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On the number of representations of n by $ax^2 + by(y-1)/2$, $ax^2 + by(3y-1)/2$ and $ax(x-1)/2 + by(3y-1)/2$

by

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1. Introduction. For $k = 3, 4, 5, \dots$ the k-gonal numbers are given by $p_k(n) = (k-2)\binom{n}{2} + n$. Thus,

$$p_3(n) = \frac{n^2+n}{2}, \quad p_4(n) = n^2, \quad p_5(n) = \frac{3n^2-n}{2}.$$

Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of integers and the set of positive integers, respectively. Let $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} = \{\langle x, y\rangle : x, y \in \mathbb{Z}\}$. For $n \in \mathbb{N}$ let

$$r(n = f(x,y)) = |\{\langle x, y\rangle \in \mathbb{Z}^2 : n = f(x,y)\}|.$$

For 143 values of $\langle a, b\rangle$ the formula for $r(n = ax(x-1)/2 + by(y-1)/2)$ is known (see [3, 4]). In [2] the author determined $r(n = (3x^2-x)/2 + b(3y^2-y)/2)$ for $b = 1, 2, 5$. In this paper, using some results for binary quadratic forms in [4, 5] we determine $r(n = x^2 + by(y-1)/2)$ in the cases $b = 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 14, 15, 16, 21, 29, 30, 35, 39, 51, 65, 95$, $r(n = x^2 + b(3y^2-y)/2)$ in the cases $b = 1, 2, 3, 4, 5, 7, 8, 13, 17$ and $r(n = (x^2-x)/2 + b(3y^2-y)/2)$ in the cases $b = 1, 2, 3, 5, 7, 10, 11, 14, 15, 19, 26, 31, 34, 35, 55, 59, 91, 115, 119, 455$. For example, we have

$$r\Big(n = x^2 + b\frac{y(y-1)}{2}\Big) = 2\sum_{k\mid 8n+b}\Big(\frac{-2b}{k}\Big) \qquad \text{for } b = 3, 5, 11, 29,$$

$$r\Big(n = x^2 + b\frac{y(3y-1)}{2}\Big) = \sum_{k\mid 24n+b}\Big(\frac{-6b}{k}\Big) \qquad \text{for } b = 5, 7, 13, 17,$$

$$r\Big(n = \frac{x(x-1)}{2} + b\frac{y(3y-1)}{2}\Big) = \sum_{k\mid 12n+(b+3)/2}\Big(\frac{-3b}{k}\Big) \qquad \text{for } b = 7, 11, 19, 31, 59,$$

where $(\frac{a}{m})$ is the Legendre–Jacobi–Kronecker symbol.

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A negative integer d with $d \equiv 0,1 \pmod 4$ is called a *discriminant*. Let $d<0$ be a discriminant. The *conductor* of d is the largest positive integer $f=f(d)$ such that $d/f^2 \equiv 0,1 \pmod 4$. As usual we set $w(d)=2,4,6$ according as $d<-4, d=-4$ or $d=-3$. For $a,b,c\in\mathbb{Z}$ we denote by $[a,b,c]$ the equivalence class containing the form $ax^2+bxy+cy^2$. It is known ([1]) that

$$(1.1)\qquad [a,b,c]=[c,-b,a]=[a,2ak+b,ak^2+bk+c] \quad \text{for } k\in\mathbb{Z}.$$

For $n\in\mathbb{N}$ and $a,b,c\in\mathbb{Z}$ with $a,c>0$ and $b^2-4ac<0$ let

$$(1.2)\qquad R([a,b,c],n)=|\{\langle x,y\rangle\in\mathbb{Z}^2 : n=ax^2+bxy+cy^2\}|.$$

That is, $R([a,b,c],n)=r(n=ax^2+bxy+cy^2)$. It is known that $R([a,b,c],n)=R([a,-b,c],n)$. If $R([a,b,c],n)>0$, we say that n is *represented* by $[a,b,c]$ or $ax^2+bxy+cy^2$. Let $H(d)$ be the form class group consisting of classes of primitive, integral binary quadratic forms of discriminant d. For $n\in\mathbb{N}$, following [5] we define

$$(1.3)\qquad \delta(n,d)=\sum_{k\mid n}\left(\frac{d}{k}\right), \qquad N(n,d)=\sum_{K\in H(d)} R(K,n).$$

When n is odd and $a\in\mathbb{Z}$, we also define $\delta(n,a)=\sum_{k\mid n}(\frac{a}{k})$, where $(\frac{a}{k})$ is the Jacobi symbol. In the paper we mainly use the formula for $N(n,d)$ and reduction formulas for $R(K,n)$ in [5] to determine $r(n=ax^2+by(y-1)/2)$, $r(n=ax^2+by(3y-1)/2)$ and $r(n=ax(x-1)/2+by(3y-1)/2)$ for some special values of $\langle a,b\rangle$.

In addition to the above notation, throughout this paper $[x]$ denotes the greatest integer not exceeding x and (a,b) denotes the greatest common divisor of integers a and b. For a prime p and $n\in\mathbb{N}$, $\mathrm{ord}_p n$ denotes the unique nonnegative integer α such that $p^\alpha \parallel n$ (i.e. $p^\alpha \mid n$ but $p^{\alpha+1}\nmid n$).

Throughout this paper, p denotes a prime and products over p run through all distinct primes p satisfying any restrictions given under the product symbol. For example the condition $p\equiv 1 \pmod 4$ under a product restricts the product to those distinct primes p which are of the form $4k+1$.

2. Basic lemmas. Let $d<0$ be a discriminant and $n\in\mathbb{N}$. Recall that $\delta(n,d)=\sum_{k\mid n}(\frac{d}{k})$. By [5, Lemma 4.1], we have

$$(2.1)\qquad \delta(n,d)=\begin{cases}\prod_{(\frac{d}{p})=1}(1+\mathrm{ord}_p n) & \\ \quad \text{if } 2\mid \mathrm{ord}_q n \text{ for every prime } q \text{ with } (\frac{d}{q})=-1, & \\ 0 \quad \text{otherwise.} & \end{cases}$$

When n is odd and d is not a discriminant, (2.1) is also true. By (2.1), $(\frac{d}{n})=-1$ implies $\delta(n,d)=0$.

LEMMA 2.1 ([5, Theorem 4.1]). *Let $d < 0$ be a discriminant with conductor f. Let $n \in \mathbb{N}$ and $d_0 = d/f^2$. Then*

$$N(n,d) = \begin{cases} 0 & \text{if } (n,f^2) \text{ is not a square,} \\ m \prod_{p \mid m} \left(1 - \frac{1}{p}\left(\frac{d/m^2}{p}\right)\right) \cdot w(d)\delta\left(\frac{n}{m^2}, d_0\right) & \\ & \text{if } (n,f^2) = m^2 \text{ for } m \in \mathbb{N}. \end{cases}$$

In particular, when $(n,f) = 1$ we have $N(n,d) = w(d)\delta(n,d_0)$.

LEMMA 2.2 ([4, Lemma 2.2]). *Let $a,b,n \in \mathbb{N}$ with $2 \nmid n$.*

(i) *If $2 \nmid a$ and $4 \nmid (a-b)b$, then*

$$R([a,0,4b],n) = \begin{cases} R([a,0,b],n) & \text{if } n \equiv a \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

If $2 \nmid a$, $2 \mid b$ and $8 \nmid b$, then

$$R([a,0,4b],n) = \begin{cases} R([a,0,b],n) & \text{if } n \equiv a \pmod 8, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If $2 \nmid (a+b)$ and $8 \nmid ab$, then*

$$R([4a,4a,a+b],n) = \begin{cases} R([a,0,b],n) & \text{if } n \equiv a+b \pmod 8, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2.3 ([4, Theorem 2.1]). *Let $b \in \{6,10,12,22,28,58\}$, $b = 2^r b_0$ $(2 \nmid b_0)$, $n \in \mathbb{N}$ and $2 \nmid n$. Then*

$$R([1,0,4b],n) = \begin{cases} 2\delta(n,-b) & \text{if } n \equiv 1 \pmod 8, \\ 0 & \text{otherwise,} \end{cases}$$

$$R([4,4,b+1],n) = \begin{cases} 2\delta(n,-b) & \text{if } n \equiv b+1 \pmod 8, \\ 0 & \text{otherwise,} \end{cases}$$

$$R([2^{r+2},0,b_0],n) = \begin{cases} 2\delta(n,-b) & \text{if } n \equiv b_0 \pmod 8, \\ 0 & \text{otherwise,} \end{cases}$$

$$R([2^{r+2},2^{r+2},2^r+b_0],n) = \begin{cases} 2\delta(n,-b) & \text{if } n \equiv 2^r + b_0 \pmod 8, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2.4 ([4, Theorem 2.6]). *Let $n \in \mathbb{N}$ and $n = 5^\alpha n_0$ with $5 \nmid n_0$. Then*

$$R([1,0,45],n) = \begin{cases} 2\delta(n_0/9,-20) & \text{if } 9 \mid n \text{ and } n_0 \equiv \pm 1 \pmod 5, \\ 2\delta(n_0,-20) & \text{if } 3 \mid n-1 \text{ and } n_0 \equiv \pm 1 \pmod 5, \\ 0 & \text{otherwise,} \end{cases}$$

$$R([5,0,9],n) = \begin{cases} 2\delta(n_0/9,-20) & \text{if } 9 \mid n \text{ and } n_0 \equiv \pm 1 \pmod 5, \\ 2\delta(n_0,-20) & \text{if } 3 \mid n-2 \text{ and } n_0 \equiv \pm 1 \pmod 5, \\ 0 & \text{otherwise.} \end{cases}$$

