Estimation of some exponential sums containing 
the fractional part function 
and some other "non-standard" exponential sums

by

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1. Introduction. Some problems in number theory and some other branches of mathematics can be reduced to the estimation of exponential sums

$$\sum_{X_1 < x \leq X_2} e(F(x)) \quad \text{with } X = X_2 - X_1 \leq X_1.$$ 

If $F(x)$ is a polynomial or a function which can be reduced to a polynomial then the sum can be evaluated by using Vinogradov’s method; if $F(x)$ is "van der Corput" type function then one uses van der Corput’s method or Bombieri–Iwaniec method. Here by \textit{van der Corput} (v.d.c.) type function of order \(k\) we mean a real-valued \(k\) times continuously differentiable function \(F(x)\) such that \(F^{(j)}(x) \sim F_j(x)/x^j\) \((j = 1, \ldots, k)\) with piecewise monotone \(F_j(x)\) such that if \(k > 1\), then

$$1 \ll F_{j+1}(x)/F_j(x) \ll 1 \quad \text{and} \quad \liminf x^{1-2/K} F^{(k)}(x) \ll 1;$$

if \(k = 1\), then

$$\lim_{x \to \infty} F_1(x) = \infty \quad \text{and} \quad \limsup |F'(x)| < 1$$

(see the notation below).

Note that if \(k > 1\) is the smallest integer such that \(F(x)\) is a v.d.c. function of order \(k\) and \(K = 2^k\) then

$$F^{(k)}(x) \ll x^{2/K-1} \quad \text{and} \quad F^{(k-1)}(x) \gg x^{4/K-1}$$

so that

$$(1) \quad x^{A/K-2} \ll F^{(k)}(x) \ll x^{2/K-1}.$$ 

If \(X\) is “not small”, the above mentioned methods give non-trivial estimates. We call such sums \textit{standard} exponential sums. If \(X\) is “small”, the
sum is called short and the well-known van der Corput’s estimates may be larger than the trivial estimates. Also, if $F(x) \text{ contains an oscillating term, van der Corput’s method cannot be used directly. We call such sums non-standard exponential sums. In the past we studied short sums [2] and sums containing an oscillating term [1], [2].}

Wenguang Zhai has recently introduced [4] a method of evaluation of exponential sums with $F(x) = f(x) + g(x)h(x)$. He applied the method to prove that for any $k \neq 0$ and any $c > 0$ the sequence $\{[n^c]\log^k n\}$ is uniformly distributed modulo 1 by proving that the discrepancy of the sequence satisfies

$$D(X) \ll X^{-\delta(c) \log X} \quad \text{for some } \delta(c) > 0.$$  

His result improved the result of Rieger [3] who proved the uniform distribution of the sequence for $1 < c < 3/2$ and $0 < k < 1$.

The method of Zhai gives a non-trivial estimate if $f(x), g(x)$ and $h(x)$ are v.d.c. functions and $g(x) \ll x^{3/4-\alpha}$ for any fixed $\alpha > 0$. One can evaluate such sums (and more general sums) with $g(x) \ll x^{1-\alpha}$ using our method of evaluation of short sums and

**Lemma 1.** Let $f(t, x)$ be a real-valued function such that

$$|f(t_1, x) - f(t_2, x)| \leq \lambda|t_1 - t_2|.$$  

Then for any real function $g(x)$, any positive integer $r$ and any $M > 0$ we have

$$S = \sum_x a(x)e(f(g(x), \{h(x)\})) = \frac{1}{M} \sum_{m=0}^{M-1} \sum_{j=0}^{\infty} b_{j,m} \sum_x a(x)e(f(g(x), m/M) + jh(x))$$

$$+ O\left(\frac{\lambda r + r}{M} \sum_x |a(x)|\right) + O\left(\frac{r}{M} \sum_{j=0}^{\infty} \frac{\sin(2\pi j/M)}{\sin(\pi j/M)} a_j \sum_x |a(x)|e(jh(x))\right),$$

where

$$a_j = (\sin(\pi j/M)/(\pi j/M))^{r+1}, \quad a_0 = 1,$$

$$b_{j,m} = a_j e(-(2m + 1)j/(2M)).$$

This lemma is also simpler to use than the corresponding lemma of Zhai. Using Lemma 1, we prove

