1. Introduction. For a positive integer \( e \geq 2 \), the Jacobi sums of order \( e \) are algebraic integers in the cyclotomic field \( \mathbb{Q}(\zeta_e) \), where \( \zeta_e = \exp(2\pi i/e) \). They are defined in terms of a finite field \( \mathbb{F}_q \) with \( q = p^r \) where \( q \equiv 1 \pmod{e} \), \( p \) prime. (See Section 2.) Jacobi sums are important objects in the theory of cyclotomy and their congruences have been studied by many authors. Earlier authors (e.g. [4]) obtained congruences for Jacobi sums defined in terms of \( \mathbb{F}_p \), \( p \equiv 1 \pmod{e} \), and later authors (e.g. [7]) considered \( q \equiv 1 \pmod{e} \).

(1) It is well known (see [4], [12]) that for Jacobi sums of odd prime order \( l \),
\[
J(1,j)_l \equiv -1 \pmod{(1 - \zeta_l)^2}.
\]
This congruence also holds modulo \((1 - \zeta_l)^3\). (See [9], [13].)

(2) Congruences for Jacobi sums of order \( 2l \) (\( l \) odd prime) were obtained by V. V. Acharya and S. A. Katre [4]. They showed that
\[
J(1,n)_{2l} \equiv -\zeta_2^{m(n+1)} \pmod{(1 - \zeta_l)^2},
\]
where \( n \) is an odd integer such that \( 1 \leq n \leq 2l - 3 \) and \( m = \text{ind} 2 \).

(3) A congruence for the Jacobi sum \( J(1,1)_9 \) of order 9 was obtained by S. A. Katre and A. R. Rajwade [10]. They showed that
\[
J(1,1)_9 \equiv -1 - (\text{ind} 3)(1 - \omega) \pmod{(1 - \zeta_9)^4},
\]
where \( \omega = \zeta_3^3 \).

(4) If \( k \) is an odd prime power \( > 3 \), then (see [8])
\[
J(i,j)_k \equiv -1 \pmod{(1 - \zeta_k)^3}.
\]
R. J. Evans [7] generalised this result to all \( k > 2 \) by elementary methods, getting sharper congruences in some cases, especially when \( k > 8 \) is a power of 2.

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It may be noted that an element $\alpha$ coprime to $l$ in the cyclotomic ring $\mathbb{Z}[\zeta_l]$, $l$ prime, can be uniquely determined if we know its prime ideal decomposition, absolute value and congruence modulo $(1 - \zeta_l)^2$. To determine an element in the ring $\mathbb{Z}[\zeta_{l^2}]$ which is coprime to $l$, the congruence is required modulo $(1 - \zeta_{l^2})^{l+1}$. In this sense, the congruences in (1), (2) and (3) above are appropriate congruences which determine the Jacobi sums.

In this paper (see Section 5) for $q = p^r \equiv 1 \pmod{l^2}$, $l > 3$ and $p$ primes, we obtain congruences for Jacobi sums of order $l^2$ modulo $(1 - \zeta_l)^{l+1}$ in terms of cyclotomic numbers of order $l$. These are the determining congruences for Jacobi sums of order $l^2$ and they sharpen the congruences in (4). In Section 6, we obtain cyclotomic numbers of order $l^2$ in terms of coefficients of Jacobi sums of order $l$ and $l^2$.

2. Preliminaries. Let $e$ be a positive integer $\geq 2$ and $q = p^r \equiv 1 \pmod{e}$, $p$ prime. Let $\mathbb{F}_q$ be a finite field with $q$ elements. Write $p^r = q = ef + 1$. Let $\zeta$ be a complex primitive $e$th root of unity. If $\gamma$ is a generator of $\mathbb{F}_q^*$ then define the multiplicative character $\chi : \mathbb{F}_q \to \mathbb{Q}(\zeta)$ by $\chi(\gamma) = \zeta$, $\chi(0) = 0$. Given a generator $\gamma$ of $\mathbb{F}_q^*$ define the Jacobi sum by

$$J(i, j) = J(i, j)_e = \sum_{v \in \mathbb{F}_q} \chi^i(v)\chi^j(1 + v), \quad 0 \leq i, j \leq e - 1.$$ 

Here $\chi^0(0) = 0$. Also, $i$ and $j$ can be considered modulo $e$, with the understanding that $\chi^i(0) = 0$ for any integer $i$. Note that $J(i, j)_e \in \mathbb{Z}[\zeta]$, the ring of integers of $\mathbb{Q}(\zeta)$.

A variation of the Jacobi sum is defined as

$$J(\chi^i, \chi^j)_e = \sum_{v \in \mathbb{F}_q} \chi^i(v)\chi^j(1 - v), \quad 0 \leq i, j \leq e - 1.$$ 

Observe that $J(i, j)_e = \chi^i(-1)J(\chi^i, \chi^j)_e$. When $q = 2^r$, $\chi^i(-1) = \chi^i(1) = 1$ and both the Jacobi sums coincide. Otherwise $\chi^i(-1) = (-1)^{ij}$ and hence the two Jacobi sums differ at most in sign. For multiplicative characters $\chi$ and $\psi$ on $\mathbb{F}_q$, $J(\chi, \psi)$ can be analogously defined. The prime ideal decomposition of Jacobi sums is well-known. See [3] p. 346, Corollary 11.2.4] for details.

In the following theorem we state some standard results about Jacobi sums.

**Theorem 2.1 (Elementary properties of Jacobi sums).**

1. If $i$ and $j$ are congruent to 0 modulo $e$ then $J(\chi^i, \chi^j)_e = q - 2$.
2. If exactly one of $i$ and $j$ is congruent to 0 modulo $e$ then $J(\chi^i, \chi^j)_e = -1$. 
(3) If $i$ is nonzero modulo $e$ and $i+j$ is congruent to 0 modulo $e$ then
\[ J(\chi^i, \chi^j)_e = -\chi^i(-1). \]
(4) $J(\chi^i, \chi^j)_e = J(\chi^j, \chi^i)_e = \chi^i(-1)J(\chi^{-i-j}, \chi^i)_e$.
(5) If $e$ divides neither $i$, $j$ nor $i+j$ then $|J(\chi^i, \chi^j)_e| = \sqrt{q}$.


