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On simultaneous diophantine equations

by

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1. Introduction. Let a and b be distinct positive integers which are not perfect squares. In [Be1], M. A. Bennett investigated the number of integer solutions of the following simultaneous diophantine equations:

$$\begin{cases} x^2 - az^2 = u, \\ y^2 - bz^2 = v, \end{cases} \tag{1.1}$$

where u and v are nonzero integers with $av \neq bu$. He gave more precise results for the following specific equations:

$$\begin{cases} x^2 - az^2 = 1, \\ y^2 - bz^2 = 1, \end{cases} \tag{1.2}$$

which are often called *simultaneous Fermat–Pell equations*. Without loss of generality, we may consider the solvability of (1.1) and (1.2) only in positive integers. We will denote the number of positive integer solutions of (1.1) by $N_+(a, b, u, v)$.

M. A. Bennett [Be1] proved that

$$N_+(a, b, 1, 1) \leq 3,$$

and that in general

$$N_+(a, b, u, v) \ll 2^{\min\{\omega(u), \omega(v)\}}(\log |u| + \log |v|),$$

where the implied constant is absolute, $\omega(m)$ being the number of distinct prime factors of m. In [M–R], D. W. Masser and J. H. Rickert gave a specific infinite parametrized family of (a, b) with $N_+(a, b, 1, 1) \geq 2$. M. A. Bennett (see [Be1] and [Be2]) suggested that $N_+(a, b, 1, 1) \leq 1$ except for those couples (a, b) (for which he assumed a, b squarefree and $b > a \geq 2$).

In this paper, we shall investigate the number of solutions of the following simultaneous diophantine equations:

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$$(1.3)\qquad \begin{cases} x^2-(M^2+4)dz^2=M^2, \\ y^2-dz^2=1, \end{cases} \quad \text{i.e.,} \quad \begin{cases} x^2-(M^2+4)y^2=-4, \\ y^2-dz^2=1, \end{cases}$$

where M is assumed to be an *odd positive integer* and where, without loss of generality, we may assume d to be a squarefree integer.

Any natural number n can be written in the form $n_1n_2^2$, where n_1 is squarefree. We write

$$(1.4)\qquad s(n)=n_1 \quad \text{and} \quad q(n)=n_2,$$

and following [W1] we call $s(n)$ the *square class* of n.

For any positive integer M and $e\in\{1,-1\}$, we define two binary recurrence sequences $g_n(M,e)$ and $h_n(M,e)$ by

$$g_{n+2}(M,e)=Mg_{n+1}(M,e)-eg_n(M,e),$$
$$h_{n+2}(M,e)=Mh_{n+1}(M,e)-eh_n(M,e),$$

with initial terms $g_0(M,e)=2$, $g_1(M,e)=M$ and $h_0(M,e)=0$, $h_1(M,e)=1$. In the following, we will consider only the case $e=-1$ unless it is clearly stated. If there is no risk of confusion, we will simply write g_n and h_n for $g_n(M,e)$ and $h_n(M,e)$. When $M=1$ and $e=-1$, $\{h_n\}_{n\in\mathbb{N}}$ is the classical Fibonacci sequence and $\{g_n\}_{n\in\mathbb{N}}$ is the usual Lucas sequence.

The purpose of this paper is to show that (1.3) has at most one positive integer solution when $\omega(d)\le 4$. More precisely, we will prove the following theorem, in which by definition,

$$(1.5)\qquad d_k=s(h_{2k+1}^2-1).$$

In particular, the first values of d_k are the following ones:

$$\begin{aligned}
d_1&=s(M^2+2),\\
d_2&=s((M^2+1)(M^2+2)(M^2+3)),\\
d_3&=s((M^2+1)(M^2+2)(M^2+3)(M^4+4M^2+2)),\\
d_4&=s((M^2+2)(M^4+4M^2+2)(M^4+3M^2+1)(M^4+5M^2+5)),\\
d_5&=s((M^2+1)(M^2+2)(M^2+3)(M^4+3M^2+1)\\
&\qquad\times(M^4+4M^2+1)(M^4+5M^2+5)),\\
d_6&=s((M^2+1)(M^2+2)(M^2+3)(M^4+4M^2+1)\\
&\qquad\times(M^6+5M^4+6M^2+1)(M^6+7M^4+14M^2+7)).
\end{aligned}$$