3. Formulas for $r(n = ax^2 + by(y-1)/2)$

LEMMA 3.1. *Let* $a, b, n \in \mathbb{N}$. *Then*

$$\begin{aligned} r(n = ax^2 + by(y-1)/2) \\ = \begin{cases} R([8a, 0, b], 8n + b) & \text{if } 8 \nmid b, \\ R([a, 0, b/8], n + b/8) - R([a, 0, b/2], n + b/8) & \text{if } 8 \mid b. \end{cases} \end{aligned}$$

Moreover, if $4 \nmid a$ *and* $2 \nmid b$, *then*

$$r(n = ax^2 + by(y-1)/2) = R([2a, 0, b], 8n + b);$$

if $2 \nmid a$, $2 \parallel b$ *and* $4 \nmid a - b/2$, *then*

$$r(n = ax^2 + by(y-1)/2) = R([a, 0, b/2], 4n + b/2).$$

Proof. It is clear that

$$\begin{aligned} r(n = ax^2 + by(y-1)/2) &= r(8n = 8ax^2 + b(2y-1)^2 - b) \\ &= |\{\langle x, y\rangle \in \mathbb{Z}^2 : 8n + b = 8ax^2 + by^2, 2 \nmid y\}| \\ &= R([8a, 0, b], 8n + b) - R([8a, 0, 4b], 8n + b). \end{aligned}$$

When $8 \nmid b$, we have $(8a, 4b) \nmid 8n + b$ and so $R([8a, 0, 4b], 8n + b) = 0$.

If $4 \nmid a$ and $2 \nmid b$, by the above and Lemma 2.2 we have

$$r(n = ax^2 + by(y-1)/2) = R([8a, 0, b], 8n + b) = R([2a, 0, b], 8n + b);$$

if $2 \nmid a$, $2 \parallel b$ and $4 \nmid a - b/2$, by the above and Lemma 2.2 we have

$$\begin{aligned} r(n = ax^2 + by(y-1)/2) &= R([8a, 0, b], 8n + b) = R([4a, 0, b/2], 4n + b/2) \\ &= R([a, 0, b/2], 4n + b/2). \end{aligned}$$

This completes the proof.

THEOREM 3.1. *Let* $n \in \mathbb{N}$. *Then*

$$r(n = x^2 + 16y(y-1)/2) = \begin{cases} 2\delta\left(\dfrac{n+2}{2}, -2\right) & \text{if } n \equiv 0 \pmod 4, \\ 2\delta(n+2, -2) & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 2, 3 \pmod 4, \end{cases}$$

$$r(n = x^2 + 4y(y-1)/2) = 2\delta(2n+1, -2),$$

$$r(n = x^2 + y(y-1)/2) = 2\delta(8n+1, -2).$$

Proof. If $n = x^2 + 16y(y-1)/2$ for some $x, y \in \mathbb{Z}$, then $n \equiv x^2 \pmod 8$ and so $n \not\equiv 2, 3 \pmod 4$. Now assume $n \equiv 0, 1 \pmod 4$. As $x^2 \equiv n+2 \pmod 4$ is insolvable, we see that $R([1, 0, 8], n+2) = 0$. Thus, by Lemmas 2.1 and 3.1 we have

$$\begin{aligned} r(n&=x^2+16y(y-1)/2)\\ &= R([1,0,2],n+2)-R([1,0,8],n+2)=R([1,0,2],n+2)\\ &= N(n+2,-8)=2\delta(n+2,-8)\\ &= \begin{cases} 2\delta((n+2)/2,-2) & \text{if } n\equiv 0 \pmod 4,\\ 2\delta(n+2,-2) & \text{if } n\equiv 1 \pmod 4.\end{cases}\end{aligned}$$

As $r(4n=x^2+16y(y-1)/2)=r(n=x^2+4y(y-1)/2)$ and $r(16n=x^2+16y(y-1)/2)=r(n=x^2+y(y-1)/2)$, by the above we deduce the remaining result. So the theorem is proved.

THEOREM 3.2. *Let $n\in\mathbb{N}$. Then*

$$r(n=x^2+8y(y-1)/2)=\begin{cases} 2\delta(n+1,-1) & \textit{if } 4\mid n,\\ 4\delta((n+1)/2,-1) & \textit{if } 8\mid n-1,\\ 0 & \textit{otherwise},\end{cases}$$

$$r(n=x^2+2y(y-1)/2)=2\delta(4n+1,-1),$$

$$r(n=2x^2+y(y-1)/2)=2\delta(8n+1,-1).$$

Proof. If $n=x^2+8y(y-1)/2$ for some $x,y\in\mathbb{Z}$, then $n\equiv x^2 \pmod 8$ and so $4\mid n$ or $8\mid n-1$. Now assume $4\mid n$ or $8\mid n-1$. By Lemmas 2.1 and 3.1 we have

$$\begin{aligned} r(n=x^2+8y(y-1)/2)&=R([1,0,1],n+1)-R([1,0,4],n+1)\\ &=N(n+1,-4)-N(n+1,-16)\\ &=\begin{cases} 4\delta(n+1,-4)-2\delta(n+1,-4) & \text{if } 4\mid n,\\ 4\delta(n+1,-4)-0 & \text{if } 8\mid n-1.\end{cases}\end{aligned}$$

This yields the result on $r(n=x^2+8y(y-1)/2)$. Since $r(4n=x^2+8y(y-1)/2)=r(n=x^2+2y(y-1)/2)$ and $r(8n=x^2+8y(y-1)/2)=r(n=2x^2+y(y-1)/2)$, from the above we deduce the remaining result. So the theorem is proved.

THEOREM 3.3. *Let $n\in\mathbb{N}$, $m\in\{3,7\}$ and $a\in\{1,2,m,2m\}$. Then*

$$r\Big(n=ax^2+\frac{2m}{a}\cdot\frac{y(y-1)}{2}\Big)=2\sum_{k\mid 4an+m}\Big(\frac{k}{m}\Big).$$

Proof. It is known that $H(-4m)=\{[1,0,m]\}$ and $f(-4m)=2$. Thus, by Lemmas 3.1 and 2.1 we have

$$\begin{aligned} r(n=x^2+2my(y-1)/2)&=R([1,0,m],4n+m)=N(4n+m,-4m)\\ &=2\sum_{k\mid 4n+m}\Big(\frac{-m}{k}\Big)=2\sum_{k\mid 4n+m}\Big(\frac{k}{m}\Big).\end{aligned}$$

Now replacing n with an we obtain the result.

THEOREM 3.4. *Let $a, n \in \mathbb{N}$, $a \mid 30$ and $4an + 15 = 3^\alpha n_0$ $(3 \nmid n_0)$. Then*

$$r\Big(n = ax^2 + \frac{30}{a} \cdot \frac{y(y-1)}{2}\Big) = \Big(1 + (-1)^\alpha \Big(\frac{n_0}{3}\Big)\Big) \sum_{k \mid n_0} \Big(\frac{k}{15}\Big).$$

Proof. If $a = 1$, then by Lemma 3.1, [5, Theorem 9.3] and (2.1) we have

$$\begin{aligned} r(n = x^2 + 30y(y-1)/2) &= R([1,0,15], 4n+15) \\ &= \Big(1 + (-1)^\alpha \Big(\frac{n_0}{3}\Big)\Big) \sum_{k \mid 3^\alpha n_0} \Big(\frac{-15}{k}\Big). \end{aligned}$$

Observe that $\sum_{k \mid 3^\alpha n_0} (\frac{-15}{k}) = \sum_{k \mid n_0} (\frac{k}{15})$. We see that the result is true for $a = 1$. Since $r(an = x^2 + 30y(y-1)/2) = r\big(n = ax^2 + \frac{30}{a} y(y-1)/2\big)$, the result follows.

