ACTA ARITHMETICA
108.4 (2003)

Zeta function and Zharkovskaya's theorem on Siegel modular forms of half-integral weight

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Introduction. The relation between the spinor L-function of Siegel modular forms of integral weight and Siegel Φ operator was studied by Zharkovskaya [5]. She showed the commutation relation between the Siegel Φ operator and the Hecke operators acting on the space of Siegel modular forms of integral weight; moreover she showed that the homomorphic map from the Hecke ring of degree n to that of degree $n-1$ is surjective; finally, she showed that the quotient part of the spinor L-function of a Siegel modular form F of degree n can be written by using the quotient part of the spinor L-function of the Siegel modular form $\Phi(F)$ of degree $n-1$ where $\Phi(F)$ is the image of F under the Siegel Φ operator. This theorem of Zharkovskaya was generalized to arbitrary levels by Andrianov [1].

The even zeta function of Siegel modular forms of half-integral weight was studied by Zhuravlev [6], [7]. Oh–Koo–Kim [3] showed the commutation relation between the Siegel Φ operator and Hecke operators acting on the space of Siegel modular forms of half-integral weight, and also that the map from a suitable Hecke ring of degree n to that of degree $n-1$ is surjective.

In this article we show the relation between even zeta functions of Siegel modular forms of half-integral weight of degree n and of degree $n-1$ (Theorem 2). Note that the case of degree $n=2$, level $q=4$ and character $\chi \equiv 1$ of our Theorem 2 has already been treated by Hayashida–Ibukiyama [2]. Our main result is deduced from the theorem of Oh–Koo–Kim [3].

Notation. We let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ have the usual meaning. Let $M_{m,n}(A)$ be the set of all $m \times n$ matrices over a commutative ring with unit A, and put $M_n(A) = M_{n,n}(A)$. For matrices $N \in M_n(A)$ and $M \in M_{n,m}(A)$, we define $N[M] = {}^{\mathrm{t}}MNM$ where ${}^{\mathrm{t}}M$ is the transpose of M. We put $M^* = {}^{\mathrm{t}}M^{-1}$. Let E_n be the identity matrix and let $\mathrm{GL}_n(A)$ be the group of invertible matrices in $M_n(A)$ and $\mathrm{SL}_n(A)$ the subgroup consisting of matrices with determinant 1. If $A \subset \mathbb{R}$, and $A_+^\times$ is the group of positive units of A, then

2000 *Mathematics Subject Classification*: Primary 11F37.

we put

$$\mathrm{GSp}_n^+(A) = \{M \in M_{2n}(A) \mid {}^{\mathrm{t}}MJ_nM = \gamma(M)J_n,\ \gamma(M) \in A_+^\times\},$$

where $J_n = \left(\begin{smallmatrix} 0 & -E_n \\ E_n & 0 \end{smallmatrix}\right)$. We denote the positive determinant matrices in $M_n(A)$ by $M_n^+(A)$. We define $\mathrm{Sp}_n(A)$ as follows:

$$\mathrm{Sp}_n(A) = \{M \in \mathrm{GSp}_n^+(A) \mid \gamma(M) = 1\}.$$

We put $e(M) = \exp(2\pi i\sigma(M))$, and $\sigma(M)$ is the trace of the matrix M. Let

$$\mathfrak{Z}_n = \{Z = X + iY \in M_n(\mathbb{C}) \mid Z = {}^{\mathrm{t}}Z,\ Y > 0\}$$

be the Siegel upper half-space of degree n. We denote the action of $\mathrm{Sp}_n(\mathbb{R})$ on $\mathfrak{Z}_n$ by

$$M\langle Z\rangle = (AZ+B)(CZ+D)^{-1} \quad \text{for } M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R}),\ Z \in \mathfrak{Z}_n.$$

For a positive integer q,

$$\Gamma_0^n(q) = \left\{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{Z}) \;\middle|\; C \equiv 0 \pmod q\right\}$$

is the congruence-subgroup of the symplectic group $\mathrm{Sp}_n(\mathbb{Z})$.