THEOREM 1.1. *Let d be a positive squarefree integer with $\omega(d)\le 4$ and suppose M to be an odd positive integer.*

(i) *For $M\in\{1,3,5\}$, the simultaneous diophantine equations in* (1.3) *have no integer solution except for*

$$d = \begin{cases} 3, 6, 42, 55, 377, 1155 & \text{when } M = 1, \\ 11, 330 & \text{when } M = 3, \\ 3, 546 & \text{when } M = 5, \end{cases}$$

where in each case there is exactly one positive integer solution.

(ii) *Suppose $M \neq 1, 3, 5$ and let $\mathcal{T} = \{d_1, d_2, d_3, d_4\}$. If $d \notin \mathcal{T}$, then the simultaneous diophantine equations in* (1.3) *have no positive integer solution. If* (1.3) *has a positive solution, then there exists $k \in \{1, 2, 3, 4\}$ such that $d = d_k$ and the solution is uniquely given by*

$$x = g_{2k+1}, \quad y = h_{2k+1}, \quad z = q(h_{2k+1}^2 - 1).$$

Our results generalize those of P. G. Walsh in the following sense: P. G. Walsh [W1] considered the case where $M = 2$ in (1.3), namely,

$$\begin{cases} x^2 - 8dz^2 = 4, \\ y^2 - dz^2 = 1, \end{cases} \quad \text{i.e.,} \quad \begin{cases} w^2 - 2dz^2 = 1, \\ y^2 - dz^2 = 1. \end{cases}$$

As far as we are concerned, we deal with the case where M is any odd integer. Note that in the special case $M = 1$, Theorem 1.1 takes care of the simultaneous Fermat–Pell equations

$$\begin{cases} x^2 - 5dz^2 = 1, \\ y^2 - dz^2 = 1. \end{cases}$$

Let us mention in passing that P. G. Walsh obtained in [W2] a family of couples (a, b) for which (1.2) has only the trivial solution $(x, y, z) = (1, 1, 0)$.

Under a very strong hypothesis, we can prove the following.

Theorem 1.2. *Assume the ABC conjecture. For any positive integer M* (*odd or even*), *there exists $d(M)$, such that for any squarefree $d > d(M)$, the simultaneous diophantine equations in* (1.3) *have at most one positive integer solution. Moreover, if (x, y, z) is a positive integer solution of the above simultaneous diophantine equations with $d > d(M)$, then on the one hand, $x = g_{2k+1}$, $y = h_{2k+1}$, $z = q(h_{2k+1}^2 - 1)$ for some k, and on the other hand, the algebraic integer $y + z\sqrt{d}$, of norm $+1$, is the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{d})$.*

Corollary 1.3. *Assume the ABC conjecture. If the fundamental unit ε_d of the real quadratic field $\mathbb{Q}(\sqrt{d})$ has norm -1, then the simultaneous diophantine equations in* (1.3) *have no positive integer solution except for finitely many d.*

2. Preliminary results. For any natural numbers a and b, we will use the notation $a \sim b$ to mean $s(a) = s(b)$, i.e., ab is a square.

Let us recall the following fundamental properties of g_n and h_n (see Propositions 2.1, 2.2, 4.5, 4.6, 4.7 and 4.8 of [K–L–N]).

LEMMA 2.1. *Let M be odd. For $n \geq 1$, we have:*

(i) $(g_n, g_{n+1}) = 1$,
(ii) $(h_n, h_{n+1}) = 1$,
(iii) $(g_n, h_{n+1}) = (g_n, h_{n-1}) = \begin{cases} 1 & \text{if } n \text{ is even}, \\ M & \text{if } n \text{ is odd}, \end{cases}$
(iv) $(g_n, h_n) = (h_n, 2) = \begin{cases} 1 & \text{if } 3 \nmid n, \\ 2 & \text{if } 3 \mid n, \end{cases}$
(v) $h_{2n+1}^2 - 1 = h_{2n} h_{2n+2}$,
(vi) $h_{2n} = g_n h_n$.

