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Regular positive ternary quadratic forms

by

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1. Introduction. A positive definite integral quadratic form f is called *regular* if f represents all integers that are represented by the genus of f. Regular quadratic forms were first studied systematically by Dickson in [4] where the term "regular" was coined. Jones and Pall in [9] classified all primitive positive definite diagonal regular ternary quadratic forms. In the last chapter of his doctoral thesis [15], Watson showed by arithmetic arguments that there are only finitely many equivalence classes of primitive positive definite regular ternary forms. More generally, a positive definite integral quadratic form f is called *n-regular* if f represents all quadratic forms of rank n that are represented by the genus of f. It was proved in [2] that there are only finitely many positive definite primitive n-regular forms of rank $n+3$ for $n \geq 2$. See also [13] for the structure theorem for n-regular forms in higher rank cases.

The problem of enumerating the equivalence classes of the primitive positive definite regular ternary quadratic forms was recently resurrected by Kaplansky and his collaborators [8]. They provided a list of 913 candidates for primitive positive definite regular ternary forms and stated that there are no others. All but 22 of 913 are already verified to be regular. In fact, the algorithm of [8] relies on the complete list of those regular ternary quadratic forms with square free discriminant [17] and a method of descent set forth by Watson in [15]. This method of descent involves a collection of transformations which change a regular ternary form to another one with smaller discriminant and simpler local structure, and it is this method which enables Watson to obtain the explicit discriminant bounds for regular ternary quadratic forms.

There are 794 primitive positive definite ternary quadratic forms having class number 1, and those forms are regular. If a positive ternary form f has class number greater than 1, then as far as the author knows, there is no

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general method of determining the set of all integers that are represented by f. In 1990, Duke and Schulze-Pillot proved in [5] that for any positive definite ternary form f, there is a constant C depending only on f such that every integer a greater than C is represented by f if a is primitively represented by the spinor genus of f. However, there is no known effective method of computing the constant C explicitly.

There are some methods of proving regularity of a particular ternary form f having class number greater than 1. One method uses some other form having class number 1 related to f, and some specific modularity depending on the form f (cf. [4], [7], [8], [9] and [17]). Another method is to prove that the spinor class number of f is one and there are no spinor exceptional integers (cf. [6], [10] and [11]). These two methods provide the proof of the regularity of $913 - (794 + 22) = 97$ ternary forms. Note that the second method is not available for proving regularity of the remaining 22 candidates.

In this paper, we show that the ternary form $L(i)$ (for the definition, see Table 4.1) is regular for every $i = 6, 11, 17, 18, 19, 20, 21$ and 22. Our method is quite similar to the former one explained above. However we use a ternary lattice representing the candidate, whereas the traditional method uses a genus mate, that is, a lattice in the genus of the candidate, or a sublattice of the candidate. We also use the fact that the number of representations of a by f is always finite, for any integer a and any positive definite quadratic form f.

The term *lattice* will always refer to an integral $\mathbb{Z}$-lattice on an n-dimensional positive definite quadratic space over $\mathbb{Q}$. The scale and the norm ideal of a lattice L are denoted by $\mathfrak{s}(L)$ and $\mathfrak{n}(L)$ respectively. Let $L = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n$ be a $\mathbb{Z}$-lattice of rank n. We write

$$L \simeq (B(x_i, x_j)).$$

The right hand side matrix is called a *matrix presentation* of L.

Throughout this paper, we always assume that every $\mathbb{Z}$-lattice L is *positive definite* and is *primitive* in the sense that $\mathfrak{s}(L) = \mathbb{Z}$. In particular, the $\mathbb{Z}$-lattice $L(i)$ denotes one of 22 candidates for regular ternary forms, which are defined in Table 4.1. A $\mathbb{Z}$-lattice L is called *odd* if $\mathfrak{n}(L) = \mathbb{Z}$, and *even* otherwise.

For any $\mathbb{Z}$-lattice L, $Q(\mathrm{gen}(L))$ (respectively $Q(L)$) denotes the set of all integers that are represented by the genus of L (L itself, respectively). In particular, following Kaplansky we call an integer a *eligible* if $a \in Q(\mathrm{gen}(L))$.