We set $(a,b) = \gcd(a,b)$ and $\langle n\rangle = n(n+1)/2$ for $a,b,n \in \mathbb{Z}$.

1. Hecke rings. The Hecke ring $\widehat{L}_p^n(\kappa)$ was introduced by Zhuravlev [6], [7] to interpret the even zeta function of Siegel modular forms of half-integral weight for general degree. The aim of this section is to describe this ring following [6], [7].

1.1. *Hecke pair and Hecke ring.* Let Γ be a group and let S be a semigroup in the multiplicative group G. (Γ, S) is called a *Hecke pair* if $\Gamma S = S\Gamma = S$ and if for any $g \in S$ the quotient sets $\Gamma\backslash\Gamma g\Gamma$ and $\Gamma g\Gamma/\Gamma$ are finite. Let $L(\Gamma,S)$ be the $\mathbb{C}$-module spanned by the left cosets (Γg), $g \in S$. By the *Hecke ring* $D(\Gamma,S)$ we mean the Γ-invariant submodule of $L(\Gamma,S)$ consisting of $X = \sum_i a_i(\Gamma g_i)$ such that $X\cdot\gamma = X$ for any $\gamma \in \Gamma$, where $X\cdot\gamma = \sum_i a_i(\Gamma g_i\gamma)$. For $X = \sum_i a_i(\Gamma g_i)$ and $Y = \sum_j b_j(\Gamma h_j)$ in $D(\Gamma,S)$, we define $X\cdot Y = \sum_{i,j} a_ib_j(\Gamma g_ih_j)$. Then $D(\Gamma,S)$ is an associative ring with generators $(\Gamma g\Gamma) = \sum_i(\Gamma g_i)$, where $g \in S$ and $\Gamma g\Gamma = \bigcup_i \Gamma g_i$ is the left coset decomposition.

We define subgroups $\Gamma_0^n(q)$ and Γ_0^n of $\mathrm{Sp}(n,\mathbb{Z})$ as follows:

$$\Gamma_0^n(q) = \left\{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n,\mathbb{Z}) \;\middle|\; C \equiv 0 \pmod q\right\},$$
$$\Gamma_0^n = \left\{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n,\mathbb{Z}) \;\middle|\; C = 0\right\}.$$

We set $\mathbb{Z}[p^{-1}] = \{a/p^r \in \mathbb{Q} \mid a, r \in \mathbb{Z}\}$. We define multiplicative sets S_p^n, $S_{p^2}^n$ and $S_{0,p}$ in $\mathrm{GSp}_n^+(\mathbb{Z}[p^{-1}])$ as follows:

$$S_p^n = \left\{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GSp}_n^+(\mathbb{Z}[p^{-1}]) \;\middle|\; C \equiv 0 \pmod q,\ \gamma(M) = p^\delta\right\},$$

$$S^n_{p^2} = \bigl\{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GSp}^+_n(\mathbb{Z}[p^{-1}]) \bigm| C \equiv 0 \pmod q,\ \gamma(M) = p^{2\delta}\bigr\},$$
$$S^n_{0,p} = \bigl\{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in S^n_p \bigm| C = 0\bigr\}.$$

We put

$$\Lambda_n = \mathrm{SL}(n,\mathbb{Z}) \quad \text{and} \quad G^n_p = \{D \in M_n(\mathbb{Z}) \mid \det D = p^\delta,\ \delta = 0,1,\ldots\}.$$

It is known that $(\Gamma^n_0(q), S^n_p)$, $(\Gamma^n_0(q), S^n_{p^2})$, $(\Gamma^n_0, S^n_{0,p})$ and (Λ_n, G^n_p) are Hecke pairs. We denote the corresponding Hecke rings by $L^n_p(q) = D(\Gamma^n_0(q), S^n_p)$, $L^n_{p^2}(q) = D(\Gamma^n_0(q), S^n_{p^2})$, $L^n_{0,p} = D(\Gamma^n_0, S^n_{0,p})$ and $H^n_p = D(\Lambda_n, G^n_p)$.