THEOREM 3.5. *Let $n \in \mathbb{N}$ and $b \in \{3, 5, 11, 29\}$. Then*

$$\begin{aligned} r(n = x^2 + by(y-1)/2) &= 2\delta(8n + b, -2b), \\ r(n = bx^2 + y(y-1)/2) &= 2\delta(8n + 1, -2b). \end{aligned}$$

Proof. By Lemma 3.1 we have $r(n = x^2 + by(y-1)/2) = R([8,0,b], 8n+b)$ and $r(n = bx^2 + y(y-1)/2) = R([1,0,8b], 8n+1)$. Thus applying Lemma 2.3 we deduce the result.

THEOREM 3.6. *Let $n \in \mathbb{N}$ and $a \in \{1, 9\}$. Then*

$$r\Big(n = ax^2 + \frac{9}{a} \cdot \frac{y(y-1)}{2}\Big) = \begin{cases} 2\delta(8n + 9/a, -2) & \text{if } 3 \mid n - a, \\ 2\delta\Big(\dfrac{8n + 9/a}{9}, -2\Big) & \text{if } 9 \mid n - 9/a, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From Lemma 3.1 we have $r(n = x^2 + 9y(y-1)/2) = R([2,0,9], 8n+9)$ and $r(n = 9x^2 + y(y-1)/2) = R([1,0,18], 8n+1)$. Since $H(-72) = \{[1,0,18], [2,0,9]\}$ and $f(-72) = 3$, applying [5, Theorem 9.3] and (2.1) we deduce the result.

LEMMA 3.2 ([4, Theorem 4.2]). *Let $m \in \{5, 7, 13, 17\}$, $i \in \{1, 2, 3, 6\}$, $n \in \mathbb{N}$ and $in = 2^{\alpha_i} 3^{\beta_i} n_0$ with $(6, n_0) = 1$. Then $R([i, 0, 6m/i], n) > 0$ if and only if $2 \mid \mathrm{ord}_q n_0$ for every prime q with $(\frac{-6m}{q}) = -1$ and*

$$n_0 \equiv \begin{cases} 1, 6m+1 \pmod{24} & \text{if } 2 \mid \alpha_i \text{ and } 2 \mid \beta_i, \\ 2m+3, 8m+3 \pmod{24} & \text{if } 2 \mid \alpha_i \text{ and } 2 \nmid \beta_i, \\ 3m+2, 3m+8 \pmod{24} & \text{if } 2 \nmid \alpha_i \text{ and } 2 \mid \beta_i, \\ m, m+6 \pmod{24} & \text{if } 2 \nmid \alpha_i \text{ and } 2 \nmid \beta_i. \end{cases}$$

Moreover, if $R([i, 0, 6m/i], n) > 0$, then

$$R([i, 0, 6m/i], n) = 2 \prod_{(\frac{-6m}{p}) = 1} (1 + \mathrm{ord}_p n_0).$$

THEOREM 3.7. *Let $m \in \{5, 7, 13, 17\}$, $a \in \{1, 3, m, 3m\}$, $n \in \mathbb{N}$ and $8an + 3m = 3^{\beta} n_0$ with $3 \nmid n_0$. Then $r\big(n = ax^2 + \frac{3m}{a} y(y-1)/2\big) > 0$ if and only if $2 \mid \mathrm{ord}_q n_0$ for every prime q with $(\frac{-6m}{q}) = -1$ and*

$$n_0 \equiv \begin{cases} 1 \pmod 3 & \text{if } m \in \{7, 13\} \text{ and } 2 \nmid \beta, \\ 2 \pmod 3 & \text{otherwise.} \end{cases}$$

Moreover, if the above conditions hold, then $r\big(n = ax^2 + \frac{3m}{a} y(y-1)/2\big) = 2\prod_{(\frac{-6m}{p})=1}(1 + \mathrm{ord}_p n_0)$.

Proof. Clearly $n_0 \equiv (2 + (-1)^{\beta})m \pmod 8$. By Lemma 3.1 we have $r(n = x^2 + 3my(y-1)/2) = R([2, 0, 3m], 8n + 3m)$. Now applying Lemma 3.2 we deduce the result for $a = 1$. Noting that $r(an = x^2 + 3my(y-1)/2) = r\big(n = ax^2 + \frac{3m}{a} y(y-1)/2\big)$ we deduce the remaining result.

LEMMA 3.3 ([4, Theorem 4.3]). *Let $m \in \{7, 13, 19\}$, $i \in \{1, 2, 5, 10\}$, $n \in \mathbb{N}$ and $in = 2^{\alpha_i} 5^{\beta_i} n_0$ with $(10, n_0) = 1$. Then $R([i, 0, 10m/i], n) > 0$ if and only if $2 \mid \mathrm{ord}_q n_0$ for every prime q with $(\frac{-10m}{q}) = -1$ and*

$$n_0 \equiv \begin{cases} 1, 9, 1 + 10m, 9 + 10m \pmod{40} & \text{if } 2 \mid \alpha_i \text{ and } 2 \mid \beta_i, \\ 5 + 2m, 5 + 8m, 5 + 18m, 5 + 32m \pmod{40} & \text{if } 2 \mid \alpha_i \text{ and } 2 \nmid \beta_i, \\ 5m + 2, 5m + 8, 5m + 18, 5m + 32 \pmod{40} & \text{if } 2 \nmid \alpha_i \text{ and } 2 \mid \beta_i, \\ m, 9m, 10 + m, 10 + 9m \pmod{40} & \text{if } 2 \nmid \alpha_i \text{ and } 2 \nmid \beta_i. \end{cases}$$

Moreover, if $R([i, 0, 10m/i], n) > 0$, then

$$R([i, 0, 10m/i], n) = 2 \prod_{(\frac{-10m}{p})=1} (1 + \mathrm{ord}_p n_0).$$

THEOREM 3.8. *Let $m \in \{7, 13, 19\}$, $a \in \{1, 5, m, 5m\}$, $n \in \mathbb{N}$ and $8an + 5m = 5^{\beta} n_0$ with $5 \nmid n_0$. Then $r\big(n = ax^2 + \frac{5m}{a} y(y-1)/2\big) > 0$ if and only if $2 \mid \mathrm{ord}_q n_0$ for every prime q with $(\frac{-10m}{q}) = -1$ and*

$$n_0 \equiv \begin{cases} \pm 1 \pmod 5 & \text{if } m = 19 \text{ and } 2 \nmid \beta, \\ \pm 2 \pmod 5 & \text{otherwise.} \end{cases}$$

Moreover, if the above conditions hold, then $r\big(n = ax^2 + \frac{5m}{a} y(y-1)/2\big) = 2\prod_{(\frac{-10m}{p})=1}(1 + \mathrm{ord}_p n_0)$.

Proof. Clearly $n_0 \equiv 5m$ or $m \pmod 8$ according as $2 \mid \beta$ or $2 \nmid \beta$. By Lemma 3.1 we have $r(n = x^2 + 5my(y-1)/2) = R([2, 0, 5m], 8n + 5m)$. Now applying Lemma 3.3 we deduce the result for $a = 1$. Noting that $r(an = x^2 + 5my(y-1)/2) = r\big(n = ax^2 + \frac{5m}{a} y(y-1)/2\big)$ we deduce the remaining result.

Theorem 3.9. *Let $n \in \mathbb{N}$. Then*

$$r(n = x^2 + 10y(y-1)/2) = \sum_{k \mid 4n+5} (-1)^{(k-1)/2}\left(\frac{k}{5}\right) - \phi_4(4n+5),$$

where $\phi_4(m)$ is given by

$$q\prod_{k=1}^{\infty}(1-q^{4k})(1-q^{20k}) = \sum_{m=1}^{\infty}\phi_4(m)q^m \quad (|q|<1)$$

or by [6, Theorem 4.5(ii)].