LEMMA 2.2. *Let M be odd and let $r \geq 1$.*

(i) *If $M = 1$, then*

$$g_r \sim 1\ (= M) \ \Leftrightarrow\ r = 1 \text{ or } 3, \qquad g_r \sim 2\ (= 2M) \ \Leftrightarrow\ r = 6.$$

(ii) *If $M = 3$, then $g_r \not\sim 2$, $g_r \not\sim 2M$, and*

$$g_r \sim 1 \ \Leftrightarrow\ r = 3, \qquad g_r \sim M \ \Leftrightarrow\ r = 1.$$

(iii) *If $M = 5$, then $g_r \not\sim 1$, $g_r \not\sim 2M$, and*

$$g_r \sim 2 \ \Leftrightarrow\ r = 6, \qquad g_r \sim M \ \Leftrightarrow\ r = 1.$$

(iv) *Elsewhere, $g_r \not\sim 2$, $g_r \not\sim 2M$, and*

$$g_r \sim 1 \ \Leftrightarrow\ r = 1 \text{ (with } M \sim 1), \qquad g_r \sim M \ \Leftrightarrow\ r = 1.$$

LEMMA 2.3. *Let M be odd and let $r \geq 1$.*

(i) *If $M = 1$, then*

$$h_r \sim 1\ (= M) \ \Leftrightarrow\ r = 1, 2 \text{ or } 12, \qquad h_r \sim 2\ (= 2M) \ \Leftrightarrow\ r = 3 \text{ or } 6.$$

(ii) *If $M = M_0^2$ with $M_0 > 1$, then $h_r \not\sim 2$, $h_r \not\sim 2M$, and*

$$h_r \sim 1\ (\sim M) \ \Leftrightarrow\ r = 1 \text{ or } 2.$$

(iii) *If $M = \sqrt{2M_0^2 - 1} > 1$, then $h_r \not\sim 2M$, and*

$$h_r \sim 1 \ \Leftrightarrow\ r = 1, \qquad h_r \sim 2 \ \Leftrightarrow\ r = 3, \qquad h_r \sim M \ \Leftrightarrow\ r = 2.$$

(iv) *Elsewhere, $h_r \not\sim 2$, $h_r \not\sim 2M$, and*

$$h_r \sim 1 \ \Leftrightarrow\ r = 1, \qquad h_r \sim M \ \Leftrightarrow\ r = 2.$$

We want to secure the fact that for $k \geq 5$ and for $M \neq 1, 3, 5$, d_k has at least 5 distinct prime factors. For this, we will need the following result.

PROPOSITION 2.4. *Let $M = \sqrt{2M_0^2 - 1} > 1$. Then*

$$s(g_6) = s((M^2 + 2)(M^4 + 4M^2 + 1))$$

contains at least two odd primes.

Proof. Since M is odd, and since $M^4 + 4M^2 + 1 \equiv 6 \pmod{16}$, we respectively have

$$M^2 + 2 \not\sim 1 \text{ nor } 2, \qquad M^4 + 4M^2 + 1 \not\sim 1 \text{ nor } 2. \tag{2.1}$$

Moreover, since $3 \nmid M_0$, we have $M^2 + 2 = 2M_0^2 + 1 \equiv 0 \pmod 3$, so

$$(M^2 + 2, M^4 + 4M^2 + 1) = (M^2 + 2, (M^2 + 2)^2 - 3) = 3. \tag{2.2}$$

Now it is obvious that $s(g_6)$ contains at least two odd primes when $s(M^2+2)$ contains at least two odd primes p, q with $p \neq 3$ and $q \neq 3$. We can then suppose that $s(M^2 + 2)$ does not contain two odd primes p, q both strictly greater than 3. Then one can easily see that the odd integer $M^2 + 2$ must be of one of the following three forms:

$$M^2 + 2 = \begin{cases} 9py^2 & (\text{with } p \neq 3), \\ 3y^2, \\ 3py^2 & (\text{with } p \neq 3) \end{cases}$$

for some integer y.