1.2. *Universal covering group.* The universal covering group $\mathfrak{G}$ for $\mathrm{GSp}^+_n(\mathbb{R})$ consists of the pairs $(M, \varphi(Z))$, where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is in $\mathrm{GSp}^+_n(\mathbb{R})$, $\varphi(Z)$ is holomorphic on $\mathfrak{Z}_n$ and $|\varphi(Z)|^2 = \det M^{-1/2}|\det(CZ+D)|$, with the group operation

$$(M, \varphi(Z)) \cdot (L, \psi(Z)) = (ML, \varphi(L\langle Z\rangle)\psi(Z)).$$

We define the standard theta series and a function j by

$$\Theta^n(Z) = \sum_{m \in \mathbb{Z}^n} e(Z[m]) \qquad (Z \in \mathfrak{Z}_n),$$
$$j(M,Z) = \frac{\Theta^n(M\langle Z\rangle)}{\Theta^n(Z)} \qquad (M \in \Gamma^n_0(4),\ Z \in \mathfrak{Z}_n).$$

We define an injective homomorphism $j : \Gamma^n_0(q) \to \mathfrak{G}$ by setting $j(M) = (M, j(M,Z))$. We set $\widehat{\Gamma}^n_0(q) = j(\Gamma^n_0(q))$, $\widehat{\Gamma}^n_0 = j(\Gamma^n_0)$; these are subgroups of $\mathfrak{G}$.

We define the projection $P : \mathfrak{G} \ni (M, \varphi(Z)) \mapsto M \in \mathrm{GSp}^+_n(\mathbb{R})$ and we put $\widehat{S}^n_p = P^{-1}(S^n_p)$, $\widehat{S}^n_{p^2} = P^{-1}(S^n_{p^2})$, $\widehat{S}^n_{0,p} = P^{-1}(S^n_{0,p})$. It is known that $(\widehat{\Gamma}^n_0(q), \widehat{S}^n_p)$, $(\widehat{\Gamma}^n_0(q), \widehat{S}^n_{p^2})$ and $(\widehat{\Gamma}^n_0, \widehat{S}^n_{0,p})$ are also Hecke pairs. The corresponding Hecke rings are denoted by $\widehat{L}^n_p(q) = D(\widehat{\Gamma}^n_0(q), \widehat{S}^n_p)$, $\widehat{L}^n_{p^2}(q) = D(\widehat{\Gamma}^n_0(q), \widehat{S}^n_{p^2})$ and $\widehat{L}^n_{0,p} = D(\widehat{\Gamma}^n_0, \widehat{S}^n_{0,p})$.

1.3. *The reduction of the Hecke ring.* We define a homomorphism $\widehat{\varepsilon}_{q,0} : \widehat{L}^n_p(q) \to \widehat{L}^n_{0,p}$, as follows: for any $\xi \in \widehat{S}^n_p$, there exist $\gamma = \widehat{\Gamma}^n_0(q)$ and $\xi_0 \in \widehat{S}^n_0$ such that $\xi = \gamma\xi_0$; then we define

$$\widehat{\varepsilon}_{q,0}(\widehat{\Gamma}^n_0(q)\xi\widehat{\Gamma}^n_0(q)) = \widehat{\Gamma}^n_0\xi_0\widehat{\Gamma}^n_0.$$

For an odd integer $2k-1$, we define a homomorphism $P_{2k-1} : \widehat{L}^n_{0,p} \to L^n_{0,p}$ by

$$P_{2k-1}(\widehat{\Gamma}^n_0\xi_0\widehat{\Gamma}^n_0) = \left(\frac{\varphi(Z)}{|\varphi(Z)|}\right)^{-2k+1}(\Gamma^n_0 M \Gamma^n_0),$$

where $\xi_0 = (M, \varphi(Z)) \in \widehat{S}^n_{0,p}$ and the function $\varphi(Z)|\varphi(Z)|^{-1}$ does not depend on the choice of ξ_0 because of the definition of $\widehat{\Gamma}_0^n$; moreover $\varphi(Z)|\varphi(Z)|^{-1}$ is a constant function of $Z \in \mathfrak{Z}_n$ because of the definition of $\widehat{S}_{0,p}^n$.