**Remark.** If $f$ is even or $q = 2^r$ then $J(i,j)_e = J(\chi^i, \chi^j)_e$, so (4) gives $J(i,j)_e = J(j,i)_e = J(-i-j,j)_e = J(j,-i-j)_e = J(-i-j,i)_e = J(i,-i-j)_e$. In particular $J(i,i)_e = J(-2i,i)_e = J(i,-2i)_e$.

**3. Cyclotomy.** Let $\gamma$, $\zeta$ and $\chi$ be as in Section 2. For $0 \leq i, j \leq e - 1$ $(i, j \mod e)$, define the $e^2$ cyclotomic numbers $(i,j)_e$ by $(i,j)_e = \text{Card}(X_{ij})$ where
\[ X_{ij} = \{ v \in \mathbb{F}_q \mid \chi(v) = \zeta^i, \chi(v+1) = \zeta^j \} = \{ v \in \mathbb{F}_q - \{0,-1\} \mid \text{ind}_\gamma v \equiv i \mod e, \text{ind}_\gamma (v+1) \equiv j \mod e \}. \]

We state some basic properties of the cyclotomic numbers. (See [5] for $q = p$, and [14].) For $q = p^r$,
\[
(i,j)_e = (i',j')_e \quad \text{if } i \equiv i' \text{ and } j \equiv j' \mod e.
\]
\[
(i,j)_e = (e-i,j-i)_e = \begin{cases} (j,i)_e & \text{if } f \text{ is even or } q = 2^r, \\ (j+e/2,i+e/2)_e & \text{otherwise}. \end{cases}
\]
Thus if $f$ is even or $q = 2^r$ with $r \geq 2$ then
\[
(i,j)_e = (j,i)_e = (i-j,-j)_e = (j-i,-i)_e = (-i,j-i)_e = (-j,i-j)_e.
\]

For $e$ odd $> 3$, the equation (3.1) partitions the $e^2$ cyclotomic numbers into classes (groups). $(0,0)_e$ forms a singleton class. For $1 \leq i \leq e - 1$, $(i,i)_e$, $(0,-i)_e$, and $(-i,0)_e$ form classes of three elements. The remaining cyclotomic numbers are grouped into classes of six elements. $(e = 3$ is exceptional; $(1,2)_3 = (2,1)_3$ is a class of only two elements.) We also have the following properties. For $e \geq 2$,
\[
\sum_{i=0}^{e-1} (i,j)_e = \begin{cases} f - 1 & \text{if } j = 0, \\ f & \text{if } 1 \leq j \leq e - 1. \end{cases}
\]

If $q = p^r$, $p$ odd prime,
\[
\sum_{j=0}^{e-1} (i,j)_e = \begin{cases} f - 1 & \text{if } f \text{ is even and } i = 0, \\ f - 1 & \text{if } f \text{ is odd and } i = e/2, \\ f & \text{otherwise}. \end{cases}
\]
Also, if \( q = 2^r \) then \( e \) is odd. In this case
\[
(3.4) \quad \sum_{j=0}^{e-1} (i, j)_e = \begin{cases} f - 1 & \text{if } i = 0, \\ f & \text{otherwise.} \end{cases}
\]
In any case,
\[
(3.5) \quad \sum_{i=0}^{e-1} \sum_{j=0}^{e-1} (i, j)_e = q - 2.
\]
Let \( q = p^r \equiv 1 \pmod{e} \) and \( d \) be any divisor of \( e \). Write \( E = e/d \). A cyclotomic number of order \( E \) can be expressed as the sum of \( d^2 \) cyclotomic numbers of order \( e \) by
\[
(3.6) \quad (k, h)_E = \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} (k + rE, h + sE)_e.
\]
See L. E. Dickson ([6 eq. (2)]) for \( q = p \). We will use this formula in Section 5.

4. Relation between Jacobi sums and cyclotomic numbers. The \( e^2 \) Jacobi sums and the \( e^2 \) cyclotomic numbers are related by
\[
(4.1) \quad \sum_i \sum_j \zeta^{-(ai+bj)} J(i, j)_e = e^2(a, b)_e,
\]
(4.2)
\[
\sum_i \sum_j (i, j)_e \zeta^{ai+bj} = J(a, b)_e.
\]
Jacobi sums and cyclotomic numbers are related to Dickson–Hurwitz sums. The latter are defined for \( i, j \pmod{e} \) by (for \( q = p \), see [4])
\[
(4.3) \quad B(i, j) = B(i, j)_e = \sum_{h=0}^{e-1} (h, i - jh)_e.
\]
They satisfy the relation \( B(i, j)_e = B(i, e - j - i)_e \). Also,
\[
(4.4) \quad B(i, 0)_e = \begin{cases} f - 1 & \text{if } i = 0, \\ f & \text{if } 1 \leq i \leq e - 1, \end{cases}
\]
and
\[
(4.5) \quad \sum_{i=0}^{e-1} B(i, j)_e = q - 2.
\]
Dickson–Hurwitz sums and Jacobi sums \( J(\chi, \chi^j)_e \) are related by (for \( q = p \), see [4])
\[
(4.6) \quad \chi^j(-1) J(\chi, \chi^j)_e = \chi^j(-1) \chi(-1) J(1, j)_e = \sum_{i=0}^{e-1} B(i, j)_e \zeta^i.
\]
Hence if \( f \) is even or \( q = 2^r \) then \( J(1, j)_e = \sum_{i=0}^{e-1} B(i, j)_e \zeta^i \).
5. Congruences for Jacobi sums \( J(1, n)_{l^2} \) of order \( l^2 \). Let \( l \geq 3 \) be a prime and \( q = p^r \equiv 1 \pmod{l^2} \), \( p \) prime. Let \( \mathbb{F}_q \) be a finite field with \( q \) elements. Write \( q = l^2 f + 1 = lf' + 1 \). Hence \( f' \equiv 0 \pmod{l} \). Note also that if \( p \) is an odd prime then \( f \) and \( f' \) are even. Let \( \zeta \) be a complex primitive \( l^2 \)th root of unity and \( \omega = \zeta^l \). Recall that \( (l) = (1 - \zeta)^{(l-1)} \), where \( (1 - \zeta) \) is a prime ideal in the ring \( \mathbb{Z}[\zeta] \). The following lemma determines an element in the ring \( \mathbb{Z}[\zeta] \) uniquely.