Assume first $M^2 + 2 = 9py^2$ with $p \neq 3$. We can conclude from (2.1) and (2.2) that $s(g_6)$ contains 3 and $p \neq 3$.

Suppose next that $M^2 + 2 = 3y^2$, where by hypothesis $M^2 - 2M_0^2 = -1$. Since $M > 1$, we know that $M = g_{2k+1}(2, -1)/2$ for some $k \geq 1$, where $g_0 = 2$, $g_1 = 2$, $g_2 = 6$, $g_3 = 14$, $\ldots$, $g_{s+1} = 2g_s + g_{s-1}$, and one easily proves the following two formulas:

$$g_s - 8(-1)^s = g_{s-1}g_{s+1}, \qquad (g_s, g_{s+2}) = 2 \quad \text{for } s \geq 1.$$

We deduce that $M^2 + 2$ is the product of two coprime integers:

$$M^2 + 2 = \frac{g_{2k+1}^2 + 8}{4} = \left(\frac{g_{2k}}{2}\right)\left(\frac{g_{2k+2}}{2}\right). \tag{2.3}$$

Since $M^2 + 2 = 3y^2$, either $g_{2k}/2 \sim 1$ or $g_{2k+2}/2 \sim 1$. Writing $w^2 = g_{2k}/2$ or $g_{2k+2}/2$, we conclude that $w^4 - 2z^2 = 1$ for some integer z; this forces $w^2 + 1$ or $w^2 - 1$ to be a square, whereupon $w = 1$ and $z = 0$, which contradicts $M > 1$.

Finally, suppose $M^2 + 2 = 3py^2$ with $p \neq 3$. From (2.3), we deduce that at least one of g_{2k} and g_{2k+2} must be ~ 2 or 6. We saw above that $g_{2k} \not\sim 2$ and $g_{2k+2} \not\sim 2$, whereupon

$$\text{either} \quad \frac{g_{2k}}{2} \sim 3 \quad \text{or} \quad \frac{g_{2k+2}}{2} \sim 3.$$

So there exist positive integers A and B such that $9A^4 - 2B^2 = 1$. Using Corollary 1.3 of [B–W], we obtain $A = 1$, $Y = 2$, i.e., $M = 7$, whereupon $s(g_6) = 2 \cdot 17 \cdot 433$ contains at least two odd primes. ∎

Proposition 2.5. *Let M be an odd integer different from $1, 3, 5$ and let $k \geq 5$. Then $d_k = s(h_{2k+1}^2 - 1)$ has at least 5 distinct prime factors.*

Proof. Suppose $k = 4m + r$ for $0 \leq r \leq 3$. Then for $m \geq 1$, we have

$$h_{2k+1}^2 - 1 \sim \begin{cases} g_{4m}\left(g_{2m}\,\dfrac{h_{2m}}{M}\right)\dfrac{g_{4m+1}}{M}\,h_{4m+1} & \text{if } r = 0, \\ \dfrac{g_{4m+1}}{M}\,h_{4m+1}g_{4m+2}\left(\dfrac{g_{2m+1}}{M}\,h_{2m+1}\right) & \text{if } r = 1, \\ g_{4m+2}\left(\dfrac{g_{2m+1}}{M}\,h_{2m+1}\right)\dfrac{g_{4m+3}}{M}\,h_{4m+3} & \text{if } r = 2, \\ \dfrac{g_{4m+3}}{M}\,h_{4m+3}g_{4m+4}\left(g_{2m+2}\,\dfrac{h_{2m+2}}{M}\right) & \text{if } r = 3, \end{cases} \tag{2.4}$$

i.e., $h_{2k+1}^2 - 1 \sim A_1A_2A_3A_4A_5$, a product of five factors in the same order as in (2.4) with $(A_i, A_j) = 1$ or 2 for $1 \leq i < j \leq 5$.

