3^2+a_4X_4^2$, *where* $a_i\in\mathbb{Z}\setminus\{0\}$ *are square-free and no three have a factor in common, then* f *represents zero if and only if the following three conditions hold*:

1. *Not all coefficients have the same sign.*
2. *If* p *is an odd prime dividing two coefficients and* $(d/p^2\,|\,p)=1$, *then* $(-a_ia_j\,|\,p)=1$, *where* $\mathrm{GCD}(a_ia_j,p)=1$ *and* $d=a_1a_2a_3a_4$ *is the discriminant of the form* f.
3. *If* $d\equiv 1 \pmod 8$ *or* $d/4\equiv 1 \pmod 8$ *then* $(-a_1a_2,-a_2a_3)_2=1$.

Here $(\alpha,\beta)_2$ takes two values: $+1$ or -1, depending on whether the equation $\alpha x_1^2+\beta x_2^2=1$ has a solution in $\mathbb{Q}_2$ or not. If $\alpha=2^u\alpha_1$, $\beta=2^v\beta_1$ and $\mathrm{GCD}(2,\alpha_1\beta_1)=1$, then $(\alpha,\beta)_2=(2\,|\,\alpha_1)^v(2\,|\,\beta_1)^u(-1)^{(\alpha_1-1)(\beta_1-1)/4}$, where $(\cdot|\cdot)$ is the usual Legendre symbol.

In order to finish the proof of Theorem 2.5 we apply the above procedure to the quadratic form $X_1^2+2X_2^2+3X_3^2+5aX_4^2$. We have to consider four cases depending on the values of $\mathrm{GCD}(a,6)$. Because this reasoning is very simple we leave it to the reader. ∎

EXAMPLE 2.7. Let $f(X)=X^5-X^3+cX$ and consider the equation $f(p)+f(q)=f(r)+f(s)$. We will show how to use the previous theorem in practice.

Consider the equation $(*)$ $x^2+2y^2+3z^2-5=0$. It has a rational solution $(x,y,z)=(0,1,1)$. Set $x=uT, y=vT+1, z=T+1$. Next solve the equation $(uT)^2+2(vT+1)^2+3(T+1)^2-5=0$ with respect to T. After some simplifications we get a parametrization of rational solutions of equation $(*)$ in the form

$$x = -\frac{2u(2v+3)}{u^2+2v^2+3}, \quad y = \frac{u^2-2v^2-6v+3}{u^2+2v^2+3}, \quad z = \frac{u^2+2v^2-4v-3}{u^2+2v^2+3}.$$

Hence we get a solution of the equation $f(p)+f(q)=f(r)+f(s)$:

$$p = \frac{2(2u^2+(2v+3)u+2v^2-9v-3)}{5(u^2+2v^2+3)},$$

$$q = \frac{u^2-4(2v+3)u-2v^2-6v+3}{5(u^2+2v^2+3)},$$

$$r = \frac{3(u^2-2v^2-6v+3)}{5(u^2+2v^2+3)}, \qquad s = \frac{2(u^2-(2v+3)u+4v^2-3(v+2))}{5(u^2+2v^2+3)}.$$

Using the method of proof of Theorem 2.5 we will show the following:

COROLLARY 2.8. *The answer to Question* 2.4 *is positive.*

Proof. This is a simple consequence of the fact that for any number N we can find a negative number a_N such that the equation $x^2+2y^2+3z^2=-5a_N$ has at least N solutions in positive integers x,y,z all divisible by 5. To prove this, we set $g_N=\prod_{k=1}^{N}(k^2+2)$ and $a_N=-(5g_N)^2$. Next, we define

$$x_k = \frac{5g_N}{k^2+2}\,(2k+3), \quad y_k = \frac{5g_N}{k^2+2}\,(k^2+3k-2), \quad z_k = \frac{5g_N}{k^2+2}\,(k^2-2k-1),$$

for $k=1,\ldots,N$. Note that x_k,y_k,z_k are integers divisible by 5.

Since

$$\left(\frac{2k+3}{k^2+2}\right)^2 + 2\left(\frac{k^2+3k-2}{k^2+2}\right)^2 + 3\left(\frac{k^2-2k-1}{k^2+2}\right)^2 = 5,$$

we see that

$$x_k^2+2y_k^2+3z_k^2 = -5a_N \quad \text{for } k=1,\ldots,N.$$

Now define $f_N(x)=x^5+a_Nx^3+cx$, where c is an integer. From our reasoning we see that on the hypersurface $\mathcal{V}_{f_N}$ there are at least N integer points given by

$$p_k = \frac{-x_k+y_k+3z_k}{5}, \quad q_k = \frac{2x_k+y_k}{5}, \quad r_k = \frac{3y_k}{5}, \quad s_k = \frac{x_k-y_k+3z_k}{5},$$

for $k=1,\ldots,N$ and the c is independent of N. ∎

The results of this section suggest the following:

CONJECTURE 2.9. *Let $f(x)=x^5+ax^3+cx$, where $a,c\in\mathbb{Z}\setminus\{0\}$. Then the set $\mathcal{V}_f(\mathbb{Q})\setminus T_f$ is infinite.*

3. Construction of $\mathbb{Q}(i)$-rational points on $\mathcal{V}_f$. In this section we will construct $\mathbb{Q}(i)$-rational points on the hypersurface $\mathcal{V}_f$.