A surjective homomorphism $\Omega_n : L_{0,p}^n \to H_p^n[t^{\pm 1}]$, where t is transcendental over H_p^n, is defined as follows: for $X \in L_{0,p}^n$ written in the form $X = \sum_i a_i\big(\Gamma_0^n\big(\begin{smallmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{smallmatrix}\big)\big)$, we set

$$\Omega_n(X) = \sum_i a_i t^{\delta_i}(\Lambda_n D_i).$$

Let $x_0, \ldots, x_n$ be algebraically independent over $\mathbb{C}$, let $h = \sum_i a_i t^{\delta_i}(\Lambda_n D_i)$ be in $H_p^n[t^{\pm 1}]$, and suppose that for D_i we take upper triangular matrices with diagonal elements $p^{d_{i1}}, \ldots, p^{d_{in}}$. Then we define an injective homomorphism $\varphi : H_p^n[t^{\pm 1}] \to \mathbb{C}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$ by setting

$$\varphi(h) = \sum_i a_i x_0^{\delta_i} \prod_{j=1}^{n} (x_j p^{-j})^{d_{ij}}.$$

Altogether, the above maps are as follows:

$$\widehat{L}_p^n(q) \xrightarrow{\widehat{\varepsilon}_{q,0}} \widehat{L}_{0,p}^n \xrightarrow{P_{2k-1}} L_{0,p}^n \xrightarrow{\Omega_n} H_p^n[t^{\pm 1}] \xrightarrow{\varphi} \mathbb{C}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}].$$

1.4. *The Hecke ring* $\widehat{L}_p^n(\kappa)$. We consider the commutative subring $\widehat{L}_p^n(\kappa)$ ($\subset \widehat{L}_{p^2}^n(q)$) which is generated over $\mathbb{C}$ by $\widehat{T}(K_0), \ldots, \widehat{T}(K_{n-1}), \widehat{T}(K_n)^{\pm 1}$, where $\widehat{T}(K_s) = (\widehat{\Gamma}_0^n(q)\widehat{K}_s\widehat{\Gamma}_0^n(q))$, $K_s = \operatorname{diag}(E_{n-s}, pE_s; p^2 E_{n-s}, pE_s)$ and $\widehat{K}_s = (K_s, p^{(n-s)/2})$ are the corresponding elements of $\mathfrak{G}$. We define $\mathbf{L}_p^n(\kappa) = P_{2k-1}\widehat{\varepsilon}_{q,0}(\widehat{L}_p^n(\kappa))$. Let $\mathbb{C}^{W_2}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the ring of W_2-invariant polynomials, where W_2 is the automorphism group generated by the permutations of $x_1, \ldots, x_n$, the transformations $x_0 \mapsto x_0 x_i$, $x_i \mapsto x_i^{-1}$, $x_j \mapsto x_j$ ($j \neq 0, i; i = 1, \ldots, n$), and the transformation $x_0 \mapsto -x_0$, $x_i \mapsto x_i$ ($i \neq 0$). Then the homomorphism $\varphi \circ \Omega_n$ gives an isomorphism of $\mathbf{L}_p^n(\kappa)$ with the polynomial ring $\mathbb{C}^{W_2}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$ (see Zhuravlev [7]).

Namely, there exist isomorphisms

$$\widehat{L}_p^n(\kappa) \simeq \mathbf{L}_p^n(\kappa) \simeq \mathbb{C}^{W_2}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]. \tag{1}$$

2. Siegel modular forms of half-integral weight and Hecke operators. In the theory of modular forms, the Hecke rings have a representation on the space of modular forms, and this representation is important when considering the multiplicative property of Fourier coefficients of modular forms. In this section we describe the representation of Hecke rings on the space of Siegel modular forms of half-integral weight according to Zhuravlev [7].