Let $M \neq \sqrt{2M_0^2 - 1}$ for any M_0 and $M \neq 1, 3, 5$. Then from Lemmas 2.2 and 2.3, one can verify $A_i \not\sim 1$ nor 2 for any $1 \leq i \leq 5$ and for any $k \geq 5$. Thus we see that in this case $\omega(d_k) \geq 5$ for $k \geq 5$.

Let $M = \sqrt{2M_0^2 - 1} > 1$. Then from Lemmas 2.2 and 2.3, one can also verify $A_i \not\sim 1$ nor 2 for any $1 \leq i \leq 5$ and for any $k \geq 7$. In the cases $k = 5$ and 6, one sees

$$d_5 = s(h_{11}^2 - 1) \sim (g_5/M)h_5g_6(g_3/M)h_3,$$
$$d_6 = s(h_{13}^2 - 1) \sim (g_7/M)h_7g_6(g_3/M)h_3,$$

where $h_3 \sim 2$ and otherwise $A_i \not\sim 1$ nor 2. Moreover, from Proposition 2.4, we know that $s(g_6)$ contains at least two odd primes, which completes the proof. ∎

3. Proof of the main result. Suppose now that (x, y, z) is a positive integer solution of (1.3). One sees that $x = g_{2k+1}$ and $y = h_{2k+1}$ for some $k > 0$ and $dz^2 = h_{2k+1}^2 - 1 = h_{2k}h_{2k+2} = g_kh_kg_{k+1}h_{k+1}$ by Lemma 2.1. In order to make the proof easier to read, we will proceed in steps.

Case A. Firstly, we consider the case $M = 1$. Then the assumption $\omega(d) \leq 4$ implies that at least one of h_s or g_s which appears in the decomposition (2.4) must be ~ 1 or 2, which forces $s \leq 12$, and $2k + 1 \leq 49$. Since

$$h_3^2 - 1 = 3, \quad h_5^2 - 1 = 6 \cdot 2^2, \quad h_7^2 - 1 = 42 \cdot 2^2,$$
$$h_9^2 - 1 = 1155, \quad h_{11}^2 - 1 = 55 \cdot 12^2, \quad h_{13}^2 - 1 = 377 \cdot 12^2,$$

and since $\omega(s(h_{2k+1}^2 - 1)) \geq 5$ for all k with $7 \leq k \leq 24$, we have proved Theorem 1.1 in this case.

Case B. Next, we consider the case $M = 3$. Then from Lemmas 2.2 and 2.3, we see that one of g_s and h_s, which appear in the decomposition (2.4), must satisfy $s \le 3$ and so $2k+1 \le 13$. Since $h_3^2 - 1 = 11\cdot 3^2$, $h_5^2 - 1 = 330\cdot 6^2$ involve the only indices k with $\omega(s(h_{2k+1}^2 - 1)) \le 4$, we can conclude.

Case C. Let us treat the case $M = 5$. From Lemmas 2.2 and 2.3, one of g_s and h_s, which appear in the decomposition (2.4), must satisfy $s \le 6$ and hence $2k+1 \le 25$. Since $h_3^2 - 1 = 3 \cdot 13^2$, $h_5^2 - 1 = 546 \cdot 30^2$ involve the only indices k with $\omega(s(h_{2k+1}^2 - 1)) \le 4$, we have the required result.

Case D. Next we shall treat the case $M \neq 1, 3, 5$. From Proposition 2.5, we see that $\omega(s(d_k)) \ge 5$ for any $k \ge 5$. One can also easily verify that $d_i \neq d_j$ for any $1 \le i < j \le 4$. Hence the conclusion.

Remark 3.1. In Theorem 1.1, one can determine the exceptional d satisfying $\omega(d) \le 4$ and such that $N_+((M^2+4)d, d, M^2, 1) = 1$ for any odd integer M. For example, when $M = 7$, then the exceptional d's are $d = 51 = 3\cdot 17$ and $d = 1326 = 2 \cdot 3 \cdot 13 \cdot 17$. One can also verify that for any odd integer $1 \le M \le 1827$, the simultaneous diophantine equations of (1.3) have at least one exceptional d such that $\omega(d) \le 4$ and $N_+((M^2+4)d, d, M^2, 1) = 1$ and there is no exceptional case for $M = 1829$.