**Lemma 5.1.** Let \( l \) be an odd rational prime and \( \zeta \) be a complex primitive \( l^2 \)th root of unity. If \( \alpha, \beta \in \mathbb{Z}[\zeta] \) are coprime to \( (1 - \zeta) \) and

(i) \( (\alpha) = (\beta) \),

(ii) \( |\alpha| = |\beta| \),

(iii) \( \alpha \equiv \beta \pmod{(1 - \zeta)^{l+1}} \),

then \( \alpha = \beta \).

**Proof.** \( (\alpha) = (\beta) \) implies that \( \alpha = \beta u \), where \( u \) is a unit in \( \mathbb{Z}[\zeta] \). Also \( |\alpha| = |\beta| \) gives \( \omega u = 1 \). Let \( u = f(\zeta) \), a polynomial in \( \zeta \) with coefficients from \( \mathbb{Z} \). Therefore \( f(\zeta) f(\zeta) = 1 \) and hence \( f(\zeta)^i f(\zeta^i) = 1 \) for every \( i \) relatively prime to \( l^2 \). From this it follows that \( u \) is a root of unity. But the only roots of unity in \( \mathbb{Z}[\zeta] \) are \( \pm \zeta^i \). So \( u = \pm \zeta^i \), \( 0 \leq i \leq l^2 - 1 \). From condition (iii), \( \pm \beta \zeta^i \equiv \beta \pmod{(1 - \zeta)^{l+1}} \). Hence

\[
\pm \zeta^i \equiv 1 \pmod{(1 - \zeta)^{l+1}} \quad \text{as } \gcd(\beta, (1 - \zeta)) = 1.
\]

The \( - \) sign in the above congruence does not hold as \( 1 + \zeta^i \equiv 2 \pmod{(1 - \zeta)} \). Hence \( \zeta^i \equiv 1 \pmod{(1 - \zeta)^{l+1}} \).

Now, by the binomial theorem, \( \zeta^l \equiv 1 + (\zeta - 1)^l \pmod{(1 - \zeta)^{l+1}} \). Hence \( \zeta^l \not\equiv 1 \pmod{(1 - \zeta)^{l+1}} \). However \( \zeta^{l^2} = 1 \). Therefore the order of \( \zeta \pmod{(1 - \zeta)^{l+1}} \) is \( l^2 \). Hence \( i = 0 \). Thus the result follows. \( \blacksquare \)

From (4.6), the Jacobi sum \( J(1, n)_{l^2} = \sum_{i=0}^{l(l-1)-1} b_{i,n} \zeta^i \) \((b_{i,n} \in \mathbb{Z} \text{ uniquely determined})\) of order \( l^2 \) is given in terms of Dickson–Hurwitz sums by

\[
J(1, n)_{l^2} = \sum_{i=0}^{l^2-1} B(i, n)_{l^2} \zeta^i.
\]

(5.1)

Here

\[
b_{i,n} = B(i, n)_{l^2} - B(l(l-1) + j, n)_{l^2},
\]

(5.2)

where \( 0 \leq j \leq l - 1 \), and \( j \equiv i \pmod{l} \).

**Lemma 5.2.** Let \( 1 \leq u \leq l - 1 \) and \( 1 \leq n \leq l^2 - 1 \). Write \( n = dl + n' \), \( 0 \leq n' \leq l - 1 \). Then

\[
\sum_{i=0}^{l-2} b_{i+u,n} \equiv B(u, n')_{l} \pmod{l}.
\]

Further this sum is zero modulo \( l \) if \( \gcd(l, n) = l \).
Proof. From (5.2),
\[
\sum_{i=0}^{l-2} b_{li+u,n} = \sum_{i=0}^{l-2} B(li + u, n)_{l^2} - (l - 1)B(l(l - 1) + u, n)_{l^2}
\]
\[\equiv \sum_{i=0}^{l-1} B(li + u, n)_{l^2} \pmod{l}
\]
\[= \sum_{i=0}^{l-1} \sum_{a=0}^{l^2-1} (a, li + u - an)_{l^2}
\]
\[= \sum_{i=0}^{l-1} \sum_{s,t=0} (ls + t, li + u - (ls + t)n)_{l^2}
\]
\[= \sum_{t=0}^{l-1} \sum_{s,i=0} (ls + t, l(i - sn) + u - nt)_{l^2}
\]
\[= \sum_{t=0}^{l-1} \sum_{s,i=0} (ls + t, l(i - sn' - dt) + u - n't)_{l^2}
\]
\[= \sum_{t=0}^{l-1} (t, u - n't)_{l} \quad \text{using (3.6)}
\]
\[= B(u, n')_{l}.
\]
If \(\gcd(l, n) = l\) then \(n' = 0\), and by (4.4), \(B(u, 0)_{l} = f' \equiv 0 \pmod{l}\).

**Lemma 5.3.** Let \(l > 3\) be a prime and \(1 \leq n \leq l^2 - 1\). Write \(n = dl + n'\) as before. For \(1 \leq h \leq l - 1\), let
\[
\lambda_h = \lambda_h(n) = \left[ \frac{n'h}{l} \right] + \left[ \frac{-h(n' + 1)}{l} \right],
\]
and for \(1 \leq h, k \leq l - 1\), \(h \neq k\), let
\[
\lambda_{h,k} = \lambda_{h,k}(n) = \left[ \frac{h + n'k}{l} \right] + \left[ \frac{k + n'h}{l} \right] + \left[ \frac{n'k - h(n' + 1)}{l} \right]
\]
\[+ \left[ \frac{n'h - k(n' + 1)}{l} \right] + \left[ \frac{k - h(n' + 1)}{l} \right] + \left[ \frac{h - k(n' + 1)}{l} \right].
\]
For a given \(n\), \(\lambda_{h,k}\) depends only on the class of six elements (cf. (3.1)) to which \((h, k)_{l}\) belongs. Define
\[
S(n) := \sum_{t=0}^{l-1} \sum_{j=0}^{l-1} tB(lt + j, n)_{l^2}.
\]
Then
\[ S(n) \equiv \sum_{h=1}^{l-1} \lambda_h(h,0)_l + \sum_c \lambda_{h,k}(h,k)_l \pmod{l} \]

where \( \sum_c \) is taken over a set of representatives of classes of six elements of cyclotomic numbers of order \( l \), obtained in view of (3.1). Furthermore \( S(n) \equiv 0 \pmod{l} \) if \( \gcd(l,n) = l \).