Let us go back to the equation of the surface $\mathcal{S}$ from the proof of Theorem 2.1 and note that the polynomial Δ has $\deg_x \Delta = 2$. Now we view $\mathcal{S}$ as a curve defined over the field $\mathbb{Q}(i)(y)$, where $i^2 + 1 = 0$. It is easy to see that it is a rational curve. Indeed, on $\mathcal{S}$ there is a $\mathbb{Q}(i)(y)$-rational point $[x : v : w] = [i : 10y : 0]$ (it is a point at infinity). Setting $x = ip$, $w = 10yp + u$ and solving the resulting equation for p we get the parametrization of our curve given by

$$x = i\,\frac{u^2 + 75y^4 + 60ay^2 + 40by}{20y(5iy^2 - u)}, \quad w = \frac{u^2 - 10iuy^2 - 75y^4 - 60ay^2 - 40by}{2(u - 5iy^2)}.$$

Hence a two-parameter solution of the equation defining $\mathcal{V}_f$ is

$$\begin{aligned}
p &= -i\,\frac{u^2 + 75y^4 + 60ay^2 + 40by}{20y(u - 5iy^2)},\\
q &= i\,\frac{u^2 - 20iy^2u - 25y^4 + 60ay^2 + 40by}{20y(u - 5iy^2)},\\
r &= \frac{u^2 + 10(1-i)y^2u - 25(3+2i)y^4 - 60ay^2 - 40by}{20y(u - 5iy^2)},\\
s &= -\frac{u^2 - 10(1+i)y^2u - 25(3-2i)y^4 - 60ay^2 - 40by}{20y(u - 5iy^2)}.
\end{aligned}$$

We sum up the discussion concerning the existence of $\mathbb{Q}(i)$-rational points on $\mathcal{V}_f$ in the following:

THEOREM 3.1. *Let $f(X) = X^5 + aX^3 + bX^2 + cX \in \mathbb{Z}[X]$. If $a = b = 0$ then there exists a $\mathbb{Q}(i)$-rational curve contained in $\mathcal{V}_f$. If $a \neq 0$ or $b \neq 0$, then there exists a $\mathbb{Q}(i)$-rational surface contained in $\mathcal{V}_f$.*

Note that in the above expressions for p, q, r, s the number c does not appear explicitly, and the solution obtained is nontrivial for all $a, b, c \in \mathbb{Z}$. If we put $a = b = c = 0$, then we get a parametric solution (defined over $\mathbb{Q}(i)$) of the diophantine equation $p^5 + q^5 = r^5 + s^5$. After simplifications the solution is (in homogeneous form)

$$\begin{aligned}
p &= u^2 + 75v^2,\\
q &= -u^2 + 20iuv + 25v^2,\\
r &= iu^2 + 10(1+i)uv + 25(2-3i)v^2,\\
s &= -iu^2 - 10(1-i)uv + 25(2+3i)v^2.
\end{aligned}$$

This solution is probably well known but we have not been able to find it in the literature. Note that this solution can be used to construct a parametric solution (over $\mathbb{Z}[i]$) of the diophantine equation

$$p^{5n} + q^5 = r^5 + s^5,$$

where n is a given positive integer. Indeed, it is easy to see that the diophantine equation $u^2 + 75v^2 = X^n$ has a parametric solution given by the solution of the system

$$u+\sqrt{-75}\, v = (t_1+\sqrt{-75}\, t_2)^n, \ \ u-\sqrt{-75}\, v = (t_1-\sqrt{-75}\, t_2)^n, \ \ X = t_1^2+75t_2^2.$$

It is clear that the solutions u, v, X lead to a polynomial solution of the equation $p^{5n} + q^5 = r^5 + s^5$.

4. Possible generalizations. In this section we consider natural generalizations of the equation defining the hypersurface $\mathcal{V}_f$.

The first natural generalization which comes to mind is

$$\mathcal{V}_{F,G}: \ F(p) + G(q) = F(r) + G(s),$$

where $F(x) = x^5 + ax^3 + bx^2 + cx$, $G(x) = x^5 + dx^3 + ex^2 + fx$ and $F(x) - F(0) \neq G(x) - G(0)$. It is clear that in order to find rational points on $\mathcal{V}_{F,G}$ we can assume that $a, b, \ldots, e, f \in \mathbb{Z}$.