Any unexplained notation and terminology can be found in [12] or [14].

2. General tools. Let L be a $\mathbb{Z}$-lattice. For any positive integer m, define

$$\Lambda_m(L) = \{x \in L : Q(x+z) \equiv Q(z) \pmod m \ \text{ for all } z \in L\}.$$

The $\mathbb{Z}$-lattice $\lambda_m(L)$ denotes the primitive lattice obtained from $\Lambda_m(L)$ by scaling $L \otimes \mathbb{Q}$ by a suitable rational number. For the properties of this transformation, see [3] or [16].

LEMMA 2.1. *Let p be a prime and L be a $\mathbb{Z}$-lattice. If p is odd and a unimodular component of L_p is anisotropic, or $p = 2$ and L is odd, or $L_2 \simeq \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right) \perp \langle 4\alpha \rangle$ for some $\alpha \in \mathbb{Z}_2$, then*

$$Q(L) \cap \delta p\mathbb{Z} = Q(\Lambda_{\delta p}(L)) \quad \textit{and} \quad Q(\mathrm{gen}(L)) \cap \delta p\mathbb{Z} = Q(\mathrm{gen}(\Lambda_{\delta p}(L))),$$

where $\delta = 2$ if $p = 2$ and L is even, and $\delta = 1$ otherwise.

Proof. The proof is quite straightforward. See, for example, [3]. ∎

Under the same assumption as above, the lemma implies the following: If L is regular then $\lambda_{\delta p}(L)$ is also regular, and conversely if $\lambda_{\delta p}(L)$ is regular, then $(Q(\mathrm{gen}(L)) - Q(L)) \cap \delta p\mathbb{Z} = \emptyset$. For each $i = 1, \ldots, 22$, one may easily show that $\lambda_{\delta p}(L(i))$ is regular or $\lambda_{\delta p}(L(i)) = L(j)$ for some j, where p is any prime satisfying the condition given in the lemma. For example,

$$\lambda_3(L(17)) \simeq \begin{pmatrix} 3 & 1 & 1 \\ 1 & 7 & 3 \\ 1 & 3 & 7 \end{pmatrix},$$

which is a regular form, and

$$\lambda_3(L(8)) \simeq L(4), \qquad \lambda_3(L(1)) \simeq L(4) \quad \text{and} \quad \lambda_3(L(4)) \simeq L(1).$$

Hence if $L(8)$ is regular, then both $L(1)$ and $L(4)$ are also regular.

From now on we will use the matrix presentation of each $\mathbb{Z}$-lattice. Let M and N be any quadratic forms of rank m and n respectively and ℓ be any positive integer. We denote by $R(N, M)$ the set of all representations from N to M, that is,

$$R(N, M) = \{T \in M_{m,n}(\mathbb{Z}) \mid T^t M T = N\}.$$

Let r be any nonnegative integer less than ℓ. We define

$$R_\ell(r, N) = \{x \in M_{n,1}(\mathbb{Z}/\ell\mathbb{Z}) \mid x^t N x \equiv r \pmod{\ell}\}.$$

For any subset $S \subset M_{n,1}(\mathbb{Z})$, we define

$$\overline{S}_\ell = \{\overline{x}_\ell = (\phi(x_1), \ldots, \phi(x_n))^t \mid x = (x_1, \ldots, x_n)^t \in S\},$$

where $\phi : \mathbb{Z} \to \mathbb{Z}/\ell\mathbb{Z}$ is the natural projection map.

The following simple observation is the starting point of our method.