Proof. It is known that $f(-80) = 2$ and $H(-80) = \{[1,0,20],[4,0,5],[3,2,7],[3,-2,7]\}$. Thus

$$R([1,0,20],4n+5)+R([4,0,5],4n+5) = N(4n+5,-80)-2R([3,2,7],4n+5).$$

If $4n+5 = 3x^2+2xy+7y^2$ for some $x,y \in \mathbb{Z}$, then $4n+5 \equiv -x^2+2xy-y^2 = -(x-y)^2 \pmod 4$. This is impossible. Thus $R([3,2,7],4n+5) = 0$. Hence, using the above and Lemma 2.1 we see that

$$\begin{aligned} &R([1,0,20],4n+5) + R([4,0,5],4n+5) \\ &\quad = N(4n+5,-80) = 2\sum_{k\mid 4n+5}\left(\frac{-20}{k}\right) = 2\sum_{k\mid 4n+5}(-1)^{(k-1)/2}\left(\frac{k}{5}\right). \end{aligned}$$

On the other hand, by [6, Theorem 2.2] we have

$$R([1,0,20],4n+5) - R([4,0,5],4n+5) = 2\phi_4(4n+5).$$

Hence

$$R([4,0,5],4n+5) = \sum_{k\mid 4n+5}(-1)^{(k-1)/2}\left(\frac{k}{5}\right) - \phi_4(4n+5).$$

By Lemma 3.1 we have

$$r(n = x^2 + 10y(y-1)/2) = R([8,0,10],8n+10) = R([4,0,5],4n+5).$$

Thus the result follows.

4. Formulas for $r(n = ax^2 + b(3y^2 - y)/2)$

Lemma 4.1. *Let $a,b,n \in \mathbb{N}$. Then*

$$\begin{aligned} &2r(n = ax^2 + b(3y^2-y)/2) \\ &\qquad = R([24a,0,b],24n+b) - R([24a,0,4b],24n+b) \\ &\qquad\quad - R([24a,0,9b],24n+b) + R([24a,0,36b],24n+b). \end{aligned}$$

In particular, if $3 \nmid b$ and $8 \nmid b$, we have

$$2r(n = ax^2 + b(3y^2-y)/2) = R([24a,0,b],24n+b).$$

Moreover, if $4 \nmid a$, $2 \nmid b$ *and* $3 \nmid b$, *then*

$$2r(n = ax^2 + b(3y^2 - y)/2) = R([6a, 0, b], 24n + b).$$

Proof. Clearly

$$\begin{aligned}
2r(n = ax^2 + b(3y^2 - y)/2) &= 2r(24n + b = 24ax^2 + b(6y-1)^2) \\
&= |\{\langle x, y\rangle \in \mathbb{Z}^2 : 24n + b = 24ax^2 + by^2,\ 2 \nmid y,\ 3 \nmid y\}| \\
&= |\{\langle x, y\rangle \in \mathbb{Z}^2 : 24n + b = 24ax^2 + by^2,\ 2 \nmid y\}| \\
&\quad - |\{\langle x, y\rangle \in \mathbb{Z}^2 : 24n + b = 24ax^2 + b(3y)^2,\ 2 \nmid y\}| \\
&= R([24a, 0, b], 24n + b) - R([24a, 0, 4b], 24n + b) \\
&\quad - (R([24a, 0, 9b], 24n + b) - R([24a, 0, 36b], 24n + b)).
\end{aligned}$$

If $3 \nmid b$ and $8 \nmid b$, we then have $2r(n = ax^2 + b(3y^2 - y)/2) = R([24a, 0, b], 24n+b)$. If $4 \nmid a$, $2 \nmid b$ and $3 \nmid b$, by Lemma 2.2 we have $R([24a, 0, b], 24n+b) = R([6a, 0, b], 24n + b)$. So the lemma is proved.

Theorem 4.1. *Let* $n \in \mathbb{N}$. *Then*

$$r(n = x^2 + 4(3y^2 - y)/2) = \delta(6n + 1, -6),$$
$$r(n = x^2 + (3y^2 - y)/2) = \delta(24n + 1, -6).$$

Proof. As $H(-24) = \{[1, 0, 6], [2, 0, 3]\}$ and $m = 2x^2 + 3y^2$ implies $m \not\equiv 1 \pmod 6$, using Lemma 2.1 we see that $R([24, 0, 4], 24n + 4) = R([1, 0, 6], 6n + 1) = N(6n + 1, -24) = 2\delta(6n + 1, -6)$. Now putting $a = 1$ and $b = 4$ in Lemma 4.1 and applying the above we obtain $r(n = x^2 + 4(3y^2 - y)/2) = \delta(6n + 1, -6)$. Replacing n by $4n$ we deduce the remaining result.

Theorem 4.2. *Let* $n \in \mathbb{N}$. *Then*

$$r(n = x^2 + 8(3y^2 - y)/2) = \begin{cases} \delta(3n + 1, -3) & \text{if } 4 \mid n, \\ 2\delta((3n + 1)/4, -3) & \text{if } 8 \mid n - 1, \\ 0 & \text{otherwise,} \end{cases}$$
$$r(n = x^2 + 3y^2 - y) = \delta(12n + 1, -3).$$

Proof. If $n = x^2 + 4(3y^2 - y)$ for some $x, y \in \mathbb{Z}$, then clearly $4 \mid n$ or $8 \mid n - 1$. Now assume $4 \mid n$ or $8 \mid n - 1$. From Lemma 4.1 we have

$$\begin{aligned}
2r(n = x^2 + 8(3y^2 - y)/2) &= R([24, 0, 8], 24n + 8) - R([24, 0, 32], 24n + 8) \\
&= R([1, 0, 3], 3n + 1) - R([3, 0, 4], 3n + 1) \\
&= N(3n + 1, -12) - R([3, 0, 4], 3n + 1).
\end{aligned}$$

If $4 \mid n$, then $3n + 1 = 3x^2 + 4y^2$ is insolvable. Thus, by Lemma 2.1 we have

$$N(3n + 1, -12) - R([3, 0, 4], 3n + 1) = N(3n + 1, -12) = 2\delta(3n + 1, -3).$$

If $8\mid n-1$, then $3n+1\equiv 4 \pmod 8$. By Lemma 2.1, [5, Theorem 9.3] and (2.1) we have

$$\begin{aligned}&N(3n+1,-12)-R([3,0,4],3n+1)\\&\qquad=6\delta((3n+1)/4,-3)-2\delta((3n+1)/4,-3)=4\delta((3n+1)/4,-3).\end{aligned}$$

Now combining the above we obtain the result for $r(n=x^2+4(3y^2-y))$. As $r(4n=x^2+4(3y^2-y))=r(n=x^2+3y^2-y)$, applying the above we deduce the remaining result.

THEOREM 4.3. *Let $n\in\mathbb{N}$ and $b\in\{5,7,13,17\}$. Then*

$$r(n=x^2+b(3y^2-y)/2)=\delta(24n+b,-6b),$$
$$r(n=bx^2+(3y^2-y)/2)=\delta(24n+1,-6b).$$

Proof. From Lemma 4.1 we have

$$2r(n=x^2+b(3y^2-y)/2)=R([b,0,6],24n+b).$$

It is easily seen that $H(-24b)=\{[1,0,6b],[2,0,3b],[3,0,2b],[6,0,b]\}$ and $24n+b$ cannot be represented by x^2+6by^2, $2x^2+3by^2$ and $3x^2+2by^2$. Thus, using the fact that $f(-24b)=1$ and Lemma 2.1 we deduce

$$R([b,0,6],24n+b)=N(24n+b,-24b)=2\delta(24n+b,-24b).$$

Now combining the above we obtain $r(n=x^2+b(3y^2-y)/2)=\delta(24n+b,$ $-6b)$. Replacing n with bn we deduce the remaining result.

THEOREM 4.4. *Let $n\in\mathbb{N}$. Then*

$$\begin{aligned}&r(n=x^2+3(3y^2-y)/2)\\&\qquad=\begin{cases}2\delta(n_0,-2) & \text{if } 9\mid n-1 \text{ and } 8n+1=3^{\alpha}n_0\ (3\nmid n_0),\\ \delta(8n+1,-2) & \text{if } 3\mid n \text{ or } n\equiv 4,7 \pmod 9,\\ 0 & \text{if } n\equiv 2 \pmod 3\end{cases}\end{aligned}$$

and

$$r(n=3x^2+(3y^2-y)/2)=\delta(24n+1,-2).$$

Proof. From Lemmas 4.1 and 2.2 we see that

$$\begin{aligned}2r(n=x^2+3(3y^2-y)/2)&=R([24,0,3],24n+3)-R([24,0,27],24n+3)\\&=R([8,0,1],8n+1)-R([8,0,9],8n+1)\\&=R([2,0,1],8n+1)-R([2,0,9],8n+1).\end{aligned}$$

Since $H(-8)=\{[1,0,2]\}$, by Lemma 2.1 we have $R([2,0,1],8n+1)=2\delta(8n+1,-2)$. It is known that $H(-72)=\{[1,0,18],[2,0,9]\}$ and $f(-72)$ $=3$. Thus, by [5, Theorem 9.3] and (2.1) we have

$$R([2,0,9],8n+1)=\begin{cases}2\delta(8n+1,-2) & \text{if } 3\mid n-2,\\ 2\delta((8n+1)/9,-2) & \text{if } 9\mid n-1,\\ 0 & \text{otherwise.}\end{cases}$$

When $9 \mid n-1$ and $8n+1 = 3^\alpha n_0$ $(3 \nmid n_0)$, we have

$$\sum_{k \mid 8n+1} \left(\frac{-2}{k}\right) - \sum_{k \mid \frac{8n+1}{9}} \left(\frac{-2}{k}\right) = \sum_{k \mid n_0} \left(\left(\frac{-2}{3^\alpha k}\right) + \left(\frac{-2}{3^{\alpha-1} k}\right)\right) = 2 \sum_{k \mid n_0} \left(\frac{-2}{k}\right).$$

Now combining all the above we obtain the result for $r(n = x^2 + 3(3y^2 - y)/2)$. Note that $r(3n = x^2 + 3(3y^2 - y)/2) = r(n = 3x^2 + (3y^2 - y)/2)$. We then obtain the remaining result.