Proof. Let \((a,b)_{l^2}\) be a cyclotomic number of order \( l^2 \). We count the number of times \((a,b)_{l^2}\) appears in the expression for \( S(n) \), and consider this count modulo \( l \). If \((a,b)_{l^2}\) appears in \( S(n) \) (in some \( B(i,n)_{l^2} \)) then it is of the form \((h,i-\lambda n)_l \) for some \( 0 \leq h, i \leq l^2 - 1 \). Therefore \( a \equiv h \pmod{l^2} \) and \( b \equiv i - \lambda n \pmod{l^2} \). Hence we see that \( b + \lambda n \equiv i \pmod{l^2} \).

Thus, \((a,b)_{l^2} = (h,i-\lambda n)_{l^2}\) comes from exactly one \( B(i,n)_{l^2} \) and it is counted as many times as \( B(i,n)_{l^2} \) is counted in \( S(n) \), i.e. \([i/l]\) times. As \([i/l] \equiv [(b + \lambda n)/l] \pmod{l} \), \((a,b)_{l^2}\) is counted \([(b + \lambda n)/l] \) times (modulo \( l \)) in \( S(n) \).

CASE (i). Consider the cyclotomic number \((lx,ly)_{l^2}\), where \( 0 \leq x, y \leq l-1 \). Now we count the number of times this cyclotomic number appears in \( S(n) \) in all its different forms with respect to (3.1). \((0,0)_{l^2}\) appears \( 0 \) times in \( S(n) \).

SUBCASE (1). If \( x = y \neq 0 \) then \((lx,ly)_{l^2}\) forms a group of three, namely \((lx,lx)_{l^2} = (0,-lx)_{l^2} = (-lx,0)_{l^2}\). Hence the number of times \((lx,ly)_{l^2}\) will be counted in these three different forms in \( S(n) \) is
\[ \equiv \left[ \frac{lx + \lambda n l x}{l} \right] + \left[ \frac{-lx n}{l} \right] + \left[ \frac{-lx}{l} \right] \pmod{l} \equiv 0 \pmod{l} .\]

SUBCASE (2). If \( x \neq y \), \( x, y \neq 0 \) then \((lx,ly)_{l^2}\) forms a group of six (cf. (3.1)), viz.
\[(lx,ly)_{l^2} = (l(x-y),-y)_{l^2} = (l(y-x),-lx)_{l^2} = (ly,lx)_{l^2} = (-ly,l(x-y))_{l^2} = (-lx,l(y-x))_{l^2} .\]

So the number of times this cyclotomic number will be counted in all its six forms is
\[ \equiv \left[ \frac{lx + \lambda n ly}{l} \right] + \left[ \frac{(x-y)l - n ly}{l} \right] + \left[ \frac{l(y-x) - n lx}{l} \right] + \left[ \frac{ly + n lx}{l} \right] \]
\[ + \left[ \frac{-ly + n(lx - ly)}{l} \right] + \left[ \frac{-lx + n(ly - lx)}{l} \right] \pmod{l} \equiv 0 \pmod{l} .\]

This shows that the contribution to \( S(n) \) from all the cyclotomic numbers \((lx,ly)_{l^2}\) corresponding to the cyclotomic number \((0,0)_{l}\) (cf. (3.6)) is \( 0 \pmod{l} \).
CASE (ii). Consider a cyclotomic number of the type \((lx + h, ly)\) where \(1 \leq h \leq l-1\) and is fixed, and \(0 \leq x, y \leq l-1\), together with two of its other forms, viz. \((l(y - x) - h, -h - lx)\) and \((-ly, l(x - y) + h)\). The number of times \((lx + h, ly)\) appears in \(S(n)\) in these three forms is

\[
\equiv \left[\frac{ly + n(lx + h)}{l}\right] + \left[\frac{-h - lx + n(l(y - x) - h)}{l}\right] \\
+ \left[\frac{l(x - y) + h - yn}{l}\right] \pmod{l}
\]

\[
\equiv \left[\frac{h}{l}\right] + \left[\frac{-h(n + 1)}{l}\right] \pmod{l}
\]

\[
\equiv \lambda_h \pmod{l}, \quad \text{putting } n = dl + n'.
\]

Note that if \(y \neq 0\), by (3.1) there are six forms of \((lx + h, ly)\), but we are content with only three mentioned above. The other three forms correspond to \((l(x - y) + h, -ly)\). Hence the contribution to \(S(n)\) of \((lx + h, ly)\) with two of its other forms as mentioned is \(\lambda_h(lx + h, ly)\pmod{l}\). Hence the total contribution of \((lx + h, ly)\), \((lx - h, ly - h)\) and \((lx, ly + h)\) for all \(0 \leq x, y \leq l-1\) is \(\equiv \lambda_h(h, 0)\pmod{l}\).

CASE (iii). Let \(1 \leq h, k \leq l-1\) with \(h \neq k\) be fixed. For any \(0 \leq x, y \leq l-1\) a cyclotomic number \((lx + h, ly + k)\) forms a group of six. Six different forms of this cyclotomic number are

\[
(lx + h, ly + k) \equiv (l(x - y) + h - k, -ly - k) = (l(y - x) + k - h, -lx - h) = (ly + k, lx + h) = (-ly - k, l(x - y) + h - k) = (-lx - h, l(y - x) + k - h)\]

So the number of times this cyclotomic number is counted in all its six different forms in \(S(n)\) is

\[
\equiv \left[\frac{ly + k + n(lx + h)}{l}\right] + \left[\frac{-ly - k + n(l(x - y) + h - k)}{l}\right] \\
+ \left[\frac{-lx - h + n(l(y - x) + k - h)}{l}\right] + \left[\frac{lx + h + n(ly + k)}{l}\right] \\
+ \left[\frac{l(x - y) + h - k - n(ly + k)}{l}\right] + \left[\frac{l(y - x) + k - h - n(lx + h)}{l}\right] \pmod{l}
\]

\[
= \left[\frac{k + nh}{l}\right] + \left[\frac{-k(n + 1) + nh}{l}\right] + \left[\frac{-h(n + 1) + nk}{l}\right] \\
+ \left[\frac{h + nk}{l}\right] + \left[\frac{h - k(n + 1)}{l}\right] + \left[\frac{k - h(n + 1)}{l}\right].
\]