2.1. *Siegel modular forms of half-integral weight.* Let k be an integer, let χ be a Dirichlet character modulo q, and let $4\,|\,q$. Then a holomorphic function $F(Z)$ on $\mathfrak{Z}_n$ is said to be a *Siegel modular form* of weight $k-1/2$ and character $\chi \in \Gamma_0^n(q)$ if

$$F(M\langle Z\rangle) = \chi(\det D) j(M,Z)^{2k-1} F(Z) \quad \text{for any } M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^n(q),$$

and in the case $n=1$ the function $F(Z)$ is holomorphic at all cusps of $\Gamma_0^1(q)$. We denote the set of such functions by $\mathfrak{M}_{k-1/2}^n(q,\chi)$. If $n=0$ then we set $\mathfrak{M}_{k-1/2}^0(q,\chi) = \mathbb{C}$ for $k>0$. Siegel modular forms have a Fourier expansion

$$F(Z) = \sum_{N \in \mathfrak{N}_n} f(N) e(NZ),$$

where $\mathfrak{N}_n$ is the set of symmetric positive semi-definite half-integral matrices of size n. From the definition of $\mathfrak{M}_{k-1/2}^n(q,\chi)$ it follows that $f(N[U]) = f(N)$ for $U \in \mathrm{SL}_n(\mathbb{Z})$.

For any function $F(Z)$ on $\mathfrak{Z}_n$ and for $\xi = (M, \varphi(Z)) \in \widehat{\Gamma}_0^n(q)$ we set

$$F|_{k-1/2,\chi}\xi = \gamma(M)^{n(2k-1)/4-\langle n\rangle} \chi(\det A) \varphi(Z)^{-2k+1} F(M\langle Z\rangle).$$

It follows from the definition that $F|_{k-1/2,\chi}\xi_1|_{k-1/2,\chi}\xi_2 = F|_{k-1/2,\chi}\xi_1\xi_2$, and if $F \in \mathfrak{M}_{k-1/2}^n(q,\chi)$, then $F|_{k-1/2,\chi}\xi = F$ for any $\xi \in \widehat{\Gamma}_0^n(q)$.

2.2. *Representations of Hecke rings on Siegel modular forms of half-integral weight.* For $F \in \mathfrak{M}_{k-1/2}^n(q,\chi)$, we define a representation of the Hecke ring $\widehat{L}_{p^2}^n(q)$ by setting

$$F|_{k-1/2,\chi}\widehat{X} = \sum_i a_i F|_{k-1/2,\chi}\widehat{M_i}$$

where $\widehat{X} = \sum_i a_i(\widehat{\Gamma}_0^n(q)\widehat{M_i}) \in \widehat{L}_{p^2}^n(q)$. We define a representation of the Hecke ring $L_{0,p}^n$ by setting

$$F|_{k-1/2,\chi}X = \sum_j b_j F|_{k-1/2,\chi}M_j$$

where $X = \sum_j b_j(\Gamma_0^n M_j) \in L_{0,p}^n$, and where

$$F|_{k-1/2,\chi}M = F|_{k-1/2,\chi}\widehat{M} \quad \text{and} \quad \widehat{M} = (M, \gamma(M)^{-n/4}|\det D|^{1/2})$$

for $M = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in \mathrm{GSp}_n^+(\mathbb{Q})$.

The following equation was shown by Zhuravlev [7]: for $F \in \mathfrak{M}_{k-1/2}^n(q,\chi)$ and $\widehat{X} \in \widehat{L}_{p^2}^n(q)$, we have

$$F|_{k-1/2,\chi}\widehat{X} = F|_{k-1/2,\chi}P_{2k-1}\widehat{\varepsilon}_{q,0}(\widehat{X}). \tag{2}$$

By virtue of this equation, we can consider the action of the Hecke ring

$\widehat{L}^n_{p^2}(q)$ on the Siegel modular forms $\mathfrak{M}^n_{k-1/2}(q,\chi)$ as the action of the corresponding Hecke ring in $L^n_{0,p}$.