**Theorem 1.** Let $k$ be a sufficiently large positive integer such that $f(x), g(x)$ and $h(x)$ are v.d.c. functions of order $k$ and let $k_1 \in [2, k - 1]$ and
\( k_2 \in [2, k-2] \) be the smallest integers such that \( f(x), g(x) \) and \( h(x) \) are v.d.c. functions of orders \( k_1, 1 \) and \( k_2 \) respectively. Assume that \( g(x) \ll x^{1-\alpha} \) for some \( \alpha > 0 \) and that for any \( m \) the functions \( f_m(x)/h_m(x) \) are piecewise monotone on \( \ll 1 \) intervals and

\[
|f_m(x)h_{m+1}(x)/(f_{m+1}(x)h_m(x)) - 1| \gg 1.
\]

Define

\[
\varphi_j(y) = f^{(j)}(g^{-1}(y)), \quad \phi_j(y) = h^{(j)}(g^{-1}(y))
\]

and assume that for any \( m \) the functions \( \varphi_j^{(m)}(y)/\phi_j^{(m)}(y) \) are piecewise monotone on \( \ll 1 \) intervals and

\[
|\varphi_j^{(m)}(y)\phi_j^{(m+1)}(y)/(\varphi_j^{(m+1)}(y)\phi_j^{(m)}(y)) - 1| \gg 1,
\]

\[
|\varphi_1^{(p)}(y)| \ll y^{2/P-3}, \quad |\phi_1^{(p)}(y)| < y^{2/P-5} \text{ for some integer } p > 1.
\]

Then

\[
S \equiv \sum_{X \leq x \leq 2X} e(f(x) + g(x)\{h(x)\}) \ll X\Delta_0,
\]

where

\[
\Delta_0 = X^{-\alpha/(3P)} + (G + X^{1/3})^{-1/(PK)}, \quad G = g(X).
\]

Also, if \( f(x) = Ch(x) \) then the above estimate holds if \( |C| > 1 \) and

\[
|S| \ll X\Delta_0 + X/G \quad \text{if } |C| < 1.
\]

**Theorem 2.** Let \( f(x, y) \) be a real-valued function on \( [X, 2X] \times [0, 1] \) such that for any \( y \) it is a v.d.c. function of order \( k \). Assume that \( k \) is the smallest such integer. Assume also that \( g(x) \) is a v.d.c. function of order 1 such that for some \( a > 0 \) we have \( g(x) \ll x^{1-a} \) and, setting \( h(n) = f(g^{-1}(n), n) \), assume that it is a v.d.c. function of order \( j \). Let \( \lambda_k \) and \( \mu_j \) be such that

\[
|\partial^k f(x, y)/\partial x^k| \asymp \lambda_k \quad \text{and} \quad |h^{(j)}(n)| \asymp \mu_j.
\]

Then

\[
S \equiv \sum_{X \leq x \leq 2X} e(f(x, \{g(x)\})) \ll X[\lambda_k^{1/(K-2)} + X^{-a/K} + G(X)^{1/2 - K} + \mu_j^{4/(KJ + K)}].
\]

For the sequence \( \{[n^\alpha] \log^\beta n\} \) considered by Rieger and Zhai,

\[
f(x) = x^\alpha \log^\beta x, \quad g(x) = -\log^\beta x, \quad h(x) = x^\alpha,
\]

so that if \( \alpha \beta \neq 0 \) the conditions of Theorem 1 are satisfied and one can use it to prove the uniform distribution of the sequence modulo 1 and to evaluate the discrepancy. One can do the same for \( f(x) = x^\alpha, g(x) = x^\beta \) and \( h(x) = x^\gamma \) with \( \alpha \neq \gamma \) and \( \beta < 1 \), and some other functions.
2. Notation. We will use the following notation: \( e(x) = \exp(2\pi ix) \); \( f(x) \ll g(x) \) means that \( f(x) = O(g(x)) \); \( f(x) \ll g(x) \) means that \( f(x) \ll g(x)x^c \); \( f(x) \asymp g(x) \) means that \( f(x) \sim g(x) \); \( f(x) \ll g(x) \ll f(x) \); \( \{x\}, [x] \) and \( \|x\| \) are the fractional part, the integer part and the distance to the nearest integer functions; \(|S|\) is the cardinality of the set \( S \). For positive integers \( k, r \) etc., we write \( K = 2^k \), \( R = 2^r \) etc.