We will show that for given F, G as above the hypersurface $\mathcal{V}_{F,G}$ contains an elliptic surface defined over $\mathbb{Q}$. To do this, define

$$p = t - \frac{U}{V}, \quad q = \frac{U}{V}, \quad r = \frac{1}{V}, \quad s = t - \frac{1}{V}. \tag{5}$$

Then

$$F(p) + G(q) - F(r) - G(s) = -\frac{tV - U - 1}{V^4} H(U, V, t),$$

where $H(U, V, t) = \sum_{i+j\leq 3} a_{i,j} U^i V^j$ and

$$\begin{aligned} &a_{3,0} = -a_{2,0} = 5t, && a_{1,0} = -a_{0,0} = 5t,\\ &a_{2,1} = -a + d - 5t^2, && a_{1,1} = a - d,\\ &a_{0,1} = -a + d + 5t^2, && a_{1,2} = b + e + (2a + d)t + 5t^3,\\ &a_{0,2} = -b - e - (a + 2d)t - 5t^3, && a_{0,3} = f - c + (e - b)t + (d - a)t^2. \end{aligned} \tag{6}$$

Note that the surface $S_{F,G} : H(U, V, t) = 0$ can be viewed as a cubic curve defined over the field $\mathbb{Q}(t)$. This curve has a $\mathbb{Q}(t)$-rational point $P = (U, V) = (1, 0)$. We can consider P as the point at infinity and transform $S_{F,G}$ birationally onto the elliptic surface $\mathcal{E}_{F,G}$ with the Weierstrass equation

$$\mathcal{E}_{F,G}: \ Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6,$$

where $a_i \in \mathbb{Z}[t]$ depend on the coefficients of F, G. We do not give the polynomials a_i exactly as they are rather complicated. However, the computations suggest the following.

CONJECTURE 4.1. *Let $a, b, c, d, e, f \in \mathbb{Z}$ and consider the elliptic surface*

$$\mathcal{E}: \ H(U,V,t) = \sum_{i+j\le 3} a_{i,j} U^i V^j = 0$$

where the $a_{i,j}$ are given by (6). *Then the set*

$$S = \{t \in \mathbb{Q} : \mathcal{E}_t \text{ is an elliptic curve and has a positive rank}\}$$

is nonempty.

Another generalization which comes to mind is

$$\mathcal{V}^f: \ f(p,q) = f(r,s),$$

where f is a symmetric ($f(x,y) = f(y,x)$) quintic polynomial, i.e.

$$(7) \quad f(x,y) = \sum_{i=1}^{5} a_i(x^i + y^i) + xy\sum_{i=1}^{3} b_i(x^i + y^i) + x^2y^2(c_0(x+y) + c_1).$$

We will show that there are in general infinitely many $\mathbb{Q}(i)$-rational points on $\mathcal{V}^f$. Indeed, the substitution (5) yields

$$f(p,q) - f(r,s) = -\frac{(U-1)(tV-U-1)}{V^4}\, G(U,V),$$

where $G(U,V) = \sum_{i+j\le 2} b_{i,j} U^i V^j$, $b_{i,j} \in \mathbb{Z}[t]$ and

$$b_{2,0} = 2a_4 - 2b_2 + c_1 + t(5a_5 - 3b_3 + c_0), \qquad b_{1,0} = 0,\ b_{0,0} = b_{2,0},$$
$$b_{0,2} = 2a_2 + t(3a_3 - b_1) + t^2(4a_4 - b_2) + t^3(5a_5 - b_3), \quad b_{0,1} = b_{1,1} = -tb_{2,0}.$$

In order to construct $\mathbb{Q}(i)(t)$-rational points on $\mathcal{V}^f$ we must consider the quadratic $C : G(U,V) = 0$ defined over the field $\mathbb{Q}(t)$. Note that $G(i,0) = 0$, so we can use standard methods to parametrize $\mathbb{Q}(i)(t)$-rational points on C and in general we get a two-parameter solution of the equation $G(U,V) = 0$. This implies the existence of a two-parameter solution defined over the field $\mathbb{Q}(i)$ of the equation defining the hypersurface $\mathcal{V}^f$.

It is clear that this method does not always work. Indeed, if $b_{2,0} \equiv 0 \in \mathbb{Z}[t]$ then the equation $G(U,V) = 0$ reduces to the equation $b_{0,2}(t) = 0$ which has at most three solutions in $\mathbb{Q}(i)$. However, if $b_{2,0}(t) \ne 0$ for some t, then the curve C is nontrivial and we can apply our method to construct $\mathbb{Q}(i)$-rational points on $\mathcal{V}^f$. This suggests the following

QUESTION 4.2. *Consider the hypersurface $\mathcal{V}_f : f(p,q) = f(r,s)$ where f is of the form* (7). *Suppose that $2a_4 - 2b_2 + c_1 = 5a_5 - 3b_3 + c_0 = 0$. Is it possible to construct $\mathbb{Q}(i)$-rational points on $\mathcal{V}^f$?*

REMARK 4.3. Although it is possible, we do not give equations defining the parametrization of the curve C in the case when $b_{2,0} \in \mathbb{Z}[t] \setminus \{0\}$. Also note that it is very likely that for a specific choice of a_i, b_j, c_k there is a rational number t_0 such that the quadric $C_{t_0} : G(U,V) = 0$ (here we specialize the curve C which is defined over $\mathbb{Q}(t)$ at $t = t_0$) has a rational point. Then we can use a standard method of parametrization of quadrics to get rational solutions of the equation defining the hypersurface $\mathcal{V}^f$.

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