5. Formulas for $r(n = a(x^2 - x)/2 + b(3y^2 - y)/2)$

Lemma 5.1. *Let $a, b, n \in \mathbb{N}$. Then*

$$\begin{aligned}
&2r(n = a(x^2 - x)/2 + b(3y^2 - y)/2) \\
&\quad = R([12a, 12a, 3a + b], 24n + 3a + b) - R([12a, 0, 4b], 24n + 3a + b) \\
&\qquad - R([12a, 12a, 3a + 9b], 24n + 3a + b) \\
&\qquad + R([12a, 0, 36b], 24n + 3a + b).
\end{aligned}$$

If $4 \nmid a$, we also have

$$\begin{aligned}
&2r(n = a(x^2 - x)/2 + b(3y^2 - y)/2) \\
&\qquad = R([3a, 0, b], 24n + 3a + b) - R([3a, 0, 4b], 24n + 3a + b) \\
&\qquad\quad - R([3a, 0, 9b], 24n + 3a + b) + R([3a, 0, 36b], 24n + 3a + b).
\end{aligned}$$

Proof. Clearly

$$\begin{aligned}
&2r(n = a(x^2 - x)/2 + b(3y^2 - y)/2) \\
&\quad = 2r(24n + 3a + b = 3a(2x - 1)^2 + b(6y - 1)^2) \\
&\quad = |\{\langle x, y\rangle \in \mathbb{Z}^2 : 24n + 3a + b = 3ax^2 + by^2, 2 \nmid xy, 3 \nmid y\}| \\
&\quad = |\{\langle x, y\rangle \in \mathbb{Z}^2 : 24n + 3a + b = 3ax^2 + by^2, 2 \nmid xy\}| \\
&\qquad - |\{\langle x, y\rangle \in \mathbb{Z}^2 : 24n + 3a + b = 3ax^2 + b(3y)^2, 2 \nmid xy\}| \\
&\quad = |\{\langle x, y\rangle \in \mathbb{Z}^2 : 24n + 3a + b = 3a(2x + y)^2 + by^2, 2 \nmid y\}| \\
&\qquad - |\{\langle x, y\rangle \in \mathbb{Z}^2 : 24n + 3a + b = 3a(2x + y)^2 + 9by^2, 2 \nmid y\}| \\
&\quad = |\{\langle x, y\rangle \in \mathbb{Z}^2 : 24n + 3a + b = 12ax^2 + 12axy + (3a + b)y^2, 2 \nmid y\}| \\
&\qquad - |\{\langle x, y\rangle \in \mathbb{Z}^2 : 24n + 3a + b = 12ax^2 + 12axy + (3a + 9b)y^2, 2 \nmid y\}| \\
&\quad = R([12a, 12a, 3a + b], 24n + 3a + b) \\
&\qquad - R([12a, 24a, 4(3a + b)], 24n + 3a + b) \\
&\qquad - (R([12a, 12a, 3a + 9b], 24n + 3a + b) \\
&\qquad - R([12a, 24a, 4(3a + 9b)], 24n + 3a + b)).
\end{aligned}$$

Using (1.1) we see that $[12a, 24a, 4(3a+b)] = [12a, 0, 4b]$ and $[12a, 24a, 4(3a+9b)] = [12a, 0, 36b]$. Thus the first part follows.

Now assume $4 \nmid a$. If $24n+3a+b = 3ax^2 + b(6y-1)^2$ for some $x, y \in \mathbb{Z}$, then clearly $2 \nmid x$. Thus,

$$\begin{aligned}
2r(n &= a(x^2-x)/2 + b(3y^2-y)/2) \\
&= 2r(24n+3a+b = 3a(2x-1)^2 + b(6y-1)^2) \\
&= 2r(24n+3a+b = 3ax^2 + b(6y-1)^2) \\
&= |\{\langle x,y\rangle \in \mathbb{Z}^2 : 24n+3a+b = 3ax^2+by^2, 2\nmid y, 3 \nmid y\}| \\
&= |\{\langle x,y\rangle \in \mathbb{Z}^2 : 24n+3a+b = 3ax^2+by^2, 2\nmid y\}| \\
&\quad - |\{\langle x,y\rangle \in \mathbb{Z}^2 : 24n+3a+b = 3ax^2+b(3y)^2, 2\nmid y\}| \\
&= R([3a,0,b], 24n+3a+b) - R([3a,0,4b], 24n+3a+b) \\
&\quad - (R([3a,0,9b], 24n+3a+b) - R([3a,0,36b], 24n+3a+b)).
\end{aligned}$$

This completes the proof.

Theorem 5.1. *Let $n \in \mathbb{N}$. Then*

$$r(n = (x^2-x)/2 + (3y^2-y)/2) = 2\delta(6n+1, -3).$$

Proof. From Lemmas 5.1 and 2.1 we see that

$$\begin{aligned}
2r(n = (x^2-x)/2 &+ (3y^2-y)/2) \\
&= R([12,12,4], 24n+4) - R([12,0,4], 24n+4) \\
&= R([3,3,1], 6n+1) - R([3,0,1], 6n+1) \\
&= N(6n+1, -3) - N(6n+1, -12) \\
&= 6\delta(6n+1,-3) - 2\delta(6n+1,-3).
\end{aligned}$$

This yields the result.

Theorem 5.2. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned}
r(n = x^2 - x + (3y^2-y)/2) &= \delta(24n+7, -6), \\
r(n = (x^2-x)/2 + 3y^2 - y) &= \delta(24n+5, -6).
\end{aligned}$$

Proof. For $x, y \in \mathbb{Z}$ it is clear that $x^2 + 6y^2 \not\equiv 5 \pmod 6$ and $2x^2 + 3y^2 \not\equiv 1 \pmod 6$. Since $H(-24) = \{[1,0,6],[2,0,3]\}$ and $f(-24) = 1$, using Lemmas 5.1 and 2.1 we have

$$\begin{aligned}
2r(n = x^2 - x + (3y^2-y)/2) &= R([6,0,1], 24n+7) \\
&= N(24n+7, -24) = 2\delta(24n+7, -24)
\end{aligned}$$

and

$$\begin{aligned}2r(n&=(x^2-x)/2+3y^2-y)\\&=R([3,0,2],24n+5)-R([3,0,8],24n+5)=R([2,0,3],24n+5)\\&=N(24n+5,-24)=2\delta(24n+5,-24).\end{aligned}$$

This yields the result.

THEOREM 5.3. *Let $n\in\mathbb{N}$. Then*

$$r(n=(x^2-x)/2+3(3y^2-y)/2)=\begin{cases}\delta(4n+1,-1) & \text{if } 3\mid n-1,\\ 2\delta(4n+1,-1) & \text{if } 3\nmid n-1 \text{ and } 9\nmid n-2,\\ 0 & \text{if } 9\mid n-2.\end{cases}$$

Proof. Putting $a=1$ and $b=3$ in Lemma 5.1 we see that

$$\begin{aligned}2r(n&=(x^2-x)/2+3(3y^2-y)/2)\\&=R([12,12,6],24n+6)-R([12,12,30],24n+6)\\&=R([2,2,1],4n+1)-R([2,2,5],4n+1).\end{aligned}$$

By Lemma 2.1 we have $R([2,2,1],4n+1)=N(4n+1,-4)=4\delta(4n+1,-1)$. As $H(-36)=\{[1,0,9],[2,2,5]\}$, by [5, Theorem 9.3] and (2.1) we have

$$R([2,2,5],4n+1)=\begin{cases}2\delta(4n+1,-1) & \text{if } 3\mid n-1,\\ 4\delta((4n+1)/9,-1) & \text{if } 9\mid n-2,\\ 0 & \text{otherwise.}\end{cases}\tag{5.1}$$

If $9\mid n-2$ and $4n+1=3^\alpha n_0$ $(3\nmid n_0)$, then

$$\begin{aligned}&\delta(4n+1,-1)-\delta((4n+1)/9,-1)\\&\quad=\sum_{k\mid 4n+1,\,k\nmid(4n+1)/9}\left(\frac{-1}{k}\right)=\sum_{k\mid n_0}\left(\left(\frac{-1}{3^{\alpha-1}k}\right)+\left(\frac{-1}{3^\alpha k}\right)\right)=0.\end{aligned}$$

Now putting all the above together we obtain the result.