3. Proofs. To prove Lemma 1, we take

\[
\chi_{r,m}(x) \equiv \chi_{r,m}(x; \delta) = (2\delta)^{-r} \int_{-\delta}^{\delta} \ldots \int_{-\delta}^{\delta} \chi_{0,m}(x + t_1 + \ldots + t_r) \, dt_1 \ldots dt_r \quad (m = 0, \ldots, M - 1),
\]

where \( \chi_{0,m}(x) \) is the characteristic function of \([m/M, (m+1)/M)\) modulo 1. Expanding \( \chi_{0,m}(x) \) into a Fourier series, we obtain

\[
(3) \quad \chi_{r,m}(x) = (2\delta)^{-r} \int_{-\delta}^{\delta} \ldots \int_{-\delta}^{\delta} \left( \frac{1}{M} + \sum_{|j|=1}^{\infty} a_{j,m} e(x + t_1 + \ldots + t_r) \right) dt_1 \ldots dt_r
\]

\[
= \frac{1}{M} + \sum_{|j|=1}^{\infty} a_{j,m} \left( \frac{\sin(2\pi j\delta)}{2\pi j \delta} \right)^r e(jx)
\]

where

\[
a_{j,m} = \frac{\sin(\pi j/M)}{\pi j} e\left( \frac{-2m + 1}{2M} \right).
\]

We use (3) with \( \delta = 1/M \) so that \( a_{j,m}(\sin(2\pi j\delta)/(2\pi j\delta))^r = b_{j,m}/M \) from the lemma. Since \( \sum_{m=0}^{M-1} \chi_{0,m}(x) = 1 \), we have \( \sum_{m=0}^{M-1} \chi_{r,m}(x) = 1 \) and we obtain

\[
S = \sum_x a(x) \sum_{m=0}^{M-1} \chi_{r,m}(h(x)) e(f(g(x), \{h(x)\}))
\]

\[
= \sum_x a(x) \sum_{m=0}^{M-1} \chi_{r,m}(h(x)) e(f(g(x), m/M))
\]

\[
+ \sum_x a(x) \sum_{m=0}^{M-1} \chi_{r,m}(h(x)) [e(f(g(x), \{h(x)\})) - e(f(g(x), m/M))].
\]

The first sum is reduced to the first sum in (2) by using (3) with \( \delta = 1/M \). To evaluate the second sum (which we denote with \( S_1' \)), we divide it into two subsums: the first subsum, \( S_1'' \), is over all \( m \) with \( \|m/M\| > r/M \), and \( S_1'' \) is the remaining part of \( S_1 \).
If $\|m/M\| > r/M$ then $\chi_{r,m}(g(x)) = 0$ unless $|g(x) - m/M| < r/M$. Since $|e(a) - e(b)| = 2|\sin(\pi(b - a))| < 2|a - b|$, we obtain

$$S'_1 \ll \sum_x |a(x)| \sum_m \chi_{r,m}(h(x)) \frac{r \lambda}{M} = \frac{r \lambda}{M} \sum_x |a(x)|.$$

To evaluate $S''_1$, we write first

$$|S''_1| \leq \sum_x |a(x)| \sum_m 2 \chi_{r,m}(h(x)) \leq 2 \sum_x |a(x)| \chi_1(h(x); 1/(2M))$$

where

$$\chi_1(t; \delta) \equiv (2\delta)^{-r} \int_{-\delta}^{\delta} \ldots \int_{-\delta}^{\delta} \chi(t + t_1 + \ldots + t_r) dt.$$ 

and $\chi(t)$ is the characteristic function of $[-r/M, r/M)$ modulo 1. Similarly to (3), we obtain

$$\chi_1(t; 1/(2M)) = \frac{2r}{M} + 2 \sum_{|j|=1}^{\infty} \sin(2\pi r j/M) \left( \frac{\sin(\pi j/M)}{\pi j/M} \right)^r e(jt)$$

so that

$$|S''_1| \leq \frac{4r}{M} \sum_x |a(x)| + 2 \sum_{j=1}^{\infty} \frac{\sin(2\pi r j/M)}{\sin(\pi j/M)} a_j e(jh(x)).$$

To prove the theorems, we need three more lemmas.

**Lemma 2.** Let $f(x) \in C^{(k+j)}[X_1, X_2]$ with $k > 1$, $j > 0$ and $1 \leq X = X_2 - X_1 \leq X_1$. Assume that $f^{(k)}(x) \leq \lambda_k$ and $f^{(k+j)}(x) \asymp \lambda_{k+j}$.