LEMMA 2.2. *Let a be a positive integer such that $a = x^t N x$ for some $x \in M_{n,1}(\mathbb{Z})$. If there is a $T \in R(\ell^2 N, M)$ such that $Tx \in \ell M_{m,1}(\mathbb{Z})$, then a is represented by M.*

Proof. Note that

$$\left(\frac{1}{\ell}Tx\right)^t M\left(\frac{1}{\ell}Tx\right) = \frac{1}{\ell^2}x^t(T^tMT)x = x^tNx = a.$$

The lemma follows directly from this. ∎

We define

$$E_\ell^M(r,N) = \{x \in R_\ell(r,N) \mid \forall T \in R(\ell^2N, M),\ Tx \notin \ell M_{m,1}(\mathbb{Z}/\ell\mathbb{Z})\}.$$

All computations below, like that of $E_\ell^M(r,N)$ for some M, N, r and ℓ, were done by the computer program MAPLE.

The following theorem is very useful in showing that every eligible integer of M in a certain arithmetic progression is represented by a particular quadratic form M.

THEOREM 2.3. *For any integer $a \in Q(N)$ such that $a \equiv r \pmod{\ell}$, if*

$$(*) \qquad \overline{R(a,N)}_\ell - E_\ell^M(r,N) \neq \emptyset$$

then a is represented by M. In particular, if $E_\ell^M(r,N) = \emptyset$ then

$$Q(N) \cap \{a \in \mathbb{Z} \mid a \equiv r \pmod{\ell}\} \subset Q(M).$$

Proof. Assume that there is an $x \in R(a,N)$ such that $\overline{x}_\ell \notin E_\ell^M(r,N)$. Then there is a $T \in R(\ell^2N, M)$ such that $Tx \in \ell M_{m,1}(\mathbb{Z})$. Hence the theorem follows from Lemma 2.2. ∎

3. Regular ternary forms. In this section we show that all eight forms marked with bold face in Table 4.1 are regular. Note that

$$\lambda_3(L(18)) = L(20), \quad \lambda_3(L(20)) = L(18),$$
$$\lambda_5(L(19)) = L(22), \quad \lambda_5(L(22)) = L(19).$$

Hence if $L(18)$ and $L(19)$ are regular, then so are $L(20)$ and $L(22)$. Therefore it is enough to show that $L(i)$ is regular for $i = 6, 11, 17, 18, 19, 21$.

THEOREM 3.1. *The ternary form $L(17)$ is regular.*

Proof. Let

$$M = L(17) = \begin{pmatrix} 7 & 2 & 2 \\ 2 & 8 & 0 \\ 2 & 0 & 20 \end{pmatrix}, \qquad N = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 7 & 3 \\ 1 & 3 & 7 \end{pmatrix}.$$

Note that $dM = 2^4 \cdot 3^2 \cdot 7$ and $dN = 2^4 \cdot 7$. Furthermore

$$M_3 \simeq \langle 1, 1, 3^2 \rangle \quad \text{and} \quad N_3 \simeq \langle 1, 1, 1 \rangle.$$

One can easily show that $\lambda_3(M) = N$ and M is represented by N. By a direct computation, we have

$$R(9N, M) = \left\{ \begin{pmatrix} -1 & -3 & -1 \\ 1 & 0 & -2 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -3 \\ 1 & -2 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 3 \\ -1 & 2 & 0 \\ -1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 1 \\ -1 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix} \right\},$$

and

$$R_3(1, N) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\},$$

$$R_3(2, N) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Furthermore $R(9N, N)$ contains the following four isometries:

$$\begin{pmatrix} 1 & -4 & -2 \\ -2 & -1 & -2 \\ 0 & 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & -2 & -4 \\ 0 & 3 & 0 \\ -2 & -2 & -1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 & 2 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 & 4 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix}.$$

In fact, $|R(9N, N)| = 20$ (see Table 4.2), but we only need these four representations. We denote by S_i the ith matrix given above for $i = 1, \ldots, 4$.

Let a be any eligible integer of M. Since M is represented by N and $h(N) = 1$ (cf. [8]), a is represented by N. Let $x = (x_1, x_2, x_3)^t$ be a vector such that $x^t N x = a$.

Assume that $a \equiv 0 \pmod 3$. Since the unimodular component of M_3 is anisotropic, one can easily show that M represents a by Lemma 2.1.