THEOREM 5.4. *Let $n\in\mathbb{N}$. Then*

$$r(n=3(x^2-x)/2+(3y^2-y)/2)=\delta(12n+5,-1).$$

Proof. Putting $a=3$ and $b=1$ in Lemma 5.1 we have

$$\begin{aligned}2r(n&=3(x^2-x)/2+(3y^2-y)/2)\\&=R([36,36,10],24n+10)=R([18,18,5],12n+5).\end{aligned}$$

By (1.1), we have $[18,18,5]=[5,-18,18]=[5,2,2]=[2,-2,5]$. Thus $R([18,18,5],12n+5)=R([2,2,5],12n+5)$. Now applying (5.1) we deduce the result.

THEOREM 5.5. *Let $n\in\mathbb{N}$, $a\in\{1,5\}$ and $3n+(a+3)/4=2^{\alpha}n_0$ $(2\nmid n_0)$. Then*

$$r\Big(n=a\frac{x^2-x}{2}+\frac{5}{a}\cdot\frac{3y^2-y}{2}\Big)=2\sum_{k\mid n_0}\Big(\frac{k}{15}\Big).$$

Proof. Set $b=5/a$. It is known that $H(-60)=\{[1,0,15],[3,0,5]\}$ and $f(-60)=2$. For $x,y\in\mathbb{Z}$ we see that $x^2+15y^2\not\equiv 2 \pmod 3$ and $3x^2+5y^2\not\equiv 1 \pmod 3$. Thus, using Lemmas 5.1 and 2.1 we have

$$\begin{aligned}
&2r(n=a(x^2-x)/2+b(3y^2-y)/2)\\
&\quad=R([3a,0,b],24n+3a+b)-R([3a,0,4b],24n+3a+b)\\
&\quad=R([3a,0,b],24n+3a+b)-R([12a,0,4b],24n+3a+b)\\
&\quad=R([3a,0,b],24n+3a+b)-R([3a,0,b],6n+(3a+b)/4)\\
&\quad=N(24n+3a+b,-60)-N(6n+(3a+b)/4,-60)\\
&\quad=2\sum_{k\mid 6n+(3a+b)/4}\Big(\frac{-15}{k}\Big)-(1+(-1)^{n+(a+3)/4})\sum_{k\mid(6n+(3a+b)/4)/4}\Big(\frac{-15}{k}\Big)\\
&\quad=2\sum_{k\mid n_0}\Big(\Big(\frac{-15}{2^{\alpha}k}\Big)+\Big(\frac{-15}{2^{\alpha+1}k}\Big)\Big)=4\sum_{k\mid n_0}\Big(\frac{-15}{k}\Big).
\end{aligned}$$

This yields the result.

THEOREM 5.6. *Let $n\in\mathbb{N}$ and $b\in\{7,11,19,31,59\}$. Then*

$$r(n=(x^2-x)/2+b(3y^2-y)/2)=\delta(12n+(b+3)/2,-3b),$$
$$r(n=b(x^2-x)/2+(3y^2-y)/2)=\delta(12n+(3b+1)/2,-3b).$$

Proof. As $b\equiv 3 \pmod 4$ we see that $3+b\equiv 3b+1\equiv 2 \pmod 4$. Thus, $R([3,0,4b],24n+3+b)=0$ and $R([3b,0,4],24n+3b+1)=0$. Hence, using Lemma 5.1 we get

$$\begin{aligned}
2r(n&=(x^2-x)/2+b(3y^2-y)/2)\\
&=R([3,0,b],24n+3+b)-R([3,0,4b],24n+3+b)\\
&\quad-R([3,0,9b],24n+3+b)+R([3,0,36b],24n+3+b)\\
&=R([3,0,b],24n+3+b)
\end{aligned}$$

and

$$\begin{aligned}
2r(n&=b(x^2-x)/2+(3y^2-y)/2)\\
&=R([3b,0,1],24n+3b+1)-R([3b,0,4],24n+3b+1)\\
&\quad-R([3b,0,9],24n+3b+1)+R([3b,0,36],24n+3b+1)\\
&=R([1,0,3b],24n+3b+1).
\end{aligned}$$

It is known that we have $H(-12b) = \{[1,0,3b], [3,0,b], [2,2,(3b+1)/2], [6,6,(b+3)/2]\}$ and $f(-12b) = 1$. One can easily see that $24n+3+b$ cannot be represented by x^2+3by^2, $2x^2+2xy+\frac{3b+1}{2}y^2$ and $6x^2+6xy+\frac{b+3}{2}y^2$, and $24n+3b+1$ cannot be represented by $3x^2+by^2$, $2x^2+2xy+\frac{3b+1}{2}y^2$ and $6x^2+6xy+\frac{b+3}{2}y^2$. Thus, applying Lemma 2.1 we have

$$\begin{aligned} R([3,0,b], 24n+3+b) &= N(24n+3+b, -12b) \\ &= 2\delta(24n+3+b, -12b) \\ &= 2\delta(12n+(b+3)/2, -3b) \end{aligned}$$

and

$$\begin{aligned} R([1,0,3b], 24n+3b+1) &= N(24n+3b+1, -12b) \\ &= 2\delta(24n+3b+1, -12b) \\ &= 2\delta(12n+(3b+1)/2, -3b). \end{aligned}$$

Now combining all the above we deduce the result.

Theorem 5.7. *Let $n \in \mathbb{N}$, $m \in \{5,7,13,17\}$ and $a \in \{1,2,m,2m\}$. Then*

$$r\left(n = a\frac{x^2-x}{2} + \frac{2m}{a}\cdot\frac{3y^2-y}{2}\right) = \sum_{k \mid 24n+3a+2m/a} \left(\frac{-6m}{k}\right).$$

Proof. If $24n+3a+2m/a = 3ax^2 + \frac{8m}{a}y^2$ for some $x, y \in \mathbb{Z}$, then $2 \nmid x$ and so $\frac{2m}{a}(1-4y^2) = \frac{2m}{a} - \frac{8m}{a}y^2 \equiv 0 \pmod 8$. This is impossible. So $R([3a,0,8m/a], 24n+3a+2m/a) = 0$. Hence, using Lemma 5.1 we see that

$$2r\left(n = a\frac{x^2-x}{2} + \frac{2m}{a}\cdot\frac{3y^2-y}{2}\right) = R([3a,0,2m/a], 24n+3a+2m/a).$$

It is known that $H(-24m) = \{[1,0,6m], [2,0,3m], [3,0,2m], [6,0,m]\}$ and $f(-24m) = 1$. Let $n' \in \mathbb{N}$ with $(n',6) = 1$. It is easily seen that

$$\begin{aligned} n' = x^2+6my^2 &\Rightarrow n' \equiv 1, 1+6m \pmod{24}, \\ n' = 2x^2+3my^2 &\Rightarrow n' \equiv 3m+2, 3m+8 \pmod{24}, \\ n' = 3x^2+2my^2 &\Rightarrow n' \equiv 2m+3, 8m+3 \pmod{24}, \\ n' = 6x^2+my^2 &\Rightarrow n' \equiv m, m+6 \pmod{24}. \end{aligned}$$

Since $1, 1+6m, 3m+2, 3m+8, 2m+3, 8m+3, m, m+6$ are distinct modulo 24, we see that n' is represented by at most one class in $H(-24m)$. As $(24n+3a+2m/a, 6) = 1$, we see that $24n+3a+2m/a$ cannot be represented by any class $K \in H(-24m)$ with $K \neq [3a,0,2m/a]$. From the above and Lemma 2.1 we derive

$$\begin{aligned}2r\Big(n=a\frac{x^2-x}{2}+\frac{2m}{a}\cdot\frac{3y^2-y}{2}\Big)&=R([3a,0,2m/a],24n+3a+2m/a)\\&=N(24n+3a+2m/a,-24m)\\&=2\delta(24n+3a+2m/a,-24m).\end{aligned}$$

So the theorem is proved.