Then

$$\left| \sum_{X_1 \leq x \leq X_2} e(f(x)) \right| \ll X \left( \lambda_k^{1/(K-2)} + (X^{-j-2} \lambda_k/\lambda_{k+j})^{4/(K(j+4))} + (\lambda_{k+j} X^{4+j-8/K})^{-4/(K(j+2))} \right).$$

Lemma 2 is a simple generalization of van der Corput estimates (for the proof, see [1, Lemma 4.1]).

**Lemma 3** [2, Lemma 4.2]. Let $f(x) \in C^2[X_1, X_2]$ be such that $f''(x) \asymp \lambda_2$ for $X_1 \leq x \leq X_2 = X_1 + X \leq 2X_1$. Assume that $\|f'(x)\| \geq \lambda_2$. Then

$$\sum_{X_1 \leq x \leq X_2} e(f(x)) \ll X \sqrt{\lambda_2} + 1 + \min\{X; 1/\sqrt{\lambda_2}; 1/\|f'(X_2)\|; 1/\|f'(X_1)\|\}.$$ 

**Lemma 4.** Let $f(x, y)$ be a real-valued function on $\{(x, y) : Y \leq y \leq 2Y, X_1 \equiv X_1(y) \leq x \leq X_2(y) \equiv X_2\}$ such that $f(x, y)$ is a v.d.c. function
of order \( k \) as a function of \( x \) and either \( g_1(y) \equiv f^{(k-1)}(X_1, y) \) or \( g_2(y) \equiv f^{(k-1)}(X_2, y) \) is a v.d.c. function of order \( j \). Assume that

\[
\frac{\partial^k f}{\partial x^k}(x, y) \asymp \lambda_k \quad \text{and} \quad g_i(y)^{(j)} \asymp \mu_j \quad \text{for a v.d.c. function } g_i(y).
\]

Then

\[
S \equiv \sum_{Y \leq y \leq 2Y} \left| \sum_{X_1 \leq x \leq X_2} e(f(x, y)) \right| \ll XY \left( \lambda_k^{1/(K-2)} + X^{-4/(3K)} + Y^{-2/K} + \mu_j^{4/(KJ+K-8)} \right)
\]

if \( k > 1 \)

and

\[
S \ll XY(\mu_j^{1/(J-1)} + 1/Y + \log X/X) \quad \text{if } k = 1.
\]

**Proof.** If \( k = 1 \), we use van der Corput’s Lemma to get

\[
S \ll \sum_y \min\{X; 1/\|f_y(X_1, y)\| + 1/\|f_y(X_2, y)\|\}
\]

\[
\ll \sum_y \min\{X; 1/\|f_y(X_1, y)\|; 1/\|f_y(X_2, y)\|\}
\]

and proceed as below. If \( k = 2 \) then we use van der Corput’s estimates (Lemma 2 with \( j = 0 \)) to obtain

\[
S \ll XY(\sqrt{\lambda_2} + 1/\sqrt{\lambda_2}).
\]

If \( \lambda_2 \gg X^{-4/3} \), the above implies \( S \ll XY\sqrt{\lambda_2} + XY^{2/3} \).

If \( X\lambda_2 \equiv \Delta_0 \leq X^{-1/3} \), we can evaluate \( S \) differently. We define

\[
Y_i(\Delta) \equiv \Delta = |\{y \in [Y, 2Y]: \|g_i(y)\| \leq \Delta\}|.
\]

Using Lemma 3, we obtain

\[
(4) \quad S \ll XY(X\lambda_2) + XY\sqrt{\lambda_2} + \sum_r \min\{X; 1/\sqrt{\lambda_2}; 1/(2^{r}\Delta_0)\}Y(2^{r}\Delta_0).
\]

Now we need to evaluate \( Y(\Delta) \). If \( \mu_1 \) is small, we divide the interval \([Y, 2Y]\) into \( \ll Y\mu_1 + 1 \) subintervals of length \( \ll 1/\mu_1 \) such that \( \|g(y)\| \) remains constant for all \( y \) in a subinterval. Each of them contains \( \ll \Delta/\mu_1 + 1 \) integers \( y \) such that \( \|g(y)\| < \Delta \) so that

\[
Y(\Delta) \ll (Y\mu_1 + 1)(\Delta/\mu_1 + 1) \ll Y\Delta + Y\mu_1 + 1.
\]