3. The $\varPsi$ operator and Siegel $\varPhi$-operator. The $\varPsi$ operator was introduced by Andrianov [1] to generalize the theorem of Zharkovskaya [5] to arbitrary levels. It was also considered for Siegel modular forms of half-integral weight by Oh–Koo–Kim [3]. In this section we recall this operator and the theorem of [3].

Let $X = \sum_i a_i(\varGamma^n_0 g_i) \in L^n_{0,p}$ where $g_i = \left(\begin{smallmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{smallmatrix}\right)$. We can take D_i upper triangular and set $D_i = \left(\begin{smallmatrix} D_i' & * \\ 0 & p^{d_i} \end{smallmatrix}\right)$, where D_i' is also upper triangular. We define a homomorphism $\varPsi(X,u) : L^n_{0,p} \to L^{n-1}_{0,p}[u^{\pm 1}]$ by

$$\varPsi(X,u) = \sum_i a_i u^{-\delta_i}(up^{-n})^{d_i}(\varGamma^{n-1}_0 g_i'),$$

where u is an independent variable, $g_i' = \left(\begin{smallmatrix} p^{\delta_i} D_i'^* & B_i' \\ 0 & D_i' \end{smallmatrix}\right)$ and B_i' denotes the block of size $n-1$ in the upper left corner of B_i. If $n=1$, we set $\varPsi(X,u) = \sum_i a_i u^{-\delta_i}(up^{-1})^{d_i}$.

We define a $\mathbb{C}$-linear homomorphism

$$\eta_{n,u} : \mathbb{C}[x_0^{\pm 1}, \dots, x_n^{\pm 1}] \to \mathbb{C}[x_0^{\pm 1}, \dots, x_{n-1}^{\pm 1}, u^{\pm 1}]$$

by the following condition:

$$\eta_{n,u}(x_0) = x_0 u^{-1}, \qquad \eta_{n,u}(x_n) = u, \qquad \eta_{n,u}(x_i) = x_i \qquad (i = 1, \dots, n-1);$$

then the following diagram is commutative:

$$\begin{array}{ccc} L^n_{0,p} & \xrightarrow{\varphi\circ\varOmega_n} & \mathbb{C}[x_0^{\pm 1}, \dots, x_n^{\pm 1}] \\ {\scriptstyle \varPsi(-,u)}\downarrow & & \downarrow{\scriptstyle \eta_{n,u}} \\ L^{n-1}_{0,p}[u^{\pm 1}] & \xrightarrow{\varphi\circ\varOmega_{n-1}\times 1} & \mathbb{C}[x_0^{\pm 1}, \dots, x_{n-1}^{\pm 1}, u^{\pm 1}] \end{array} \tag{3}$$

where $\varphi \circ \varOmega_{n-1} \times 1$ is the ring homomorphism defined by $\varphi \circ \varOmega_{n-1} \times 1 = \varphi \circ \varOmega_{n-1}$ on $L^{n-1}_{0,p}$, $(\varphi \circ \varOmega_{n-1} \times 1)(u^{\pm 1}) = u^{\pm 1}$.

Let $F \in \mathfrak{M}^n_{k-1/2}(q,\chi)$. We define $\varPhi : \mathfrak{M}^n_{k-1/2}(q,\chi) \to \mathfrak{M}^{n-1}_{k-1/2}(q,\chi)$ by

$$\varPhi(F)(Z) = \lim_{\lambda\to\infty} F\begin{pmatrix} Z & 0 \\ 0 & i\lambda \end{pmatrix}, \qquad Z \in \mathfrak{H}_{n-1}.$$

This $\varPhi$ is called the *Siegel $\varPhi$-operator*.

The following theorem was shown by Oh–Koo–Kim [3].