Theorem 5.8. *Let $n\in\mathbb{N}$. Then*

$$r(n=15(x^2-x)/2+(3y^2-y)/2)=\delta(12n+23,-5),$$
$$r(n=3(x^2-x)/2+5(3y^2-y)/2)=\delta(12n+7,-5).$$

Proof. Suppose $12n+23=5^\alpha n_1$ with $5\nmid n_1$. From Lemma 5.1 and the fact that $24n+46\equiv 2\pmod 4$ we have $R([45,0,4],24n+46)=0$ and so

$$\begin{aligned}&2r(n=15(x^2-x)/2+(3y^2-y)/2)\\&\quad=R([45,0,1],24n+46)-R([45,0,4],24n+46)\\&\qquad-R([45,0,9],24n+46)+R([45,0,36],24n+46)\\&\quad=R([45,0,1],24n+46)=R([1,0,45],5^\alpha\cdot 2n_1).\end{aligned}$$

Since $24n+46\equiv 1\pmod 3$, applying the above and Lemma 2.4 we see that

$$r\Big(n=15\frac{x^2-x}{2}+\frac{3y^2-y}{2}\Big)=\begin{cases}\delta(2n_1,-20)&\text{if } n_1\equiv\pm2\pmod 5,\\0&\text{if } n_1\equiv\pm1\pmod 5.\end{cases}$$

If $n_1\equiv\pm1\pmod 5$, as $n_1\equiv 5^\alpha n_1=12n+23\equiv 3\pmod 4$ we see that $(\frac{-20}{n_1})=(\frac{-5}{n_1})=-(\frac{5}{n_1})=-(\frac{n_1}{5})=-1$ and so $\delta(n_1,-20)=0$ by (2.1). Since $\delta(2n_1,-20)=\delta(n_1,-20)$, we always have

$$r\Big(n=15\frac{x^2-x}{2}+\frac{3y^2-y}{2}\Big)=\sum_{k\mid n_1}\Big(\frac{-20}{k}\Big)=\sum_{k\mid 12n+23}\Big(\frac{-5}{k}\Big).$$

Now suppose $12n+7=5^\beta n_2$ with $5\nmid n_2$. From Lemma 5.1 and the fact that $24n+14\equiv 2\pmod 4$ we have $R([9,0,20],24n+14)=0$ and so

$$\begin{aligned}&2r(n=3(x^2-x)/2+5(3y^2-y)/2)\\&\quad=R([9,0,5],24n+14)-R([9,0,20],24n+14)\\&\qquad-R([9,0,45],24n+14)+R([9,0,180],24n+14)\\&\quad=R([9,0,5],24n+14)=R([5,0,9],5^\beta\cdot 2n_2).\end{aligned}$$

Since $24n+14\equiv 2\pmod 3$, applying the above and Lemma 2.4 we see that

$$r\Big(n=3\frac{x^2-x}{2}+5\frac{3y^2-y}{2}\Big)=\begin{cases}\delta(2n_2,-20)&\text{if } n_2\equiv\pm2\pmod 5,\\0&\text{if } n_2\equiv\pm1\pmod 5.\end{cases}$$

If $n_2\equiv\pm1\pmod 5$, as $n_2\equiv 5^\alpha n_2=12n+7\equiv 3\pmod 4$ we see that $(\frac{-20}{n_2})=(\frac{-5}{n_2})=-(\frac{5}{n_2})=-(\frac{n_2}{5})=-1$ and so $\sum_{k\mid n_2}(\frac{-20}{k})=0$ by (2.1).

Since $\delta(2n_2,-20)=\delta(n_2,-20)$, we always have

$$r\left(n=3\frac{x^2-x}{2}+5\frac{3y^2-y}{2}\right)=\sum_{k\mid n_2}\left(\frac{-20}{k}\right)=\sum_{k\mid 12n+7}\left(\frac{-5}{k}\right).$$

The proof is now complete.

THEOREM 5.9. *Let $n\in\mathbb{N}$ and $4n+3=5^\alpha n_0$ $(5\nmid n_0)$. If $n_0\equiv\pm1 \pmod 5$, then*

$$r\left(n=\frac{x^2-x}{2}+15\frac{3y^2-y}{2}\right)=r\left(n=5\frac{x^2-x}{2}+3\frac{3y^2-y}{2}\right)=0.$$

If $n_0\equiv\pm2 \pmod 5$, then

$$r\left(n=\frac{x^2-x}{2}+15\frac{3y^2-y}{2}\right)=\begin{cases}2\delta(n_1,-5) & \text{if } 9\mid n-6,\\ 0 & \text{if } 3\mid n-2,\\ \delta(n_0,-5) & \text{otherwise,}\end{cases}$$

$$r\left(n=5\frac{x^2-x}{2}+3\frac{3y^2-y}{2}\right)=\begin{cases}2\delta(n_1,-5) & \text{if } 9\mid n-6,\\ 0 & \text{if } 3\mid n-1,\\ \delta(n_0,-5) & \text{otherwise,}\end{cases}$$

where n_1 is given by $n_0=3^\beta n_1$ $(3\nmid n_1)$.

Proof. By Lemma 5.1 and the fact that $24n+18\equiv 2 \pmod 4$ we have

$$\begin{aligned}2r(n=(x^2-x)/2+15(3y^2-y)/2)&\\ &=R([3,0,15],24n+18)-R([3,0,60],24n+18)\\ &\quad -R([3,0,135],24n+18)+R([3,0,540],24n+18)\\ &=R([3,0,15],24n+18)-R([3,0,135],24n+18)\\ &=R([1,0,5],8n+6)-R([1,0,45],8n+6)\end{aligned}$$

and

$$\begin{aligned}2r(n=5(x^2-x)/2+3(3y^2-y)/2)&\\ &=R([15,0,3],24n+18)-R([15,0,12],24n+18)\\ &\quad -R([15,0,27],24n+18)+R([15,0,108],24n+18)\\ &=R([15,0,3],24n+18)-R([15,0,27],24n+8)\\ &=R([1,0,5],8n+6)-R([5,0,9],8n+6).\end{aligned}$$

For $m\in\mathbb{N}$, $5m=x^2+5y^2$ $(x,y\in\mathbb{Z})$ implies $5\mid x$ and $m=5(x/5)^2+y^2$. Thus $R([1,0,5],5m)=R([1,0,5],m)$. Hence

$$\begin{aligned}R([1,0,5],8n+6)&=R([1,0,5],5^\alpha\cdot 2n_0)=R([1,0,5],5^{\alpha-1}\cdot 2n_0)\\ &=\cdots=R([1,0,5],2n_0).\end{aligned}$$

If $n_0 \equiv \pm 1 \pmod 5$, then $x^2 \equiv 2n_0 \pmod 5$ is insolvable and so

$$R([1,0,5], 8n+6) = R([1,0,5], 2n_0) = 0.$$

Hence, by the above we have $r(n = (x^2-x)/2 + 15(3y^2-y)/2) = 0$ and $r(n = 5(x^2-x)/2 + 3(3y^2-y)/2) = 0$.

Now we assume $n_0 \equiv \pm 2 \pmod 5$. Then $2n_0 \equiv \pm 4 \pmod 5$ and so $2n_0$ cannot be represented by $2x^2+2xy+3y^2$. Since $H(-20) = \{[1,0,5], [2,2,3]\}$, from the above and Lemma 2.1 we see that

$$R([1,0,5], 8n+6) = R([1,0,5], 2n_0) = N(2n_0, -20) = 2\delta(2n_0, -20).$$

From Lemma 2.4 we know that

$$R([1,0,45], 8n+6) = \begin{cases} 2\delta(2n_0/9, -20) & \text{if } 9 \mid n-6, \\ 2\delta(2n_0, -20) & \text{if } 3 \mid n-2, \\ 0 & \text{otherwise}, \end{cases}$$
$$R([5,0,9], 8n+6) = \begin{cases} 2\delta(2n_0/9, -20) & \text{if } 9 \mid n-6, \\ 2\delta(2n_0, -20) & \text{if } 3 \mid n-1, \\ 0 & \text{otherwise}. \end{cases}$$

Since $\delta(2n_0, -20) = \delta(n_0, -5)$, $\delta(2n_0/9, -20) = \delta(n_0/9, -5)$ and $n_0 = 3^\beta n_1$ $(3 \nmid n_1)$, we see that for $n \equiv 6 \pmod 9$,

$$\begin{aligned} &\delta(2n_0, -20) - \delta(2n_0/9, -20) \\ &= \sum_{k \mid n_0,\, k \nmid n_0/9} \left(\frac{-5}{k}\right) = \sum_{k \mid n_1} \left(\left(\frac{-5}{3^\beta k}\right) + \left(\frac{-20}{3^{\beta-1} k}\right) \right) = 2 \sum_{k \mid n_1} \left(\frac{-5}{k}\right). \end{aligned}$$

Now putting all the above together we obtain the result.