If \( \mu_1 \) is not small but \( \mu_k \) is small for some \( k > 1 \), we use (3) with \( r = 1 \), \( M = 3/\delta \), \( m = 0 \) and \( m = M - 1 \) to obtain

\[
Y(\Delta) \leq \min_{\delta \geq \Delta} Y(\delta),
\]
where
\[ Y(\delta) \ll Y\delta + \sum_{|l|=1}^{\infty} \min\{1/M; Ml^{-2}\} \left| \sum_{y} e(lg(y)) \right| \]
\[ \ll Y\delta + \sum_{l} \min\{1/M; Ml^{-2}\} \]
\[ \times [Y(l\mu_j^{1/(J-2)} + Y^{1-2/J} \log Y + Y^{1-8/J} + 16J^{-2} \mu_j^{1/(J-2)}], \]
and \( Y(\Delta) \ll Y\Delta + Y\mu_j^{1/(J-1)} \). We substitute this into (4) to obtain
\[ (5) \quad S \ll XY \sqrt{\lambda_2} + X^2Y \lambda_2 + XY \mu_j^{1/(J-1)} + X\sqrt{Y} \]
\[ \ll XY \sqrt{\lambda_2} + X^{2/3}Y + XY \mu_j^{1/(J-1)} + X\sqrt{Y}. \]
This proves the lemma for \( k = 2 \). If \( k > 2 \), we apply H. Weyl–van der Corput inequality \( m = k - 2 \) times:
\[ \left| \frac{S}{XY} \right|^M \ll Q^{-M/2} + \frac{1}{Q^{M-1}XY} \sum_{q_1=1}^{Q} \cdots \sum_{q_m=1}^{Q} \sum_{y} e(f_1(x, y)), \]
where
\[ M = 2^m, \quad Q = \min\{\lambda_k^{-1/(2M-1)}; X^{2/M}; \mu_j^{-2/(M(J+1)-2)}\} \]
and
\[ f_1(x, y) = q_1 \ldots q_m \int_{0}^{1} \int_{0}^{1} f_{x^m}(x + t_1 q_1 + \ldots + t_m q_m, y) dt. \]
Using (5), we obtain
\[ \left| \frac{S}{XY} \right|^M \ll Q^{-M/2} + \frac{1}{Q^{M-1}XY} \]
\[ \times \sum_{q_1, \ldots, q_m} (XY \sqrt{q_1 \ldots q_m \lambda_k} + X^{2/3}Y + X\sqrt{Y} + XY(q_1 \ldots q_m \mu_j)^{1/(J-1)}) \]
\[ \ll Q^{-M/2} + \sqrt{Q^{M-1} \lambda_k} + X^{-1/3} + Y^{-1/2} + (Q^{M-1} \mu_j)^{1/(J-1)} \]
\[ \ll \lambda_k^{M/(K-2)} + X^{-1/3} + Y^{-1/2} + \mu_j^{4M/(KJ+K-8)} . \]

To prove Theorem 1, we assume first that \( G \equiv g(X) \ll X^{1/(3K)} \). We use Lemma 1 with \( r = 3 \) and \( M = \max\{X^{1/(2K)}/G; GX^{1/(4K)}\} \) to obtain
\[ S \ll \sum_{|j|=0}^{\infty} |a_j| \sum_{m=0}^{M-1} \sum_{x} e(f(x) + g(x)m/M + jh(x)) \]
\[ + \frac{XG}{M} + \sum_{j=1}^{\infty} |a_j| \sum_{x} e(jh(x)). \]
Lemma 2 with \( k = k_1 + 1 \) and \( m = 0 \) shows that the last sum is
\[
\ll X \sum_{j=1}^{\infty} \min\{1/M; M^3 j^{-4}\} \times [(j \lambda_{k_1+1})^{1/(2K_1-2)} + X^{-2/K} + (j \lambda_{k_1+1} X^{4-4/K_1})^{-1/K_1}] 
\ll X [(M \lambda_{k_1+1})^{1/(2K_1-2)} + X^{-2/K} + (M \lambda_{k_1+1} X^{4-4/K_1})^{-1/K_1}] \ll X^{1-1/(4K)}.
\]