Remark 3.2. It looks like there are not so many small odd integers M for which $d \in \mathcal{T} = \{d_1, d_2, d_3, d_4\}$ with $\omega(d) \ge 5$: we verified with a computer that the only odd integers M, $1 \le M \le 5369$, such that $\omega(s(d_1)) = \omega(s(M^2+2)) \ge 5$ are the following twelve:

$$1829, 2203, 2285, 2863, 3121, 3509, 3787, 4155, 4733, 4903, 5155, 5195.$$

Moreover, we can show that there exist infinitely many M such that under the condition $\omega(d) \le 4$, the simultaneous diophantine equations in (1.3) have no integer solutions. Let p_i $(1 \le i \le 5)$ be distinct odd primes such that $\left(\frac{-2}{p_i}\right) = 1$, i.e., $p_i \equiv 1$ or $3 \pmod 8$. Choose a_i such that $a_i^2 \equiv -2 + p_i \pmod{p_i^2}$ for $1 \le i \le 5$. By the chinese remainder theorem, there exist infinitely many odd M which satisfy $M \equiv a_i \pmod{p_i^2}$ for $1 \le i \le 5$. Since $M^2 + 2$ appears in d_1, d_2, d_3, d_4 and since $s(M^2 + 2)$ contains at least the five primes $p_1, \ldots, p_5$, we have $N_+((M^2+4)d, d, M^2, 1) = 0$ for these M. If we choose, for example, $p_1 = 3$, $p_2 = 11$, $p_3 = 17$, $p_4 = 19$, $p_5 = 41$, then under the condition $\omega(d) \le 4$, $N_+((M^2+4)d, d, M^2, 1) = 0$ for any positive integer $M \equiv 139106648843 \pmod{2 \cdot 3^2 \cdot 11^2 \cdot 17^2 \cdot 19^2 \cdot 41^2}$.

4. Using the *ABC* conjecture. In this section, we shall estimate the number of integer solutions of simultaneous diophantine equations under the *ABC* conjecture as stated in [La].

ABC Conjecture. *For any $\varepsilon > 0$, there exists a constant $K(\varepsilon)$ (depending only on ε) such that if a, b, c are nonzero relatively prime integers with $a + b + c = 0$, then*

$$\max\{|a|, |b|, |c|\} \leq K(\varepsilon) R^{1+\varepsilon},$$

where $R = \operatorname{rad}(abc) = \prod_{p|abc} p$ (p denoting a prime).

Using the results of [R–W] and assuming the *ABC* conjecture, the first author proved in [Ka] that for any fixed integer M (odd or even), $s(h_{2k+1}^2 - 1) > \sqrt{h_{2k+1}^2 - 1}$ except for finitely many indices k. Write $\varepsilon_{2k+1} = h_{2k+1} + \sqrt{h_{2k+1}^2 - 1}$, $\mathbb{Q}(\sqrt{d_k}) = \mathbb{Q}(\sqrt{h_{2k+1}^2 - 1})$, where $d_k = s(h_{2k+1}^2 - 1)$, and put

$$d(M) = \max\{d_k \mid k \in \mathbb{N} \text{ and } \varepsilon_{2k+1} \text{ is not the fundamental unit of } \mathbb{Q}(\sqrt{d_k})\}.$$

Then with the same argument as in [Ka], we have the following proposition.

Proposition 4.1. *Assume the ABC conjecture. For any positive integer M (odd or even), there exists $d(M)$ such that for any k with $d_k = s(h_{2k+1}^2 - 1) > d(M)$, $h_{2k+1} + \sqrt{h_{2k+1}^2 - 1}$ is the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{h_{2k+1}^2 - 1})$.*

We note that the number of positive integer solutions of (1.3) is the number of indices k such that $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{h_{2k+1}^2 - 1})$ for a fixed d. From this proposition, we see that the real quadratic fields $\mathbb{Q}(\sqrt{h_{2k+1}^2 - 1})$ are all different except for finitely many indices k. Thus we have proved Theorem 1.2.

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