Putting \(n = dl + n'\) we see that
\[
\lambda_{h,k} = \left[ \frac{h + n'k}{l} \right] + \left[ \frac{k + n'h}{l} \right] + \left[ \frac{n'k - h(n' + 1)}{l} \right] + \left[ \frac{n'h - k(n' + 1)}{l} \right] \\
+ \left[ \frac{k - h(n' + 1)}{l} \right] + \left[ \frac{h - k(n' + 1)}{l} \right].
\]

Hence the total contribution of \((lx + h, ly + k)l^2\) and of its five other forms for \(0 \leq x, y \leq l - 1\) is
\[
\sum_{x,y=0}^{l-1} \lambda_{h,k}(lx + h, ly + k)l^2 = \lambda_{h,k}(h,k)l.
\]

This ends Case (iii).

Hence by Cases (i)–(iii),
\[
S(n) \equiv \sum_{h=1}^{l-1} \lambda_h(h,0)_l + \sum_c \lambda_{h,k}(h,k)_l \pmod{l},
\]
where \(\sum_c\) is taken over a set of representatives of classes of six elements of cyclotomic numbers of order \(l\), obtained from (3.1).

Now let \(n' = 0\), i.e. \((l,n) = l\). Then
\[
\lambda_h = \left[ \frac{n'h}{l} \right] + \left[ \frac{-h(n' + 1)}{l} \right] = \left[ \frac{-h}{l} \right] = -1,
\]
whereas
\[
\lambda_{h,k} = \left[ \frac{h + n'k}{l} \right] + \left[ \frac{k + n'h}{l} \right] + \left[ \frac{n'k - h(n' + 1)}{l} \right] + \left[ \frac{n'h - k(n' + 1)}{l} \right] \\
+ \left[ \frac{k - h(n' + 1)}{l} \right] + \left[ \frac{h - k(n' + 1)}{l} \right] \\
= \left[ \frac{h}{l} \right] + \left[ \frac{k}{l} \right] + \left[ \frac{-h}{l} \right] + \left[ \frac{-k}{l} \right] + \left[ \frac{k - h}{l} \right] + \left[ \frac{h - k}{l} \right] = -3.
\]
We use (3.2) and (3.5) to obtain
\[
S(n) \equiv - \sum_{h=1}^{l-1} (h,0)_l - 3 \sum_c (h,k)_l \pmod{l}
\]
\[
= 1 + (0,0)_l - f' - \frac{1}{2} \sum_c 6(h,k)_l \\
= 1 - f' + (0,0)_l - \frac{1}{2} \left( q - 2 - (0,0)_l - 3 \sum_{k=1}^{l-1} (k,0)_l \right) \\
= 1 - f' + (0,0)_l - \frac{1}{2} (q - 2 - 3(f' - 1) + 2(0,0)_l) \\
= \frac{1}{2} f' - \frac{1}{2} (q - 1) \equiv 0 \pmod{l}.
\]
This completes the proof of the lemma. •
Consider the Jacobi sum of order \( l^2 \), \( J(1, n)_{l^2} = \sum_{i=0}^{l(l-1)-1} b_{i,n} \zeta^i \). Writing it in powers of \( \zeta - 1 \) we see that

\[
J(1, n)_{l^2} = \sum_{i=0}^{l(l-1)-1} c'_{i,n}(\zeta - 1)^i \quad \text{where} \quad c'_{i,n} = \sum_{m=i}^{l(l-1)-1} \begin{pmatrix} m \\ i \end{pmatrix} b_{m,n}.
\]

But from Y. Ihara \cite{Ihara8} (see also R. J. Evans \cite{Evans7}), \( J(1, n)_{l^2} \equiv -1 \pmod{(1 - \zeta)^3} \). Therefore \( c'_{0,n} \equiv -1 \pmod{l} \) and \( c'_{1,n} \equiv c'_{2,n} \equiv 0 \pmod{l} \). Hence

\[
J(1, n)_{l^2} \equiv -1 + \sum_{i=3}^{l} c'_{i,n}(\zeta - 1)^i \pmod{(1 - \zeta)^{l+1}}.
\]

We shall now get congruences for \( c'_{i,n} \) for \( 3 \leq i \leq l \). Write \( m = lt + u \), \( 0 \leq u \leq l - 1 \) and \( 0 \leq t \leq l - 2 \).

**Case 1.** Let \( 3 \leq i \leq l - 1 \). Then

\[
\begin{pmatrix} m \\ i \end{pmatrix} = \frac{m(m-1)\cdots(m-i+1)}{i!} \equiv \frac{u(u-1)\cdots(u-i+1)}{i!} = \begin{pmatrix} u \\ i \end{pmatrix} \pmod{l},
\]

where \( \binom{u}{i} = 0 \) for \( 0 \leq u < i \). Therefore

\[
c'_{i,n} \equiv \sum_{u=i}^{l-1} \left[ \begin{pmatrix} u \\ i \end{pmatrix} \left( \sum_{t=0}^{l-2} b_{lt+u,n} \right) \right] \pmod{l}.
\]

We apply Lemma \[5.2\] to obtain

\[
c'_{i,n} \equiv \sum_{u=i}^{l-1} \left[ \begin{pmatrix} u \\ i \end{pmatrix} \left( \sum_{t=0}^{l-2} b_{lt+u,n} \right) \right] \equiv \sum_{u=i}^{l-1} \begin{pmatrix} u \\ i \end{pmatrix} B(u, n')_l \pmod{l}.
\]

Define, for \( 3 \leq i \leq l - 1 \),

\[(5.3) \quad c_{i,n} := \sum_{u=i}^{l-1} \begin{pmatrix} u \\ i \end{pmatrix} B(u, n')_l.\]

Thus \( c'_{i,n} \equiv c_{i,n} \pmod{l} \), \( 3 \leq i \leq l - 1 \).