To evaluate the first sum, for a fixed \( j \), we divide the interval \([X, 2X]\) into \( \ll \log X \) subintervals with \( |f^{(p)}(x) + j h^{(p)}(x)| \sim \lambda_p \gg |f^{(p)}(X)|X^{-\varepsilon_1} \) and one interval (which we denote with \( I \)) on which the last inequality does not hold, where \( \varepsilon_1 > 0 \) is a sufficiently small number and \( p \) is the smallest integer such that
\[
|f^{(p)}(X)|X^{1-1/P} \leq 1 \quad \text{and} \quad |j h^{(p)}(X)|X^{1-1/P} \leq 1.
\]

Obviously, \( p < k \). The conditions of the theorem imply that if \( x \in I \) then
\[
|f^{(p+1)}(x) + j h^{(p+1)}(x)| \gg |f^{(p+1)}(X)|.
\]

Using Lemma 2 with \( k = p \) and \( m = 1 \) if \( x \in I \) and \( m = 0 \) otherwise, we find that the first sum is
\[
\ll \sum_{j} \min\{1; M^2 j^{-2}\} X^{1-1/K} \log X \ll X^{1-1/(4K)}
\]
so that
\[
S \ll X^{1-1/(4K)}.
\]

Now we assume that \( G \gg X^{1/(3K)} \). We take \( \varepsilon_0 = (G')^{4/(9P)} + G^{-4/(PK)} \) and define \( a(x) = 1 - \chi(x) \) where \( G' = g'(X) \) and \( \chi(x) \) is the characteristic function of \([-\varepsilon_0, \varepsilon_0]\) modulo 1. Then
\[
S = \sum_{X \leq x \leq 2X} a(g(x)) e(f(x) + g(x)\{h(x)\}) + O\left( \sum_{X \leq x \leq 2X} \chi(g(x)) \right).
\]

The \( O \)-term is \( \ll |\{x \in [X, 2X] : \|g(x)\| \leq 3\varepsilon_0\}| \cdot \ll XG' + 1 \) subintervals of length \( \ll 1/G' \) each such that \( \|g(X)\| \) remains constant on each subinterval. The number of \( x \) in each subinterval such that \( \|g(x)\| \leq 3\varepsilon_0 \) is \( \ll 1 + \varepsilon_0/G' \) so that the \( O \)-term is
\[
\ll (XG' + 1)(1 + \varepsilon_0/G') \ll X \varepsilon_0 + XG'.
\]

Now we apply Lemma 1 with \( r = 3 \) and \( M = GX^{1/(4K)} \) to obtain
\[
S \ll \sum_{|j| = 0}^{\infty} |a_j| \left| \sum_{m=0}^{M-1} \sum_{x} a(h(x)) e(f(x) + j h(x) + g(x)m/M + mj/M) \right| \ll X \varepsilon_0 + XG' + X^{1-1/(4K)} + \sum_{j=1}^{\infty} |a_j| \left| \sum_{x} e(jh(x)) \right|.
\]
As above, the last sum is \( \ll X^{1-1/(4K)} \). Now we need to evaluate the first sum. We denote it by \( \Sigma \) and denote the sum over \( m \) and \( x \) by \( S_1 \); summing over \( m \), we obtain

\[
S \ll \sum_{x} a(g(x))e(f(x) + jh(x)) \frac{e(g(x)) - 1}{e((j + g(x))/M) - 1}.
\]

Let \( G_1 \) and \( G_2 \) be the minimum and maximum of \( g(x) \) on \([X, 2X]\).

Setting

\[
I(y) = \{ x \in [X, 2X] : y + \varepsilon_0 \leq g(x) \leq y + 1 - \varepsilon_0 \} \equiv [X_1(y), X(y)]
\]

and writing \( j = u + vM \) with \(|u| < M/2\), we obtain

\[
S_1 \ll \sum_{G_1 \leq y \leq G_2} \left| \sum_{x \in I(y)} e(f(x) + jh(x)) \frac{e(g(x)) - 1}{e((u + g(x))/M) - 1} \right|.
\]

If \( X_1(y) \leq x \leq X(y) - 1 \) then

\[
\frac{1}{e((u + g(x))/M) - 1} - \frac{1}{e((u + g(x + 1))/M) - 1} \ll \frac{MG'}{(y + u)^2 + \varepsilon_0^2}.
\]