Assume that $a \equiv 1 \pmod 3$. In this case, one can easily show that $E_3^M(1, N) = \emptyset$. Hence M represents a by Theorem 2.3.

Finally, assume that $a \equiv 2 \pmod 3$. If $(*)$ holds, then Theorem 2.3 gives the desired conclusion that a is represented by M. So it is only necessary to consider the case that $(*)$ does not hold; that is,

$$\overline{R(a, N)_3} \subset E_3^M(2, N). \tag{3.1}$$

Note that

$$E_3^M(2, N) = \{(0, \pm 1, \pm 1)^t, (0, \pm 1, \mp 1)^t\}.$$

Hence we may further assume that

$$(x_1, x_2, x_3) \equiv (0, \pm 1, \pm 1) \quad \text{or} \quad (0, \pm 1, \mp 1) \pmod 3.$$

Assume that $x = (x_1, x_2, x_3)^t \equiv (0, \pm 1, \pm 1)^t \pmod 3$. Since $S_1, S_2 \in R(9N, N)$ and $S_1 x, S_2 x \in 3M_{3,1}(\mathbb{Z})$, it follows that

$$\frac{1}{3} S_1 x, \frac{1}{3} S_2 x \in R(a, N).$$

Hence, from the assumption (3.1), we have

$$\frac{x_1 - 4x_2 - 2x_3}{3} \equiv \frac{x_1 - 2x_2 - 4x_3}{3} \equiv 0 \pmod 3.$$

If we let $x_1 = 3s$ and $x_2 - x_3 = 3t$ for $s, t \in \mathbb{Z}$, then

$$s - 2x_3 - 4t \equiv s - 2x_3 - 2t \equiv 0 \pmod 3.$$

Therefore $t \equiv s + x_3 \equiv 0 \pmod 3$. From this it follows that

$$\frac{-2x_1 - x_2 - 2x_3}{3} \equiv x_3 \pmod 3,$$
$$x_2 \equiv \frac{-2x_1 - 2x_2 - x_3}{3} \pmod 3.$$

This implies that

$$\frac{1}{3} S_i x \equiv (0,1,1)^t \pmod 3 \quad \text{or} \quad (0,-1,-1)^t \pmod 3,$$

for $i = 1, 2$. Define a matrix T such that

$$9 \cdot T = S_1 S_2 = \begin{pmatrix} 5 & -10 & -2 \\ 2 & 5 & 10 \\ -6 & -6 & -3 \end{pmatrix}.$$

From the above observation, we have $T^n x \in R(a, N)$ for every nonnegative integer n. Since $R(a, N)$ is finite, there exist positive integers $n > m$ for which $T^n x = T^m x$, that is, $T^m(T^{n-m} - I)x = 0$. Note that there is a transition matrix P such that

$$T = P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} P,$$

where λ_1, λ_2 are the complex roots of $9t^2 + 2t + 9 = 0$. It follows that $\dim(\ker(T^{n-m} - I)) = 1$. Furthermore since $\langle (-3,1,1)^t \rangle = \ker(T - I) \subset \ker(T^{n-m} - I)$, we have

$$x \in \ker(T^{n-m} - I) = \langle (-3,1,1)^t \rangle.$$

If $x = (-3k, k, k)^t$, one can easily verify that

$$a = x^t N x = 35k^2 = (k, k, -k) M (k, k, -k)^t.$$

Now assume that $x = (x_1, x_2, x_3)^t \equiv (0, \pm 1, \mp 1)^t \pmod 3$. In this case, we may apply a similar argument by just replacing S_1 and S_2 by S_3 and S_4, respectively. This completes the proof. ■

For the quadratic form $L(18)$, we take $\ell = 8$ and

$$N = \lambda_4(L(18)) = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 7 & 3 \\ 0 & 3 & 7 \end{pmatrix},$$

which is a regular form. Note that

$$L(18)_2 \simeq \langle 7\rangle \perp \begin{pmatrix} 0 & 8 \\ 8 & 0 \end{pmatrix}.$$