**Case 2.** Let \( i = l \). Then for \( m = lt + u \) as above, \( \begin{pmatrix} m \\ i \end{pmatrix} \equiv t \pmod{l} \). Using this observation, from (5.2) we obtain

\[
c'_{l,n} = \sum_{m=l}^{l(l-1)-1} \begin{pmatrix} m \\ l \end{pmatrix} b_{m,n} \equiv \sum_{t=0}^{l-2} \sum_{j=0}^{l-1} t b_{lt+j,n} \pmod{l}
\]

\[
= \sum_{t=0}^{l-2} \sum_{j=0}^{l-1} t(B(lt + j, n)_{l^2} - B(l(l-1) + j, n)_{l^2})
\]

\[
= \sum_{t=0}^{l-2} \sum_{j=0}^{l-1} tB(lt + j, n)_{l^2} - \left( \sum_{t=0}^{l-2} t \right) \left( \sum_{j=0}^{l-1} B(l(l-1) + j, n)_{l^2} \right).
\]
Now, $-\sum_{t=0}^{l-2} t = -(l - 1)(l - 2)/2 \equiv l - 1 \pmod{l}$. Hence

$$c'_l,n \equiv \sum_{t=0}^{l-1} \sum_{j=0}^{l-1} tB(lt + j, n) \pmod{l}.$$

Let $\lambda_h, \lambda_{h,k}$ and $c$ be as in Lemma 5.3. Define, for $i = l$,

$$(5.4) \quad c_l,n := \sum_{h=1}^{l-1} \lambda_h(h, 0)t + \sum_c \lambda_{h,k}(h, k)t.$$

Then by Lemma 5.3

$$c'_l,n \equiv \sum_{t=0}^{l-1} \sum_{j=0}^{l-1} tB(lt + j, n) \equiv S(n) \equiv c_l,n \pmod{l}.$$

Thus,

$$J(1, n) \equiv -1 + \sum_{i=3}^{l} c_i,n(\zeta - 1)^i \pmod{(1 - \zeta)^{l+1}}.$$

Furthermore, from Lemmas 5.2 and 5.3 if $l | n$ then $c_i,n \equiv 0 \pmod{l}$ for $3 \leq i \leq l$, and we get

$$J(1, n) \equiv -1 \pmod{(1 - \zeta)^{l+1}}.$$

We conclude the above discussion in the following theorem.

**Theorem 5.4.** Let $l > 3$ be a prime and $p^r = q \equiv 1 \pmod{l^2}$. If $1 \leq n \leq l^2 - 1$, then a (determining) congruence for $J(1, n)$ for a finite field $\mathbb{F}_q$ is given by

$$J(1, n) \equiv \begin{cases} -1 + \sum_{i=3}^{l} c_i,n(\zeta - 1)^i \pmod{(1 - \zeta)^{l+1}} & \text{if } \gcd(l, n) = 1, \\ -1 \pmod{(1 - \zeta)^{l+1}} & \text{if } \gcd(l, n) = l, \end{cases}$$

where for $3 \leq i \leq l - 1$, $c_i,n$ are described by (5.3) and $c_l,n = S(n)$ is given by Lemma 5.3.

**Remark 1.** Since Dickson–Hurwitz sums are sums of cyclotomic numbers, for $3 \leq i \leq l$, $c_i,n$ are integral linear combinations of cyclotomic numbers of order $l$.

**Remark 2.** For a given $l$, the $c_i,n$ and hence the above congruence for $J(1, n)$ depends only on $n \pmod{l}$, i.e.

$$J(1, n) \equiv J(1, l + k) \pmod{(1 - \zeta)^{l+1}}.$$

**Remark 3.** For $\gcd(l, n) = l$, the result in the theorem also follows from the work of R. J. Evans ([7, Thm. 1]).

**Remark 4.** The absolute value of the Jacobi sum $J(1, n)$ (see Thm. 2.1(5)) and its prime ideal decomposition (see [3, p. 346, Corollary 11.2.4])
are known. In view of Lemma 5.1, the congruence condition for $J(1, n)_{l^2}$ obtained in Thm. 5.4 together with the absolute value and prime ideal decomposition gives an algebraic characterisation of $J(1, n)_{l^2}$ and hence of all Jacobi sums of order $l^2$.

**Remark 5.** Congruences for Jacobi sums of order $l^2 \mod (1 - \zeta)^{l+1}$ could be obtained in terms of cyclotomic numbers of order $l$. In the same fashion it is expected that the determining congruences for Jacobi sums of order $l^m$, which are required modulo $(1 - \zeta)^{m-1}+1$, can be obtained in terms of cyclotomic numbers of order $l^{m-1}$ (or of order $l^k$, $1 \leq k \leq m-1$). Also appropriate congruences for Jacobi sums of order $n$ may be obtained in terms of cyclotomic numbers of orders $d$ properly dividing $n$. These expectations are consistent with the result of P. van Wamelen (2002) who gave an algebraic characterization of Jacobi sums of order $n$ in terms of their absolute value, prime ideal decomposition and the Jacobi sums of orders $d$ properly dividing $n$. (See [15].)

### 6. Cyclotomic numbers of order $l^2$

Let $l$ be an odd prime. In this section we obtain formulae for the cyclotomic numbers $(h, k)_{l^2}$ of order $l^2$ in terms of coefficients of the Jacobi sums of order $l^2$ and $l$. Such formulae for cyclotomic numbers of order $l$, and cyclotomic numbers of order $2l$ were obtained by S. A. Katre and A. R. Rajwade [11], and V. V. Acharya and S. A. Katre [1] respectively.