Abel’s summation formula and the above inequalities yield

\[
S_1 \ll \sum_{y} \left\{ \frac{M}{|y + u| + \varepsilon_0} \left| \sum_{x \in I(y)} e(\psi(x)) \right| + \frac{MG'}{(y + u)^2 + \varepsilon_0^2} \sum_{X_1(y) \leq s \leq X(y)} \left| \sum_{s \leq x \leq X(y)} e(\psi(x)) \right| \right\},
\]

where \( \psi(x) = f(x) + jh(x) + ig(x) \) and \( i = 0 \) or \( 1 \). We set \( X_0 = 1/G' \). Then the second sum above is

\[
\ll MG' \sum_{s \leq X_0} \sum_{y} \frac{1}{(u + y)^2 + \varepsilon_0^2} \left| \sum_{X(y) - s \leq x \leq X(y)} e(\psi(x)) \right|
\]

so we get

\[
(6) \quad \Sigma \ll \sum_{|v|=0}^{\infty} \frac{1}{v^4 + 1} \sum_{u \leq M/2} \sum_{y} \left\{ \frac{1}{|y + u| + \varepsilon_0} \left| \sum_{x \in I(y)} e(\psi(x)) \right| + \frac{G'}{(y + u)^2 + \varepsilon_0^2} \sum_{X_1(y) \leq s \leq X(y)} \left| \sum_{s \leq x \leq X(y)} e(\psi(x)) \right| \right\}
\]

\[
\ll \max_{s \leq X_0} \sum_{|v|=0}^{V} \frac{1}{v^4 + 1} \sum_{|u| \leq M/2} \sum_{y} \frac{1}{|y + u| + \varepsilon_0^2} \left| \sum_{X(y) - s \leq x \leq X(y)} e(\psi(x)) \right| + R',
\]
where

\[ V = X^{1/(4K)} \quad \text{and} \quad R' \ll \sum_{v=V}^{\infty} v^{-4} X (\log X + \varepsilon_0^{-2}) \ll X^{1-1/(4K)}. \]

Let \( r \) be the smallest integer such that

\[ |f^{(r)}(x)| \leq X^{2/R-1} \quad \text{and} \quad |h^{(r)}(x)| \leq X^{2/R-1}. \]

Obviously, \( 1 < r < k \). To evaluate the sum in (6) we need to evaluate

\[ Y(\Delta) \equiv \{|y \in [G_1, G_2] : ||\varphi(y)| \leq \Delta\} \]

where \( \varphi(y) = A\psi^{(r-1)}(X(y)) \) and \( A \leq X_0^{1-1/R} \) is a fixed number. Assume that \( t \) is the smallest integer such that

\[ |(Af^{(r-1)}(X(y)))^{(t)}| \leq G^{2/T-1} \quad \text{and} \quad |(Ah^{(r-1)}(X(y)))^{(t)}| \leq G^{2/T-1}. \]

We take a small constant \( \varepsilon > 0 \) and divide the set of all \( y \) into \( \ll \log X \) intervals with

\[ |\varphi^{(t)}(y)| \lambda_t \geq A(|(f^{(r-1)}(X(y)))^{(t)}| + |(h^{(r-1)}(X(y)))^{(t)}|)X_0^{-\varepsilon_1} \]

and at most one interval, \( I \), in which the above inequality is not satisfied. The conditions of the theorem imply that if \( y \in I \) then

\[ |\varphi^{(t+1)}(y)| \gg A|(f^{(r-1)}(X(y)))^{(t+1)}|G^{-\varepsilon_1}. \]

Using Lemma 2 with \( k = t \) and \( m = 0 \) if \( y \notin I \) and \( m = 0 \) otherwise as above we obtain

\[ Y(\Delta) \leq \min_{\delta \geq \Delta} Y(\delta) \]

\[ \ll \min_{\delta} \left( G\delta + \sum_{j=1}^{\infty} \min\{\delta; 1/(\delta j^2)\} \left| \sum_{y} e(j\varphi(y)) \right| \right) \]

\[ \ll G \min_{\delta} \left( \delta + \sum_{j=1}^{\infty} \min\{\delta; 1/(\delta j^2)\} \right) \left( G^{-1/T} + (j\mu_t)^{1/(T-2)} \right) \]

\[ \ll G \min_{\delta} \left( \delta + G^{-1/T} + \left( \mu_t / \delta \right)^{1/(T-2)} \right) \]

\[ \leq G(\Delta + G^{-1/P} + \mu_t^{1/(T-1)}) \]

\[ \ll G(\Delta + G^{-1/P}). \]

To evaluate the sum in (6) we assume first that \( r = 2 \). We divide the interval \([X(y) - s, X(y)]\) into \( \ll \log X \) subintervals with \( \varphi''(x) \approx \lambda_2 \) and consider one of them, corresponding to the largest subsum. We denote it by \( S(u,v,y) \).