Therefore we may only consider an eligible integer a of $L(18)$ that is congruent to 7 modulo 8. Since there are too many isometries, for example $|R(64N, N)| = 88$, we do not write them down here. In this case, one can easily show that

$$E_8^{L(18)}(7, N) = \{(\pm 2, \pm 4, \pm 1)^t, (\pm 2, \pm 4, \pm 5)^t\}.$$

If we choose

$$S_1 = \begin{pmatrix} 7 & 4 & -6 \\ -5 & 4 & 2 \\ 5 & 4 & 6 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 4 & 0 & 8 \\ 4 & 0 & -8 \\ -4 & -8 & 0 \end{pmatrix} \in R(64N, N),$$

then $S_i x \in 8M_{3,1}(\mathbb{Z}/8\mathbb{Z})$ for any $i = 1, 2$ and $x \in E_8^{L(18)}(7, N)$. Hence if we apply the same method described above in this situation, we can easily show that a is represented by $L(18)$. For the quadratic form $L(21)$, we take $\ell = 8$ and

$$N = \lambda_4(L(21)) = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 11 & 5 \\ 0 & 5 & 15 \end{pmatrix}.$$

Note $L(21)_2 \simeq \langle 3\rangle \perp \left(\begin{smallmatrix} 0 & 8 \\ 8 & 0 \end{smallmatrix}\right)$ and $E_8^{L(21)}(3, N) = \{(\pm 1, \pm 6, \pm 7)^t, (\pm 3, \pm 2, \pm 5)^t\}$. In this case, we may use

$$\begin{pmatrix} 8 & 8 & 0 \\ -2 & -4 & -10 \\ 2 & -4 & 2 \end{pmatrix}, \begin{pmatrix} 7 & 9 & -3 \\ -3 & -5 & -9 \\ 3 & -3 & 1 \end{pmatrix} \in R(64N, N).$$

Since all computations are quite similar to the case $L(17)$, we only provide a table containing all parameters needed for the computations, for the remaining three quadratic forms (cf. Table 4.2).

Remark 3.2. The method described in this article could also be effective even if M is not regular. For example, one can show that every eligible integer of the form $6n + 5$ is represented by the Ramanujan form

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

by taking $\ell = 3$ and $N = \begin{pmatrix} 2&0&0\\ 0&2&1\\ 0&1&3 \end{pmatrix}$. In the case when

$$M = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 6 & 0 \\ 1 & 0 & 10 \end{pmatrix},$$

one also show that every eligible integer of the form $6n+4$ is represented by M by taking $\ell = 3$ and $N = \begin{pmatrix} 4&1&1\\ 1&4&1\\ 1&1&16 \end{pmatrix}$ (cf. Lemma 8.3 of [1]). In both cases, $h(M) = 2$ and N is contained in the genus of M.

4. Tables. The regularity of eight ternary forms marked with bold face in the following table was proved in this article.

Table 4.1. 22 candidates for regular ternary forms

$$L(1) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 10 & 2 \\ 1 & 2 & 26 \end{pmatrix} \quad L(2) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 12 & 3 \\ 1 & 3 & 26 \end{pmatrix} \quad L(3) = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 10 & 2 \\ 2 & 2 & 22 \end{pmatrix}$$

$$L(4) = \begin{pmatrix} 6 & 3 & 3 \\ 3 & 10 & 3 \\ 3 & 3 & 30 \end{pmatrix} \quad L(5) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 20 & 5 \\ 1 & 5 & 58 \end{pmatrix} \quad \boldsymbol{L(6)} = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 14 & -6 \\ 1 & -6 & 44 \end{pmatrix}$$

$$L(7) = \begin{pmatrix} 10 & 2 & 1 \\ 2 & 16 & -4 \\ 1 & -4 & 22 \end{pmatrix} \quad L(8) = \begin{pmatrix} 10 & 3 & 3 \\ 3 & 18 & 9 \\ 3 & 9 & 30 \end{pmatrix} \quad L(9) = \begin{pmatrix} 10 & 3 & 5 \\ 3 & 18 & 6 \\ 5 & 6 & 34 \end{pmatrix}$$