With the set up of Section 5, write Jacobi sums of order $l$ as $J(1, j)_{l} = \sum_{i=0}^{l-2} a_{i,j} \omega^i$, where $a_{i,j} \in \mathbb{Z}$. Let $G' = \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ and $G = \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$. We compute $\text{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}}(J(1, j)_{l}\omega^{-t})$. Note that $\text{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}}(\omega) = -1$. Therefore,

\[(6.1) \quad \text{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}}(J(1, j)_{l}\omega^{-t}) = \text{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}} \left( \sum_{i=0}^{l-2} a_{i,j} \omega^i \right) = \sum_{i=0}^{l-2} a_{i,j} \text{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}}(\omega^{i-t}) = la_{t,j} - \sum_{i=0}^{l-2} a_{i,j}.
\]

Similarly, we compute $\text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(J(1, n)_{l^2}\zeta^{-t})$. In this case, $\text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta) = 0$, where $\zeta$ is any primitive $l^2$th root of unity, while $\text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\omega) = -l$. Let $B(i, n) = B(i, n)_{l^2}$. Therefore, we have

\[(6.2) \quad \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(J(1, n)_{l^2}\zeta^{-t}) = \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}} \left( \sum_{i=0}^{l^2-1} B(i, n) \zeta^i \zeta^{-t} \right) = \sum_{i=0}^{l^2-1} B(i, n) \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta^{i-t}) = l(l - 1)B(t, n) - l \sum_{u=1}^{l-1} B(ul + t, n).
\]
Lemma 6.1. For $t$ and $n$ modulo $l^2$, define

$$C(t, n) := l(l - 1)B(t, n) - l \sum_{u=1}^{l-1} B(ul + t, n).$$

Let $0 \leq t \leq l^2 - 1$. Write $t = jl + s$, where $0 \leq j \leq l - 1$ and $0 \leq s \leq l - 1$. Then

$$C(t, n) = \epsilon(t)b_{t,n} - l \sum_{u=0}^{l-2} b_{ul+t,n}, \quad \text{where}$$

$$\epsilon(t) = \begin{cases} l^2 & \text{if } 0 \leq j \leq l - 2, \ i.e. \ 0 \leq t < l(l-1), \\ -l & \text{if } j = l - 1, \ i.e. \ l(l-1) \leq t \leq l^2 - 1. \end{cases}$$

Proof. (i) Let $0 \leq j \leq l - 2$. Then

$$C(t, n) = l(l - 1)B(t, n) - l \sum_{u=1}^{l-1} B(ul + t, n)$$

$$= l(l - 1)B(jl + s, n) - l \sum_{u=1}^{l-1} B((u + j)l + s, n)$$

$$= l(l - 1)B(jl + s, n) - l(l - 1)B(l(l - 1) + s, n)$$

$$+ l(l - 2)B(l(l - 1) + s, n) - l \sum_{u=1}^{l-j-2} B((u + j)l + s, n)$$

$$- l \sum_{u=l-J}^{l-1} B((u + j)l + s, n)$$

$$= l(l - 1)(B(jl + s, n) - B(l(l - 1) + s, n))$$

$$- l \sum_{u=1}^{l-1} (B((u + j)l + s, n) - B(l(l - 1) + s, n))$$

$$- l \sum_{u=l-j}^{l-1} (B((u + j)l + s, n) - B(l(l - 1) + s, n)).$$

In the first sum put $u + j = x$, and in the second put $u + j \equiv x \pmod{(l - 1)}$. Hence using (5.2) we get

$$C(t, n) = l(l - 1)(B(jl + s, n) - B(l(l - 1) + s, n))$$

$$- l \sum_{x=0}^{j-1} (B(xl + s, n) - B(l(l - 1) + s, n))$$

$$- l \sum_{x=j+1}^{l-2} (B(xl + s, n) - B(l(l - 1) + s, n))$$
\begin{align*}
&= l(l - 1)b_{jl+s,n} - l \sum_{x=j+1}^{l-2} b_{xl+s,n} - l \sum_{x=0}^{j-1} b_{xl+s,n} \\
&= l^2b_{jl+s,n} - l \sum_{x=0}^{l-2} b_{xl+s,n} = l^2b_{t,n} - l \sum_{x=0}^{l-2} b_{xl+s,n}.
\end{align*}

For every \( u \), we have \( ul + t \equiv xl + s \ (\text{mod} \ l(l-1)) \) for some \( x \in \{0, \ldots, l-2\} \). Therefore

\[
C(t, n) = l^2b_{t,n} - l \sum_{u=0}^{l-2} b_{ul+t,n}.
\]

(ii) Let \( j = l - 1 \). Then

\[
C(t, n) = l(l - 1)B(t, n) - l \sum_{u=1}^{l-1} B(u(l + t, n))
\]

\[
= l(l - 1)B(l(l - 1) + s, n) - l \sum_{u=1}^{l-1} B((u - 1 + l)l + s, n)
\]

\[
= -l \sum_{u=1}^{l-1} (B((u - 1 + l)l + s, n) - B(l(l - 1) + s, n))
\]

\[
= -l \sum_{u=1}^{l-1} (B((u - 1)l + s, n) - B(l(l - 1) + s, n))
\]

\[
= -l \sum_{x=0}^{l-2} (B(xl + s, n) - B(l(l - 1) + s, n)).
\]

Again, using (5.2),

\[
C(t, n) = -l \sum_{x=0}^{l-2} b_{xl+s,n} = -l \sum_{u=0}^{l-2} b_{ul+t,n}.
\]

So from (i) and (ii) above we get

\[
C(t, n) = \epsilon(t)b_{t,n} - l \sum_{u=0}^{l-2} b_{ul+t,n}, \quad \text{where}
\]

\[
\epsilon(t) = \begin{cases} 
  l^2 & \text{if } 0 \leq j \leq l - 2, \text{ i.e. } 0 \leq t < l(l-1), \\
  -l & \text{if } j = l - 1, \text{ i.e. } l(l-1) \leq t \leq l^2 - 1.
\end{cases}
\]

Now we observe that

\[
\sum_{i=1}^{(l^2-1)/2} (\zeta^{-it} + \zeta^{it}) = \begin{cases} 
  l^2 - 1 & \text{if } t = 0, \\
  -1 & \text{otherwise}.
\end{cases}
\]
Therefore,
\[
\frac{(l^2-1)/2}{(l^2-1)/2} \sum_{i=1}^{(l^2-1)/2} J(i, 0)(\zeta^{ih} + \zeta^{-ih} + \zeta^{-ik} + \zeta^{ik} + \zeta^{-ih+ik} + \zeta^{ih-ik})
\]
\[
= - \sum_{i=1}^{(l^2-1)/2} (\zeta^{ih} + \zeta^{-ih}) - \sum_{i=1}^{(l^2-1)/2} (\zeta^{ik} + \zeta^{-ik}) - \sum_{i=1}^{(l^2-1)/2} (\zeta^{ih-ik} + \zeta^{-ih+ik})
\]
\[
= 3 + \delta(h, k),
\]
where \(\delta(h, k)\) is given by
\[
\delta(h, k) = \begin{cases} 
-3l^2 & \text{if } h \equiv k \equiv 0 \pmod{l^2}, \\
-l^2 & \text{if exactly one of } h, k, h-k \equiv 0 \pmod{l^2}, \\
0 & \text{if } h, k, h-k \not\equiv 0 \pmod{l^2}.
\end{cases}
\]