If \( \lambda_2 \geq X_0^{-4/3} \), we use Lemma 2 with \( k = 2 \) and \( m = 0 \) and obtain

\[ S(u,v,y) \ll X_0^{2/3}. \]
If $\lambda_2 \leq X_0^{-4/3}$ we use Lemma 3 to evaluate $S(u, v, y)$ if
(8) \[ \|\psi'(X(y))\| \geq CX_0\lambda_2 \equiv \Delta_0 \]
with an appropriate $C$ or evaluate it trivially otherwise.

Note that if (8) holds then for all $x \in [X(y) - s, X(y)]$ we have
\[ \|\psi'(x)\| = \|\psi'(X(y)) + O(\lambda_2 X_0)\| \approx \|\psi'(X(y))\|. \]

Summing over all $u$ and $y$ and using (7), we obtain
(9) \[ S(v) \approx \sum_{u,y} S(u, v, y) \ll (\log X + \epsilon_0^{-2}) \]
\[ \times \left[ X_0^{2/3} + \sum_l \min\{X_0; 1/((\Delta_0 2^l))\} Y(\Delta_0 2^l) + X_0 Y(\Delta_0) \right] \log X \]
\[ \ll (\log X + \epsilon_0^{-2})(GX_0^{2/3} + X_0 G^{1-1/P}) \log^2 X. \]

If $r > 2$, we apply Hölder’s inequality to get
\[ S(v) \ll \left( \sum_{u,y} \frac{1}{|y + u| + \epsilon_0^2} \right)^{1-4/R} \]
\[ \times \left( \sum_{u,y} \frac{1}{|y + u| + \epsilon_0^2} \left| \sum_x e(\psi(x)) \right|^R \log X \right)^{4/R}. \]

Now we use H. Weyl–van der Corput inequality $r - 2$ times with $Q = X_0^{4/R}$ to obtain
\[ S(v) \ll [G(\log X + \epsilon_0^{-2})]^{1-4/R} \left[ \sum_{y,u} \frac{1}{|y + u| + \epsilon_0^2} \left( X_0^{R/4} Q^{-R/8} \right. \right. \]
\[ + X_0^{R/4 - 1} Q^{1-4/R} \sum_{q_1=1}^Q \ldots \sum_{q_{r-2}=1}^Q \left| \sum_x e(A\psi_1(x)) \right| \bigg)^{4/R} \]
where
\[ A = q_1 \ldots q_{r-2} \quad \text{and} \quad \psi_1(x) = \int_0^1 \ldots \int_0^1 \psi(x + q_1 t_1 + \ldots + q_{r-2} t_{r-2}) \, dt. \]

Using (7) we obtain, as in the proof of (9),
\[ S(v) \ll \epsilon_0^{-2} X Q^{-1/2} \log^2 X + [\epsilon_0^{-2} X]^{1-4/R} \log X \]
\[ \times \left[ Q^{1-R/4} \sum_{q_1,\ldots,q_{r-2}} \epsilon_0^{-2} (X_0^{2/3} G + X_0 G^{1-1/P}) \right]^{4/R} \]
\[ \ll \epsilon_0^{-2} X \left[ X_0^{-4/(3R)} + G^{-4/(PR)} \right] \log^2 X, \]
and
\[ \sum_v S(v) \ll \varepsilon_0^{-2} X \left[ X_0^{-4/(3R)} + G^{-4/(PR)} \right] \log^2 X \ll X^{1-a/(3K)} + XG^{-1/(PK)}. \]

To prove Theorem 2 we set \( n = [g(x)] \). Let \( G_1 \) and \( G_2 \) be the minimum and maximum of \( g(x) \) on \([X, 2X]\) and \( G' = g'(X) \). Using Lemma 4, we obtain
\[ S \ll \left( \sum_{x,n} 1 \right) \left( \lambda_k^{1/(K-2)} + (G')^{-4/(3K)} + G^{-2/K} + \mu_j^{4/(KJ+K-8)} \right). \]

Since \( \sum_{x,n} 1 \ll X \) and \( G' = g_1(X)/X \gg X^{-a} \), this completes the proof.

References


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