THEOREM 1 (Oh–Koo–Kim). *Let $F \in \mathfrak{M}^n_{k-1/2}(q,\chi)$ and $\widehat{X} \in \widehat{L}^n_{p^2}(q)$. Then*

$$\varPhi(F|_{k-1/2,\chi}\widehat{X}) = \varPhi(F)|_{k-1/2,\chi}\varPsi(X, p^{n-(k-1/2)}\chi(p)^{-1}), \tag{4}$$

where $X = P_{2k-1}\widehat{\varepsilon}_{q,0}(\widehat{X}) \in L^n_{0,p^2}$. (*If* $n = 1$, *then the right hand side of the above equation is the action of* $L^0_{0,p^2} = \mathbb{C}$ *on* $\mathfrak{M}^0_{k-1/2}(q,\chi) = \mathbb{C}$, *which is just the multiplication of complex numbers.*) *Moreover*, *the map* $\Psi(*, p^{n-(k-1/2)}\chi(p)^{-1}) : \mathbf{L}^n_p(\kappa) \to \mathbf{L}^{n-1}_p(\kappa)$ *is a surjective ring homomorphism. If* F *is an eigenfunction for the action of* $\mathbf{L}^n_p(\kappa)$ *and if* $\Phi(F)$ *is not the zero function then* $\Phi(F)$ *is also an eigenfunction for the action of* $\mathbf{L}^{n-1}_p(\kappa)$.

4. The even zeta function of Siegel modular forms of half-integral weight. The even zeta function of Siegel modular forms of half-integral weight was studied by Zhuravlev [7]; he generalized a theorem on the degree 1 case of Shimura [4] to the general degree. In this section we recall the result of [7].

Let $\gamma(z)$ be the polynomial defined by

$$\gamma(z) = \prod_{i=1}^{n}(1 - x_i z)(1 - x_i^{-1} z).$$

The right hand side of this equation has expansion

$$\gamma(z) = \sum_{i=0}^{2n}(-1)^i R^n_i z^i,$$

where $R^n_i \in \mathbb{C}^{W_2}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$. Because $\varphi \circ \Omega_n$ is an isomorphism (see (1)), there exists a Hecke operator $R^n_{i,p} \in \mathbf{L}^n_p(\kappa)$ such that $\varphi \circ \Omega_n(R^n_{i,p}) = R^n_i$.

Let $F(Z) = \sum f(M)e(MZ) \in \mathfrak{M}^n_{k-1/2}(q,\chi)$ be an eigenfunction for the action of $\mathbf{L}^n_p(\kappa)$. We denote the eigenvalues of F for the Hecke operators $R^n_{i,p}$ by $\lambda_F(R^n_{i,p})$. Since $R^n_{2n-i} = R^n_i$ and $\lambda_F(R^n_{2n,p}) = 1$, we define the p-*parameters* $\{\alpha^{\pm 1}_{i,p}\}$ of F as follows:

$$\prod_{i=1}^{n}(1 - \alpha_{i,p} z)(1 - \alpha^{-1}_{i,p} z) = \sum_{i=0}^{2n}(-1)^i \lambda_F(R^n_{i,p}) z^i.$$

Now we describe the result of Zhuravlev [7]. Let λ be a completely multiplicative function which grows no faster than some power of the argument, and let N be a positive definite matrix in $\mathfrak{N}_n$. When the real part of s is sufficiently large, the following series, called the *even zeta function*, has Euler expansion:

$$\sum_{\substack{M \in \mathrm{SL}_n(\mathbb{Z})\backslash M^+_n(\mathbb{Z}) \\ (\det M, q)=1}} \frac{\lambda(\det M) f(N[{}^{\mathrm{t}}M])}{(\det M)^{s+k-3/2}} = \prod_{p\,\mathrm{prime}} \frac{P_{F,p}(N,\lambda,p^{-s})}{Q_{F,p}(\lambda,p^{-s})}, \tag{5}$$