From (4.1), (6.1), (6.2) and Lemma 6.1 we get the following

**Theorem 6.2.** Let \(p\) be a prime and \(p^r = q \equiv 1 \pmod{l^2}\). Then the cyclotomic numbers \((h, k)_{l^2}\) of order \(l^2\) are given in terms of coefficients of the Jacobi sums of order \(l\) and order \(l^2\) by

\[
l^4(h, k)_{l^2} = q + 1 + \delta(h, k) + l \sum_{j=1}^{l-2} a_{h+jk,j} - \sum_{i=0}^{l-2-1} a_{i,j} - l \sum_{j=1}^{l-2} \sum_{i=0}^{l-2} b_{ul+h+jk,j}
\]
\[
- l \sum_{i=1}^{l-2} \sum_{u=0}^{l-2} b_{ul+h+l+k,li} + \sum_{j=1}^{l-2} \epsilon(h+jk)b_{h+jk,j} + \sum_{i=1}^{l-2} \epsilon(hil+k)b_{hil+k,li}.
\]

**Proof.** Write \(q = 1 + l^2 f\). Now either \(f\) is even and \(q = p^r\), \(p\) odd; or \(f\) is odd and \(q = 2^r\). Hence by the Remark in Section 2 we get

\[
l^4(h, k)_{l^2} = \sum_{i,j=0}^{l^2-1} J(i, j)_{l^2} \zeta^{-ih-ik} \quad \text{(from (4.1))}
\]
\[
= J(0, 0)_{l^2} + \sum_{i=1}^{(l^2-1)/2} J(i, 0)_{l^2} (\zeta^{ih} + \zeta^{-ih} + \zeta^{-ik} + \zeta^{ik} + \zeta^{-ih+ik} + \zeta^{ih-ik})
\]
\[
+ \sum_{j=1}^{l-2} \sum_{\sigma \in G'} \sigma(J(1, j)_{l^2} \omega^{-h-jk}) + \sum_{j=1}^{l^2-2} \sum_{\sigma \in G} \sigma(J(1, j)_{l^2} \zeta^{-h-jk})
\]
\[
+ \sum_{i=1}^{l-1} \sum_{\sigma \in G} \sigma(J(il, 1)_{l^2} \zeta^{-ih-k})
\]
\[= q + 1 + \delta(h, k) + \sum_{j=1}^{l-2} \text{Tr}_{Q(\omega)/Q}(J(1, j)\omega^{-h-jk}) \quad \text{(from above)}\]
\[+ \sum_{j=1}^{l^2-2} \text{Tr}_{Q(\zeta)/Q}(J(1, j)\zeta^{-h-jk}) + \sum_{i=1}^{l-1} \text{Tr}_{Q(\zeta)/Q}(J(1, li)\zeta^{-lih-k})\]
\[= q + 1 + \delta(h, k) + \sum_{j=1}^{l-2} \left(l a_{h+jk,j} - \sum_{i=0}^{l-2} a_{i,j}\right) \quad \text{(from (6.1))}\]
\[+ \sum_{j=1}^{l^2-2} \text{Tr}_{Q(\zeta)/Q} \left(\sum_{i=0}^{l^2-1} B(i, j)\zeta^{-i-h-jk}\right) + \sum_{i=1}^{l-1} \text{Tr}_{Q(\zeta)/Q} \left(\sum_{j=0}^{l^2-1} B(j, li)\zeta^{j-lih-k}\right)\]
\[= q + 1 + \delta(h, k) + l \sum_{j=1}^{l-2} a_{h+jk,j} - \sum_{i=0}^{l-2} \sum_{j=1}^{l-2} a_{i,j}\]
\[+ \sum_{j=1}^{l^2-2} \text{Tr}_{Q(\zeta)/Q} \left(\sum_{t=0}^{l^2-1} B(t + h + jk, j)\zeta^t\right)\]
\[+ \sum_{i=1}^{l^2-2} \text{Tr}_{Q(\zeta)/Q} \left(\sum_{t=0}^{l^2-1} B(t + lih + k, li)\zeta^t\right)\]
\[= q + 1 + \delta(h, k) + l \sum_{j=1}^{l-2} a_{h+jk,j} - \sum_{i=0}^{l-2} \sum_{j=1}^{l-2} a_{i,j}\]
\[+ \sum_{j=1}^{l^2-2} \left[l(l-1)B(h + jk, j) - l \sum_{x=1}^{l-1} B(xl + h + jk, j)\right]\]
\[+ \sum_{i=1}^{l-1} \left[l(l-1)B(lih + k, li) - l \sum_{x=1}^{l-1} B(xl + hil + k, li)\right] \quad \text{(from (6.2))}\]
\[= q + 1 + \delta(h, k) + l \sum_{j=1}^{l-2} a_{h+jk,j} - \sum_{i=0}^{l-2} \sum_{j=1}^{l-2} a_{i,j}\]
\[+ \sum_{j=1}^{l^2-2} C(h + jk, j) + \sum_{i=1}^{l-1} C(hil + k, li)\]
\[= q + 1 + \delta(h, k) + l \sum_{j=1}^{l-2} a_{h+jk,j} - \sum_{i=0}^{l-2} \sum_{j=1}^{l-2} a_{i,j} - l \sum_{j=1}^{l-2} \sum_{u=0}^{l^2-2} b_{ul+h+jk,j}\]
\[l \sum_{i=1}^{l-2} \sum_{u=0}^{l^2-2} b_{ul+hil+k,li} + \sum_{j=1}^{l^2-2} \epsilon(h + jk)b_{h+jk,j} + \sum_{i=1}^{l^2-2} \epsilon(hil + k)b_{hil+k,li},\]

where the last equality is obtained using Lemma 6.1. ■
Remark. For cyclotomic numbers of order 9 see also [2].

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