LEMMA 5.2 ([4, Theorem 6.1]). *Let $i, n \in \mathbb{N}$, $i \mid 30$ and $in = 2^{\alpha_i} 3^{\beta_i} 5^{\gamma_i} n_0$ with $(n_0, 30) = 1$. Let $m \in \{7, 11, 23\}$ and*

$$A_i = \begin{cases} [i, 0, 15m/i] & \text{if } 2 \nmid i, \\ \left[i, i, \frac{1}{2}\left(\frac{i}{2} + \frac{15m}{i/2}\right) \right] & \text{if } 2 \mid i. \end{cases}$$

Then $R(A_i, n) > 0$ if and only if $2 \mid \operatorname{ord}_q n_0$ for every prime q with $(\frac{-15m}{q}) = -1$, $(\frac{-1}{n_0}) = (-1)^{((m+1)/4)\alpha_i + \beta_i}$, $(\frac{n_0}{3}) = (-1)^{\alpha_i + [(m+2)/3]\beta_i + \gamma_i}$ and $(\frac{n_0}{5}) = (-1)^{\alpha_i + \beta_i + [(m-3)/5]\gamma_i}$. Moreover, if $R(A_i, n) > 0$, then

$$R(A_i, n) = 2 \prod_{(\frac{-15m}{p}) = 1} (1 + \operatorname{ord}_p n_0).$$

THEOREM 5.10. *Let $n \in \mathbb{N}$, $m \in \{7, 11, 23\}$, $a \in \{1, 5, m, 5m\}$ and $12n + (3a + 5m/a)/2 = 5^\gamma n_0$ $(5 \nmid n_0)$. Then n is represented by $a(x^2-x)/2 + \frac{5m}{a}(3y^2-y)/2$ if and only if $2 \mid \operatorname{ord}_q n_0$ for every prime q with $(\frac{-15m}{q}) = -1$*

and

$$\left(\frac{n_0}{5}\right) = \begin{cases} (-1)^{[(m-3)/5]\gamma} & \text{if } a = 1, \\ (-1)^{[(m-3)/5](\gamma+1)} & \text{if } a = 5, \\ (-1)^{1+[(m-3)/5](\gamma+1)} & \text{if } a = m, \\ (-1)^{1+[(m-3)/5]\gamma} & \text{if } a = 5m. \end{cases}$$

Moreover, if the above conditions hold, then

$$r\left(n = a(x^2 - x)/2 + \frac{5m}{a}(3y^2 - y)/2\right) = \prod_{(\frac{-15m}{p})=1} (1 + \operatorname{ord}_p n_0).$$

Proof. As $3a + 5m/a \equiv 2 \pmod 4$, we see that $R([3a, 0, 20m/a], 24n + 3a + 5m/a) = R([12a, 0, 20m/a], 24n + 3a + 5m/a) = 0$. Hence, using Lemma 5.1 we get

$$2r\left(n = a\frac{x^2 - x}{2} + \frac{5m}{a} \cdot \frac{3y^2 - y}{2}\right) = R([3a, 0, 5m/a], 24n + 3a + 5m/a).$$

From $12n + (3a + 5m/a)/2 = 5^\gamma n_0$ we deduce that $n_0 \equiv (3a + 5m/a)/2 \equiv a(3 + 5m)/2 \equiv a(m - 1)/2 \pmod 4$ and $(-1)^\gamma n_0 \equiv 5^\gamma n_0 \equiv m/a \equiv am \pmod 3$. Thus

$$\left(\frac{-1}{n_0}\right) = (-1)^{(m-1-2a)/4}, \quad \left(\frac{n_0}{3}\right) = (-1)^\gamma \left(\frac{am}{3}\right) = -(-1)^{[(m+2)/3]+\gamma} \left(\frac{a}{3}\right).$$

For $a = 1$ we have $3(24n + 3 + 5m) = 2 \cdot 3 \cdot 5^\gamma n_0$. For $a = 5$ we have $15(24n + 15 + m) = 2 \cdot 3 \cdot 5^{\gamma+1} n_0$. For $a = m$ we have $5(24n + 5 + 3m) = 2 \cdot 5^{\gamma+1} n_0$. For $a = 5m$ we have $24n + 1 + 15m = 2 \cdot 5^\gamma n_0$. Thus applying the above and Lemma 5.2 we deduce the result.

Using Lemma 5.1 and [4, Theorems 6.2 and 6.3] one can similarly prove the following result.

THEOREM 5.11. *Let $n \in \mathbb{N}$, $m \in \{13, 17\}$, $a \in \{1, 7, m, 7m\}$ and $12n + (3a + 7m/a)/2 = 7^\gamma n_0$ $(7 \nmid n_0)$. Then n is represented by $a(x^2 - x)/2 + \frac{7m}{a}(3y^2 - y)/2$ if and only if $2 \mid \operatorname{ord}_q n_0$ for every prime q with $(\frac{-21m}{q}) = -1$ and*

$$n_0 \equiv \begin{cases} 3, 5, 6 \pmod 7 & \text{if } a = 1, 7, \\ 1, 2, 4 \pmod 7 & \text{if } a = m, 7m. \end{cases}$$

Moreover, if the above conditions hold, then

$$r\left(n = a(x^2 - x)/2 + \frac{7m}{a}(3y^2 - y)/2\right) = \prod_{(\frac{-21m}{p})=1} (1 + \operatorname{ord}_p n_0).$$

Let $\mu(n)$ denote the Möbius function. Using Lemma 5.1 and [4, Theorem 8.1] one can similarly deduce the following result.

THEOREM 5.12. *Let $a, n \in \mathbb{N}$, $a \mid 455$ and $12n + (3a + 455/a)/2 = 5^\gamma 7^\delta n_0$ with $(n_0, 35) = 1$. Then n is represented by $a(x^2 - x)/2 + \frac{455}{a}(3y^2 - y)/2$*

if and only if $2\,|\,\mathrm{ord}_q\, n_0$ *for every prime* q *with* $(\frac{-1365}{q}) = -1$ *and* $(\frac{n_0}{5}) = -(\frac{n_0}{7}) = (-1)^{\gamma+\delta}\mu(a)$. *Moreover, if the above conditions hold, then*

$$r\Big(n = a(x^2-x)/2 + \frac{455}{a}(3y^2-y)/2\Big) = \prod_{(\frac{-1365}{p})=1}(1+\mathrm{ord}_p\, n_0).$$

Remark 5.1. Let $n \in \mathbb{N}$ and $4n+1 = 5^\alpha n_0 = 3^\beta n_1$ with $5 \nmid n_0$ and $3 \nmid n_1$. By [2, Theorem 4.4] we have

$$r\Big(n = \frac{3x^2-x}{2} + 5\frac{3y^2-y}{2}\Big) = \frac{1+(\frac{n_0}{5})}{2}\sum_{k\mid n_1}\Big(\frac{-5}{k}\Big). \tag{5.2}$$

Suppose $n_1 = 5^\alpha n_2$. Then $5 \nmid n_2$ and $n_0 = 3^\beta n_2$. Since $n_0 \equiv 1 \pmod 4$ and $n_1 \equiv (-1)^\beta \pmod 4$, we have $(\frac{n_0}{5}) = (\frac{5}{n_0}) = (\frac{-5}{n_0}) = (\frac{-5}{3^\beta n_2}) = (\frac{-5}{n_2})$ and $\sum_{k\mid n_1}(\frac{-5}{k}) = \sum_{k\mid n_2}(\frac{-5}{k})$. By (2.1), $\sum_{k\mid n_2}(\frac{-5}{k}) \neq 0$ implies $(\frac{n_0}{5}) = (\frac{-5}{n_2}) = \prod_{p\,\mid\, n_2}(\frac{-5}{p})^{\mathrm{ord}_p n_2} = 1$. Thus, by (5.2) we have

$$r\Big(n = \frac{3x^2-x}{2} + 5\frac{3y^2-y}{2}\Big) = \sum_{k\mid n_2}\Big(\frac{-5}{k}\Big) = \sum_{k\mid n_1}\Big(\frac{-5}{k}\Big). \tag{5.3}$$

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