$$L(10) = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 30 & 15 \\ 1 & 15 & 64 \end{pmatrix} \quad \boldsymbol{L(11)} = \begin{pmatrix} 14 & 2 & 7 \\ 2 & 16 & 6 \\ 7 & 6 & 46 \end{pmatrix} \quad L(12) = \begin{pmatrix} 10 & 3 & 3 \\ 3 & 18 & 0 \\ 3 & 0 & 54 \end{pmatrix}$$

$$L(13) = \begin{pmatrix} 10 & 1 & 3 \\ 1 & 26 & -6 \\ 3 & -6 & 66 \end{pmatrix} \quad L(14) = \begin{pmatrix} 18 & 6 & 3 \\ 6 & 22 & -4 \\ 3 & -4 & 58 \end{pmatrix} \quad L(15) = \begin{pmatrix} 22 & 3 & 6 \\ 3 & 30 & -3 \\ 6 & -3 & 78 \end{pmatrix}$$

$$L(16) = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 6 & 2 \\ 1 & 2 & 14 \end{pmatrix} \quad \boldsymbol{L(17)} = \begin{pmatrix} 7 & 2 & 2 \\ 2 & 8 & 0 \\ 2 & 0 & 20 \end{pmatrix} \quad \boldsymbol{L(18)} = \begin{pmatrix} 7 & 3 & 1 \\ 3 & 15 & -3 \\ 1 & -3 & 23 \end{pmatrix}$$

$$\boldsymbol{L(19)} = \begin{pmatrix} 11 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 19 \end{pmatrix} \quad \boldsymbol{L(20)} = \begin{pmatrix} 5 & 2 & 2 \\ 2 & 20 & -4 \\ 2 & -4 & 68 \end{pmatrix} \quad \boldsymbol{L(21)} = \begin{pmatrix} 11 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 51 \end{pmatrix}$$

$$\boldsymbol{L(22)} = \begin{pmatrix} 7 & 1 & 2 \\ 1 & 23 & 6 \\ 2 & 6 & 92 \end{pmatrix}$$