where $P_{F,p}(N,\lambda,z)$ is a polynomial of z of degree at most $2n$ (the explicit form of this polynomial, given in [7], is not needed here) and $Q_{F,p}(\lambda,z)$ is a polynomial of z of degree $2n$. In particular $Q_{F,p}(\lambda, z)$ does not depend on the choice of N. The polynomial $Q_{F,p}(\lambda, z)$ was defined as follows:

$$(6)\qquad Q_{F,p}(\lambda, z) = \prod_{i=0}^{n}(1-\alpha_{i,p}\chi(p)\lambda(p)z)(1-\alpha_{i,p}^{-1}\chi(p)\lambda(p)z),$$

where $\alpha_{i,p}^{\pm 1}$ are the p-parameters of F.

5. Main theorem. Let F be a Siegel modular form of weight $k-1/2$ belonging to $\Gamma_0^n(q)$, where $q>0$ is an integer divisible by 4. We assume that F is an eigenfunction for the action of $\widehat{L}_p^n(\kappa)$ (§1.4). Let λ be a completely multiplicative function which grows no faster than some power of the argument.

We put $L(s,\lambda,F) = \prod_{(p,q)=1} Q_{F,p}(\lambda, p^{-s+k-3/2})^{-1}$ (see (5), (6)).

Then we obtain the following theorem, an analogy of the theorem of Zharkovskaya [5].

THEOREM 2. *If $\Phi(F)\neq 0$, then*

$$L(s,\lambda,F) = L_1(s-n+1,\lambda,E_{2k-2n,\chi^2})L(s,\lambda,\Phi(F)),$$

where

$$L_1(s,\lambda,E_{2k-2n,\chi^2}) = \prod_{p,(p,q)=1}(1-\lambda(p)p^{-s})^{-1}(1-\lambda(p)\chi(p)^2p^{2k-2n-1-s})^{-1}.$$

If $k>n+1$ then $L_1(s,\lambda,E_{2k-2n,\chi^2})$ is the L-function of the Eisenstein series of degree 1 of weight $2k-2n$ with character χ^2 twisted by λ.

Proof. We define $R_p^n(z)\in \mathbf{L}_p^n(\kappa)[z]$ by

$$R_p^n(z) = \sum_{i=0}^{2n}(-1)^i R_{i,p}^n z^i,$$

where $R_{i,p}^n$ are the elements of $\mathbf{L}_p^n(\kappa)$ defined in Section 4. By using (3) and (4), we have

$$\begin{aligned}\Phi(F|_{k-1/2,\chi}R_p^n(z)) &= \Phi(F)|_{k-1/2,\chi}\Psi(R_p^n(z), p^{n-(k-1/2)}\chi(p)^{-1})\\ &= (1-p^{n-(k-1/2)}\chi(p)^{-1}z)(1-p^{(k-1/2)-n}\chi(p)z)\\ &\quad\times(\Phi(F)|_{k-1/2,\chi}R_p^{n-1}(z));\end{aligned}$$

moreover,

$$\Phi(F|_{k-1/2,\chi}R_p^n(z)) = \Big(\prod_{i=1}^{n}(1-\alpha_{i,p}^{-1}z)(1-\alpha_{i,p}z)\Big)\Phi(F),$$

where $\alpha_{i,p}^{\pm}$ are the p-parameters of F. From the above we can take $\alpha_{n,p}^{\pm 1} = (\chi(p)^{-1}p^{n-(k-1/2)})^{\pm 1}$, and we can regard $\alpha_{i,p}^{\pm 1}$ $(i=1,\ldots,n-1)$ as the p-parameters of $\Phi(F)$.

We have

$$Q_{F,p}(\lambda, p^{-s+k-3/2}) = (1 - \lambda(p)p^{-s+n-1})(1 - \lambda(p)\chi(p)^2 p^{-s+2k-n-2}) \times Q_{\Phi(F),p}(\lambda, p^{-s+k-3/2}).$$

Consequently, we have proved Theorem 2.

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Received on 1.3.2002 (4240)