Table 4.2. Some data for regular ternary forms

	$d(L(i))$	N	$L(i)_3$
$L(i)$	$\pm R(9N, L(i))$		$\pm R(9N, N)$
	$\pm R_3(1, N)$		$\pm R_3(2, N)$
	$E_3^{L(i)}(1, N)$		$E_3^{L(i)}(2, N)$
	$S_1, S_2 \in R(9N, N)$		$S_3, S_4 \in R(9N, N)$
$L(6)$	$2 \cdot 3^2 \cdot 5^3$	$\lambda_3(L(6)) = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 6 & 3 \\ 1 & 3 & 14 \end{pmatrix}$	$\langle 1, 1, 3^2 \rangle$
	$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 1 \end{pmatrix}$		$\begin{pmatrix} 1 & 2 & -4 \\ 1 & 2 & 5 \\ 1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 6 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & -4 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 6 \\ 2 & 1 & -3 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 2 \\ 2 & -1 & -2 \\ 0 & 0 & 3 \end{pmatrix},$ $\begin{pmatrix} 1 & 4 & 2 \\ 2 & -1 & 1 \\ 0 & 0 & -3 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ -2 & -3 & -3 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & -3 \end{pmatrix}$
	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$		$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$
	$\emptyset$		$(0, \pm 1, \pm 1), (\pm 1, \pm 1, 0)$
	$\begin{pmatrix} 3 & 0 & 0 \\ -2 & -3 & -3 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 2 \\ 2 & -1 & 1 \\ 0 & 0 & -3 \end{pmatrix}$		$\begin{pmatrix} 1 & 2 & -4 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 6 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{pmatrix}$
$L(11)$	$2^3 \cdot 3^2 \cdot 5^3$	$\lambda_3(L(11)) = \begin{pmatrix} 6 & 3 & 2 \\ 3 & 14 & 1 \\ 2 & 1 & 14 \end{pmatrix}$	$\langle 1, 1, 3^2 \rangle$
	$\begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & -2 \\ -1 & 0 & -1 \end{pmatrix}$		$\begin{pmatrix} 1 & -1 & -4 \\ 0 & 3 & 0 \\ -2 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & -4 \\ 0 & -3 & 0 \\ -2 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & -3 \\ 1 & 0 & 3 \\ -1 & -3 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 & -3 \\ 1 & 1 & 3 \\ -1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ -1 & 0 & -3 \\ -1 & -3 & 0 \end{pmatrix},$ $\begin{pmatrix} 3 & 3 & 0 \\ -1 & -1 & -3 \\ -1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \begin{pmatrix} 3 & 3 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 3 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$
	$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$		$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$
	$\emptyset$		$(0, 0, \pm 1), (\pm 1, \pm 1, 0)$
	$\begin{pmatrix} 3 & 3 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ -1 & 0 & -3 \\ -1 & -3 & 0 \end{pmatrix}$		$\begin{pmatrix} 3 & 3 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & -4 \\ 0 & -3 & 0 \\ -2 & -1 & -1 \end{pmatrix}$
$L(17)$	$2^4 \cdot 3^2 \cdot 7$	$\lambda_3(L(17)) = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 7 & 3 \\ 1 & 3 & 7 \end{pmatrix}$	$\langle 1, 1, 3^2 \rangle$
	$\begin{pmatrix} 1 & 1 & 3 \\ -1 & 2 & 0 \\ -1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 1 \\ -1 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix}$		$\begin{pmatrix} 1 & -4 & -2 \\ -2 & -1 & -2 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & -2 & -4 \\ -2 & -2 & -1 \\ 0 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -4 & -2 \\ 0 & 0 & 3 \\ -2 & -1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & -2 & -4 \\ 0 & 3 & 0 \\ -2 & -2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 4 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix},$ $\begin{pmatrix} 1 & 4 & 4 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 2 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \end{pmatrix}$
	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$		$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$
	$\emptyset$		$(0, \pm 1, \pm 1), (0, \pm 1, \mp 1)$
	$\begin{pmatrix} 1 & -4 & -2 \\ -2 & -1 & -2 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & -2 & -4 \\ 0 & 3 & 0 \\ -2 & -2 & -1 \end{pmatrix}$		$\begin{pmatrix} 3 & 2 & 2 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 4 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix}$

<table>
<tr><td rowspan="5">$L(19)$</td><td>$2^6 \cdot 3^2 \cdot 5$</td><td>$\lambda_3(L(19)) = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 11 & 3 \\ 1 & 3 & 11 \end{pmatrix}$</td><td>$\langle 1, 1, 3^2 \rangle$</td></tr>
<tr><td colspan="2">$\begin{pmatrix} 0 & 0 & 3 \\ 1 & 2 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{pmatrix}$</td><td>$\begin{pmatrix} 1 & -4 & -4 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & -4 & -4 \\ -1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 5 \\ 0 & 3 & 0 \\ 1 & -1 & -2 \end{pmatrix}, \begin{pmatrix} 2 & 5 & 1 \\ 0 & 0 & 3 \\ 1 & -2 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 5 \\ 1 & -1 & -2 \\ 0 & 3 & 0 \end{pmatrix},$ $\begin{pmatrix} 2 & 5 & 1 \\ 1 & -2 & -1 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 2 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \end{pmatrix}$</td></tr>
<tr><td colspan="2">$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$</td><td>$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$</td></tr>
<tr><td colspan="2">$(0, \pm 1, \pm 1), (0, \pm 1, \mp 1)$</td><td>$\emptyset$</td></tr>
<tr><td colspan="2">$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 5 & 1 \\ 1 & -2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$ for $(0, \pm 1, \pm 1)$</td><td>$\begin{pmatrix} 3 & 2 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \begin{pmatrix} 1 & -4 & -4 \\ -1 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix}$ for $(0, \pm 1, \mp 1)$</td></tr>
</table>

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